Math 636: the topology and dynamics of rational maps

Homework, etc.

Version of April 17, 2018

(1) Consider the real function $f_c : \mathbb{R} \to \mathbb{R}$ given by $f_c : x \mapsto x^2 + c$ where $c \in \mathbb{R}$ is a parameter.

(a) For $c \in [-2, 1/4]$, show that there is some closed interval $I_c \subseteq [-2, 2]$, depending on c, so that $f_c(I_c) \subseteq I_c$.

(b) Show that the real map $x \mapsto x^2 - 1$ has an attracting cycle of period 2.

(c) For $c > 1/4$, show that the orbit of every $x_0 \in \mathbb{R}$ is unbounded.

(d) What happens for $c < -2$? Is the orbit of every point $x_0 \in \mathbb{R}$ unbounded?

(2) Write down your own question from class on Day 1 and answer it.

(3) Prove that the group of conformal automorphisms $\mathbb{C} \to \mathbb{C}$, $\mathrm{Aut}(\mathbb{C})$, is the affine group $\{z \mapsto \alpha z + \beta \mid \alpha, \beta \in \mathbb{C}\}$.

(4) Draw your own version of the *bifurcation diagram* from class on Day 1. How does it compare to the one you saw in class? If there is a difference, explain it.

(5) Let $f_c : \mathbb{C} \to \mathbb{C}$ be given by $f_c(z) = z^2 + c$. Recall that the *filled Julia set* of f_c is

$$K_c := \{z_0 \in \mathbb{C} \mid \text{the orbit of } z_0 \text{ is bounded}\}.$$

Prove that K_c is a nonempty, compact subset of $\mathbb{C}$.

(6) Recall that the boundary of the filled Julia set is the *Julia set* J_c of f_c. Prove that $f_c(K_c) = K_c$, and prove that $f(J_c) = J_c$.

(7) (a) Prove that the filled Julia set of the map $z \mapsto z^2$ is the closed unit disk.

(b) Prove that the filled Julia set of $z \mapsto z^2 - 2$ is the interval $[-2, 2]$.

(c) Prove that the filled Julia set of $z \mapsto z^2 + 10$ is a Cantor set.

(8) Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$.

(a) Prove that if $c \in \mathbb{R}$, then

$$z_0 \in K_c \implies \overline{z_0} \in K_c.$$

(b) Let $A : \mathbb{C} \to \mathbb{C}$ be reflection across the imaginary axis, and let $B : \mathbb{C} \to \mathbb{C}$ be reflection across the real axis (aka complex conjugation). Prove that

$$A(B(K_c)) = B(A(K_c)) = K_c.$$

(9) Consider the following candidate definitions from class on Day 1:

$$\mathcal{M}_1 = \{c \in \mathbb{C} \mid K_c \text{ is connected}\}$$
$$\mathcal{M}_2 = \{c \in \mathbb{C} \mid J_c \text{ is connected}\}$$
$$\mathcal{M}_3 = \{c \in \mathbb{C} \mid K_c \text{ has positive Lebesgue measure}\}$$
$$\mathcal{M}_4 = \{c \in \mathbb{C} \mid K_c \text{ has nonempty interior}\}$$

How are these sets related to each other? (It might help to wait on answering this question until later in the term).

The Mandelbrot set. Recall the *Mandelbrot set*

$$\mathcal{M} = \{c \in \mathbb{C} \mid K_c \text{ is connected}\}.$$

(10) Prove that $c \in \mathcal{M} \implies \bar{c} \in \mathcal{M}$.

(11) Prove that $\mathcal{M}$ is a compact subset of $\mathbb{C}$.

(12) A complex quadratic polynomial is specified by three complex parameters $p(z) = \alpha z^2 + \beta z + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{C}$. Why can we get away with studying quadratic polynomials of the form $z \mapsto z^2 + c$? Show that any quadratic polynomial $p(z) = \alpha z^2 + \beta z + \gamma$ is conjugate (via an element of $\mathrm{Aut}(\mathbb{C})$) to a unique quadratic polynomial of the form $z \mapsto z^2 + c$ (this means that the c-plane is a *moduli space* for the quadratic family).

(13) **Noah's question, Jan 4.** The triangles $T_1 \subseteq \mathbb{C}$ and $T_2 \subseteq \mathbb{C}$ are *equivalent* if there is a real affine transformation $A : \mathbb{C} \to \mathbb{C}$ so that $A(T_1) = T_2$. A periodic cycle of period 3 for the map $f_c(z) = z^2 + c$ determines a (possibly degenerate) triangle $T \subseteq \mathbb{C}$. As c runs over $\mathbb{C}$, what is the set of triangles, up to equivalence, that arise from periodic cycles of period 3 of f_c? What is the set of triangles, up to equivalence, that arise from only the attracting cycles of period 3 of f_c, as c runs over $\mathbb{C}$?

Postcritically finite. The polynomial $f : z \mapsto z^2 + c$ is *postcritically finite* of the orbit of the critical point $z_0 = 0$ is finite. That is, f is postcritically finite if the *postcritical set*

$$P = \bigcup_{n>0} f^n(0)$$

is finite.

(14) Prove that there are infinitely many parameters $c \in \mathbb{C}$ so that $f(z) = z^2 + c$ is postcritically finite.

(15) Find all parameters c so that 0 is periodic of period 7 for the map $f(z) = z^2 + c$.

(16) Prove that for all $n \geqslant 1$, the postcritical set of f^n is equal to the postcritical set of f.

Gleason polynomials. Write $f_c(z) = z^2 + c$. For $n \in \mathbb{N}$, consider the polynomial

$$Q_n(c) = f_c^n(0).$$

The roots of Q_n are called *centers* of the Mandelbrot set. Each center is contained in a connected component of the interior of $\mathcal{M}$ called a *hyperbolic component*, and each hyperbolic component contains one and only one center.

(17) Prove that the roots of Q_n are algebraic integers.

(18) Prove that the roots of Q_n are simple. Here is a start: Note that $Q_n(c) = (Q_{n-1}(c))^2 + c$, and define $\overline{Q}_n \in \mathbb{Z}[c]$ to be the polynomial Q_n mod 2. Show that the discriminant of $\overline{Q}_n$ is nonzero, and conclude that the discriminant of Q_n must be odd, and in particular, it is nonzero. This proof is originally due to Andrew Gleason.

(19) **Open Problem.** Prove or disprove: the polynomial factor of Q_n whose roots are those values of c for which 0 is periodic of period exactly n is irreducible over $\mathbb{Z}$. What is the Galois group of this polynomial?

Theorem. The topological closure of all of the centers of $\mathcal{M}$ contains the boundary of $\mathcal{M}$.

You can find a proof of this theorem in the paper *The Mandelbrot set is universal* by C. McMullen, posted on our course website.

Kneading data.

Consider the real polynomial $f_c(x) = x^2 + c$ for $c \in \mathbb{R}$.

(20) Show that if $c > 0$, then the real polynomial $f_c(x) = x^2 + c$ is not postcritically finite. (This might be a little easier a few more classes).

Let f_c be a real postcritically finite quadratic polynomial with postcritical set P. Label the elements of P as follows:

$$p_1 := c, \quad \text{and} \quad p_i := f^i(0),\ 1 \leqslant i \leqslant n.$$

Define the set $Q := f^{-1}(P)$. Note that $P \subseteq Q$, and that $Q \subseteq \mathbb{R}$. Write the set Q as

$$Q = \{q_{-n} < q_{-n+1} < \ldots, q_{-1} = 0 = q_1 < \ldots, q_{n-1} < q_n\}.$$

Definition. The *kneading data* of f_c is the map

$$k : \{1, \ldots, n\} \to \{-n, -n+1, \ldots, -2, 1, 2, \ldots, n-1, n\}$$

given by $p_i = q_{k(i)}$ for $1 \leqslant i \leqslant n$. We encode this map k with a the vector

$$\langle k(1), \ldots, k(n) \rangle.$$

This is really a vector of information that keeps track of the relative position of the postcritical points on $\mathbb{R}$.

Basic question we will think about in our class. What kneading vectors arise from (real) postcritically finite polynomials $f_c(x) = x^2 + c$?

Necessary conditions. Let's make some immediate observations about what kneading data should be allowed:

- **Distinct postcritical points.** The map $k : \{\text{indices of } P\} \to \{\text{indices of } Q\}$ should be injective.
- **Critical value condition.** We must have $k(1) = -n$.

- **Periodic or Preperiodic?** The last entry is either 1 (in the case where the critical point is periodic), or there is a unique index $1 \leqslant j < n$ for which the last entry $k(n) = -k(j)$. This is saying that if 0 is not periodic, there is a unique point in the postcritical set that has TWO inverse images p_j and p_n contained in P. Let's say that the kneading data k is *periodic* if $k(n) = 1$, and otherwise, it is *preperiodic*.

- **Monotonicity conditions.** The kneading data satisfies

$$|k(i)| < |k(j)| \implies k(i+1) < k(j+1).$$

If we begin with an abstract map $k : \{1, \ldots, n\} \to \{-n, \ldots, -2, 1, 2, \ldots, n\}$, we can ask if it is the kneading vector $\langle k(1), \ldots, k(n) \rangle$ is associated to a polynomial $x \mapsto x^2 + c$. If yes, then k must satisfy the conditions above.

Definition. We say that the (abstract) kneading map k is *admissible* if it satisfies the necessary conditions above.

(21) Write a computer program that inputs a positive integer n, and outputs a list of all periodic kneading data that is admissible.

(22) For period 6, list all admissible periodic kneading data.

(23) Find a real quadratic polynomial whose critical point is periodic of period 5, and compute the kneading vector.

(24) Prove that for all $n \geqslant 1$, there is a real quadratic polynomial $f(x) = x^2 + c$ such that the critical point $x_0 = 0$ is periodic of period n.

(25) **David's question from class, Jan 11.** Let $f(x) = x^2 + c$ be a real postcritically finite polynomial with postcritical set P. Prove that the minimum element of $Q = f^{-1}(P)$ is equal to c.

(26) **Conner's question from class, Jan 11.** Let

$$k : \{1, \ldots, n\} \to \{-n, \ldots, -2, 1, 2, \ldots, n\}$$

be a map, and let $k = \langle k(1), \ldots, k(n) \rangle$ be the associated vector, which may or may not be admissible. What happens if you try to build a piecewise linear topological map $F : [a, b] \to [a, b]$ using the recipe from class? What happens if you do the iterative *pullback algorithm* using the map $\sigma_k : ([-2, 2] \times \cdots \times [-2, 2]) \to ([-2, 2] \times \cdots \times [-2, 2])$ with the $\pm$ signs coming from k if k is not admissible?

(27) Build a piecewise linear, continuous map $f : [a, b] \to [a, b]$ with the following kneading sequence $\langle -4, 2, -3, -2 \rangle$. Can you find a polynomial with this kneading sequence?

Periodic cycles.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial, and let $z_0 \mapsto \cdots \mapsto z_{n-1}$ be a periodic cycle of period n. The *multiplier* of the cycle is $\lambda = (f^n)'(z_i)$.

- If $|\lambda| < 1$, the cycle is *attracting*. If $\lambda = 0$, the cycle is *superattracting*.
- If $|\lambda| > 1$, the cycle is *repelling*.

- If $|\lambda| = 1$, the cycle is *indifferent*.

(There are analogous definitions if f is a real polynomial).

(28) Show that if z_i and z_j are any two points in a given periodic cycle of period n, then $(f^n)'(z_i) = (f^n)'(z_j)$.

(29) Let f be a real polynomial that maps the interval $[a, b]$ to itself homeomorphically, and suppose that a and b are not critical points of f. Show that there must be a periodic cycle of f, contained in $[a, b]$, that is either attracting or indifferent.

Preperiodic kneading data. From kneading data k, we can construct a model piecewise-linear map, by interpolating between the points

$$(1, k(1)), (k(1), k(2)), \dots, (k(n-1), k(n))$$

and, in the preperiodic case with $k(n) = -k(j)$, also $(k(n), k(j+1))$.

Definition. If k is admissible kneading data, a *cycle of intervals* is a sequence

$$([a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]),$$

each one mapping homeomorphically to the next by the model piecewise-linear map. A cycle of intervals is an *obstruction* if none of the a_i or b_i are critical points.

(30) Find more examples of obstructed, admissible kneading data.

(31) Show that periodic kneading data can never be obstructed.

(32) Remember the polynomial

$$Q_n(c) = f_c^{\circ n}(0)$$

from earlier. Suppose f_c is a preperiodic polynomial with preperiod k and period ℓ, so that c is a root of $Q_k - Q_\ell$.

(a) Show that

$$Q_k - Q_\ell = (Q_{k-1} - Q_{\ell-1})(Q_{k-1} + Q_{\ell-1})$$

so c is also a root of $Q_{k-1} + Q_{\ell-1}$.

(b) What other "boring" factors of $Q_k - Q_\ell$ can you find?

(c) What is the degree of the "interesting" part of $Q_k - Q_\ell$?

(d) Is the "interesting" part always irreducible?

(33) For preperiodic data of type $(3, 2)$, we found the "interesting" part of $Q_5 - Q_2$ was the cubic polynomial $c^3 + c^2 - c + 1$ and looked at the kneading data for the real root.

(a) Find a complex root of this polynomial and draw the associated Julia set.

(b) Locate this complex point inside of the Mandelbrot set. How does its position relate the three-eared rabbit polynomial (with $c \approx 0.282 + 0.530i$)?

(34) **Alex's question, Jan 18.** Recall the "obstructed cartoon" from class on Jan 18.

We know that there is no real quadratic polynomial $x \mapsto x^2 + c$ with this cartoon. But what about polynomials of other degrees? Find a real polynomial f of degree $d > 2$ with a critical point $\star$ whose forward orbit is given by this cartoon. Can you find an example where f is postcritically finite?

(35) $\mathbb{C}$**urt's question, after hearing Noah's question, Jan 15.**Let S be the set of all rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree 2 that have an attracting fixed point at ∞. What are the triangles (up to equivalence [see Noah's question]) that arise from attracting 3-cycles of f, as f runs over S?

Definition. A *parabolic* fixed point of a rational map f is a fixed point z_0 with $|f'(z_0)| = 1$ and $f'(z_0)$ a root of unity.

(36) Show that $f(z) = z^2 - 3/4$ has a parabolic fixed point at $z_0 = -1/2$. What is the behavior of points near z_0 under iteration? Which points get attracted to z_0 and which points get repelled from z_0?

(37) **Filled Julia set and Julia set are stable under iteration.** Let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial. We define the *filled Julia set* of f to be

$$K(f) = \{z_0 \in \mathbb{C} \mid \text{ the orbit of } z_0 \text{ is bounded}\}.$$

We define the *Julia set* of f to be $J(f) = \partial K(f)$. As in the case with quadratic polynomials, both $K(f)$ and $J(f)$ are nonempty, compact subsets of $\mathbb{C}$. Prove that for all $n \in \mathbb{N}$, we have

$$K(f) = K(f^n) \quad \text{and} \quad J(f) = J(f^n).$$

Topologically vs. Geometrically attracting.

Definition. A fixed point z of a map f is *topologically attracting* if it has a neighborhood U so that the successive iterates f^n are all defined throughout U, and so that the sequence of maps

$$n \mapsto f^n|_U : U \to U$$

converges uniformly to the constant map $U \to z$.

Proposition. A fixed point for a holomorphic map of a Riemann surface is topologically attracting if and only if its multiplier satisfies $|\lambda| < 1$.

Proof. In one direction, we can use Taylor's theorem to prove that if the multiplier $|\lambda| < 1$, then the fixed point z is topologically attracting for the map f. In the other direction, if f is topologically attracting, then for any sufficiently small disk $\mathbb{D}_\epsilon$ around z there exists an iterate f^n that maps $\mathbb{D}_\epsilon$ onto a proper subset of itself. By the Schwarz Lemma, this implies that the multiplier $|\lambda^n| < 1$, and therefore $|\lambda| < 1$.

(38) Write down what it means for a fixed point z of a map f to be *topologically repelling*, and prove that your definition is equivalent to the statement that the multiplier satisfies $|\lambda| > 1$.

Basins of attraction.

Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic map with an attracting fixed point at z_0. (Maps on $\widehat{\mathbb{C}}$ also work.)

(39) Show that z_0 has a small neighborhood U so that $f(U) \subset U$.

Let $U_0 = U$ as above and, for $i \geqslant 0$, define U_{i+1} to be the connected component of $f^{-1}(U_i)$ that contains U_i.

Definition. The *immediate basin* of z_0 is

$$\bigcup_{i=0}^{\infty} U_i.$$

(40) Show that the immediate basin of z_0 is independent of the choice of U.

The Kobayashi Metric.

Motivation. We ultimately would like to prove that if a polynomial $f : \mathbb{C} \to \mathbb{C}$ has an attracting periodic cycle, then that cycle attracts a critical point. This is a really big deal in complex dynamics, and we will get a lot of mileage out of it. And in fact, the analogous statement in higher dimensions, say for a polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$, is false. There are polynomial diffeomorphisms of $\mathbb{C}^2$ with attracting periodic cycles, and these diffeomorphisms have no critical points, of course. (Google *Hènon map.*)

Definition. For $\Omega \subset \widehat{\mathbb{C}}$ a domain and $z_0 \in \Omega$, define

$$K_{z_0,\Omega} := \sup \left\{ |g'(0)| \;\middle|\; g\colon \mathbb{D} \to \Omega \text{ holomorphic}, g(0) = z_0 \right\}$$

to be the maximum derivative of any map from the disk taking 0 to z_0. (In fact, this supremum is realized by the Riemann map to Ω.) Define the *Kobayashi metric* on Ω by setting the arc length element ds at z_0 by

$$ds^2 = \frac{4\big(dx^2 + dy^2\big)}{\big(K_{z_0,\Omega}\big)^2},$$

i.e., a multiple of the standard Euclidean metric. (Lengths scale locally by a factor of $2/K_{z_0,\Omega}$.)

(41) Let $\Omega = \mathbb{H} = \{\, x + iy \mid y > 0 \,\}$. Prove that for all $z_0 \in \Omega$, $K_{z_0,\Omega}$ is finite. Then compute the Kobayashi metric on $\mathbb{H}$ via the definition above.

(42) Let $\Omega \subseteq \mathbb{C}$. Show that there exists one $z_0 \in \Omega$ for which $K_{z_0,\Omega} = \infty$ if and only if for all $z_0 \in \Omega$, we have $K_{z_0,\Omega} = \infty$.

Definition. We say that $\Omega \subseteq \widehat{\mathbb{C}}$ is *hyperbolic* if $K_{z_0,\Omega}$ is finite, so that the Kobayashi metric is actually a metric (rather than the identically-zero pseudometric).

(43) Verify, in your preferred notation for Riemannian tensors, the assertion in class that a holomorphic map $f\colon \Omega_1 \to \Omega_2$ does not increase distances: for $z, w \in \Omega_1$,

$$d_2(f(z), f(w)) \leqslant d_1(z, w).$$

(44) Suppose Ω_1 and Ω_2 are hyperbolic with Kobayashi metrics d_1 and d_2 respectively. Assume that $\Omega_1 \subseteq \Omega_2$, and let $\iota\colon \Omega_1 \hookrightarrow \Omega_2$ denote the holomorphic inclusion map. From class (and the previous exercise), we know that for all $z, w \in \Omega_1$,

$$d_2(\iota(z), \iota(w)) \leqslant d_1(z, w).$$

Show that equality is realized if and only if $\Omega_1 = \Omega_2$.

(45) Given integers $n, m \geqslant 1$, how many real parameters c are there such that the critical point of $f : x \mapsto x^2 + c$ has *preperiod* n, and period m? This means we want $f^n(0)$ to be the first point in the critical orbit that is periodic, and it is periodic of period m. Do some experiments and then look for your answer on the online encyclopedia of integer sequences.

On January 30, we finished proving this very important theorem that we will use extensively in our class:

Schwarz-Pick Theorem. Let $f : \Omega_1 \to \Omega_2$ be a holomorphic map between two hyperbolic domains. Then:

- the map f does not increase the Kobayashi metric; that is, f does not increase hyperbolic distances, and
- f is a local isometry with respect to the Kobayashi metrics on domain and range if and only if f is a covering map.

On the way to proving the Schwarz-Pick theorem, we discussed covering maps.

Covering maps.

(46) Let X and Y be locally compact Hausdorff spaces, and suppose that the map $f : X \to Y$ is proper and a local homeomorphism. Show that f is a covering map.

(47) Give an example of a local homeomorphism that is not a covering map.

(48) Give an example of a proper map that is not a covering map.

(49) Give an example of a covering map that is not proper.

(50) Suppose $f : X \to Y$ is a proper map between locally compact Hausdorff spaces. Let $U \subseteq Y$, and prove that the restriction

$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$

is a proper map. Now let V be a connected component of $f^{-1}(U)$. Prove that the restriction

$$f|_V : V \to U$$

is proper.

(51) Let $\iota : \mathbb{D} \hookrightarrow \mathbb{C}$ denote the holomorphic inclusion map. Prove that ι is not a covering map.

(52) Let $f : X \to Y$ be a covering map. Prove that the cardinality of the set $f^{-1}(y)$ is locally constant over Y. Conclude that if Y is connected, then this cardinality is constant as y runs over Y. This is called the *degree* of the covering map f; CAUTION: it may be infinite.

(53) Let $f : U \to V$ be holomorphic. Show that if $f'(u) \neq 0$, then f is a local homeomorphism at $u \in U$; that is, there is some neighborhood $W \subseteq U$ that contains u so that

$$f|_W : W \to f(W)$$

is a homeomorphism.

Riemann mapping theorem. Let $\Omega \subseteq \mathbb{C}$ be a nonempty, simply connected open subset that is not all of $\mathbb{C}$. Then there is a biholomorphic map $\phi : \Omega \to \mathbb{D}$, called a *Riemann Map* for Ω.

One can obtain this theorem from the much more powerful statement:

Uniformization theorem. There are exactly three simply connected Riemann surfaces (up to biholomorphism). They are:

$$\mathbb{C}, \quad \widehat{\mathbb{C}}, \quad \text{and} \quad \mathbb{D}.$$

Corollary. Let X be a connected Riemann surface. Then the universal cover of X is biholomorphic to exactly one of: $\mathbb{C}$, $\widehat{\mathbb{C}}$, or $\mathbb{D}$.

This is essentially saying there there are three kinds of geometry for Riemann surfaces: hyperbolic, Euclidean, and spherical. The situation is more complicated in higher dimensions.

(54) As a sanity check, prove that no two of $\mathbb{C}$, $\widehat{\mathbb{C}}$, and $\mathbb{D}$ are biholomorphic.

Facts. See Chapter 2 of the early version of Milnor's book posted on the course website for proofs of the following facts:

- **Spherical case.** If X is a Riemann surface with universal cover $\widehat{\mathbb{C}}$, then X is biholomorphic to $\widehat{\mathbb{C}}$.
- **Euclidean case.** If X is a Riemann surface with universal cover $\mathbb{C}$, then either
  - $*$ X is biholomorphic to $\mathbb{C}$,

  * X is biholomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$, which in turn is biholomorphic to the punctured plane $\mathbb{C}^*$, or
  * X is biholomorphic to $\mathbb{C}/\Lambda$, where Λ is a lattice of rank 2 (to say that Λ is a lattice means that that Λ is an additive subgroup of $\mathbb{C}$. The rank 2 part means that Λ is generated by two elements that are linearly independent over $\mathbb{R}$). In this case, X is called a *torus*.

- **Hyperbolic case.** In ALL other cases, the universal cover of X is biholomorphic to $\mathbb{D}$.

Based on the facts above, we see that most Riemann surfaces X are *hyperbolic*. This means that their universal cover is the unit disk, and they inherit a Kobayashi metric from the Poincaré metric on $\mathbb{D}$, using the universal covering map $f : \mathbb{D} \to X$ to define the metric on X locally.

Basins are hyperbolic. In some of the arguments in our class, we have used the fact that the basin of an attracting fixed point is hyperbolic. We will prove this by showing that if U is the basin of an attracting fixed point of a rational map f, then $\widehat{\mathbb{C}} - U$ contains at least three points. By the Facts above, this means that U is hyperbolic. Before we consider $\widehat{\mathbb{C}} - U$, we need some definitions and theorems. Obtaining this result is very important and requires a few pages of discussion.

Definition. A collection $\mathcal{F}$ of holomorphic functions from a Riemann surface S to a compact Riemann surface T is called a *normal family* if every infinite sequence $n \mapsto f_n \in \mathcal{F}$ contains a subsequence which converges locally uniformly to some limit $g : S \to T$.

Definition. The sequence $n \mapsto (f_n : S \to T)$ converges *locally uniformly* to $g : S \to T$ if for every compact $K \subseteq S$, the sequence

$$n \mapsto (f_n|_K : K \to T)$$

converges uniformly to $g|K : K \to T$.

Note that if the sequence $f_n : S \to T$ converges locally uniformly to $g : S \to T$, then g is necessarily holomorphic.

The following is an incredibly useful theorem from complex analysis.

Montel's Theorem, 1927. Let S be a Riemann surface, and let $\mathcal{F}$ be a collection of holomorphic maps $f : S \to \widehat{\mathbb{C}}$ what omit three different values. That is, assume that there are distinct points $a, b, c \in \widehat{\mathbb{C}}$ so that $f(S) \subseteq \widehat{\mathbb{C}} - \{a, b, c\}$ for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family.

Definition. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. Let f^n denote the nth iterate of f. Consider the family of iterates $\mathcal{F} = \{f^n : n \in \mathbb{N}\}$. The *Fatou set* of f is the largest open subset of $\widehat{\mathbb{C}}$ on which $\mathcal{F}$ is a normal family.

Definition. The complement of the Fatou set is called the *Julia set* of f, denoted $J(f)$. It is a compact subset of $\widehat{\mathbb{C}}$.

Example. Let's show that $J(f) = S^1$ for $f(z) = z^2$. Clearly $f^n(z) \to 0$ or ∞ when $|z| \neq 1$. This means that $J(f) \subseteq S^1$.

(55) For the map $f(z) = z^2$, prove that for all $z \in S^1$, there is no neighborhood of z on which the family of iterates is normal. This establishes that $S^1 \subseteq J(f)$, and therefore $J(f) = S^1$ by the remarks above.

(56) Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. Prove that the Julia set $J(f)$ is totally invariant. That is, show that

$$f^{-1}(J(f)) = J(f) = f(J(f)).$$

Hint: show that the Fatou set is totally invariant.

(57) Let f be a rational map with an attracting fixed point at $z_0 \in \widehat{\mathbb{C}}$. Let U be the basin of z_0; that is, U is the open subset of $\widehat{\mathbb{C}}$ consisting of all w so that the orbit $n \mapsto f^n(w)$ converges to z_0. Show that U is contained in the Fatou set of f.

The Riemann-Hurwitz formula. We will establish this in class on Feb 6. Let $f : X \to Y$ be a holomorphic map between compact Riemann surfaces of degree d. Then

$$\chi(X) = d \cdot \chi(Y) - \sum_{x \in X} (\deg(f, x) - 1)$$

where $\deg(f, x)$ is the *local degree* of f at x. This number is greater than 1 if and only if x is a critical point of f.

Application: rational maps on the Riemann sphere. When we apply the Riemann-Hurwitz formula to a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ we see that

$$\sum_{x \in \widehat{\mathbb{C}}} (\deg(f, x) - 1) = 2d - 2. \tag{1}$$

One way people sum this up is to say that "the number of critical points of a rational map of degree d, counted with multiplicity, is $2d - 2$". You can verify that this number really is the number of critical points of f by writing $f(z) = p(z)/q(z)$, where p and q are polynomials of degree d with no common factor. Then use the quotient rule. Ew.

Theorem. (Finite totally invariant subsets.) Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$, and let $E \subseteq \widehat{\mathbb{C}}$ be a finite totally invariant set; that is, $f^{-1}(E) = E = f(E)$. Then E has at most two elements.

Proof. Suppose that E contains $k \geqslant 1$ elements. Because E is finite, and because E is totally invariant, the restriction

$$f|_E : E \to E$$

is a permutation. This means that there is some $m \geqslant 1$ so that

$$f^m|_E : E \to E$$

is the identity. For a given $e \in E$, consider the equation $f^m(z) = e$. The map f^m has degree d^m, so there should be d^m solutions to this equation, counted with multiplicity. But because

E is totally invariant, and because f^m is the identity on E, the only solution is $z = e$. This means that

$$\deg(f^m, e) = d^m.$$

Therefore, each $e \in E$ is a critical point of the map f^m. We now apply Equation 1 to f^m to see

$$2d^m - 2 = \sum_{x \in \widehat{\mathbb{C}}} (\deg(f^m, x) - 1) \geqslant \sum_{e \in E} (\deg(f^m, e) - 1) = |E|(d^m - 1).$$

The result follows.

Grand orbits. We say that the points z and w in $\widehat{\mathbb{C}}$ have the same grand orbit under f if their orbits merge at some point. More precisely, if are integers $n, m \geqslant 1$ so that

$$f^n(z) = f^m(w).$$

Let $\mathrm{GO}(f, z)$ denote the set of all $w \in \widehat{\mathbb{C}}$ that have the same grand orbit as z.

Proposition. The set $\mathrm{GO}(f, z)$ is totally invariant under f.

(58) Prove the proposition above.

Example. Let $f(z) = z^2$. Then $\mathrm{GO}(f, 0) = \{0\}$ and $\mathrm{GO}(f, \infty) = \{\infty\}$. We also have

$$\mathrm{GO}(f, 1) = \{z \in \widehat{\mathbb{C}} \mid \text{ there is } m \geqslant 1 \text{ so that } z^{2^m} = 1\}.$$

Definition. We say that $z \in \widehat{\mathbb{C}}$ is *grand orbit finite* if $\mathrm{GO}(f, z)$ is a finite set. We define the *exceptional set* of f to be the set of all grand orbit finite points, and we denote it as $E(f)$.

Proposition. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. Then $E(f)$ contains at most two points. If $|E(f)| = 1$, then f is Möbius conjugate to a polynomial. If $|E(f)| = 2$, then f is Möbius conjugate to $z \mapsto z^{\pm d}$.

(59) Prove the proposition above.

Corollary. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. Then the exceptional set $E(f)$ is a subset of the Fatou set of f. In particular, $E(f) \cap J(f) = \varnothing$.

(60) Prove the corollary above.

Corollary^2. The Julia set $J(f)$ is either infinite or empty.

We will now prove that $J(f)$ is never empty provided that f is a rational map of degree $d \geqslant 2$.

Theorem. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. Then $J(f)$ is nonempty.

Proof. For the sake of contradiction, assume that $J(f)$ is empty. Then the Fatou set of f is equal to $\widehat{\mathbb{C}}$, so the family of iterates $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$ is normal on the whole sphere. This means that there is a subsequence

$$k \mapsto (f^{n_k} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}})$$

that converges to a holomorphic limit $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Because the map g is holomorphic on $\widehat{\mathbb{C}}$, it is necessarily rational and so it has a finite degree. Consider the sequence of degrees

$$k \mapsto \deg(f^{n_k}) = d^{n_k}.$$

Since the maps $f^{n_k} \to g$, we must have $d^{n_k} \to \deg(g)$ as $k \to \infty$. But d^{n_k} goes to ∞ since $d \geqslant 2$.

We are finally ready for our theorem.

Theorem. (Basins are hyperbolic.) Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. Let $z_0 \in \widehat{\mathbb{C}}$ be an attracting fixed point of f with basin U. Then $U \subseteq \widehat{\mathbb{C}}$ is hyperbolic.

Proof. By one of the homework problems above, the basin U is contained in the Fatou set of f, so we have $U \cap J(f) = \varnothing$. Therefore, $J(f) \subseteq \widehat{\mathbb{C}} - U$. Because $J(f)$ is infinite, the complement $\mathbb{C} - U$ contains at least three points. Therefore, U is hyperbolic.

We are finally done with that discussion.

(61) **QR Topology Problem, afternoon session, January 2009.** Let $\ldots \subseteq X_2 \subseteq X_1$ be a nested sequence of closed, nonempty, connected subsets of a compact Hausdorff space X. Prove that

$$\bigcap_{i=1}^{\infty} X_i$$

is connected.

The locus $\mathcal{H}$ in parameter space of quadratic polynomials.

Now that we know that every attracting cycle of a rational map attracts a critical point, we can do some things. Let's return to quadratic polynomials.

Proposition. The polynomial $f_c(z) = z^2 + c$ can have at most one attracting cycle.

Let's use this to define a new subset of c-parameter space:

$$\mathcal{H} = \{c \in \mathbb{C} \mid f_c \text{ has an attracting cycle}\}.$$

This is a nonempty open subset of $\mathbb{C}$. Let's define a map

$$\mathrm{per} : \mathcal{H} \to \mathbb{N}$$

that takes a $c \in \mathcal{H}$ and returns the period of the unique attracting cycle. What does the set $\mathrm{per}^{-1}(n)$ look like?

(62) Prove that $\mathrm{per}^{-1}(n)$ is nonempty and open in $\mathbb{C}$.

(63) Compute $\mathrm{per}^{-1}(2)$.

In class, we are proving this result:

Theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial. Then the filled Julia set K_f is connected iff the set of critical points $\mathrm{Crit}(f)$ is contained in K_f.

Corollary. The Mandelbrot set $\mathcal{M}$ is equal to

$$\{c \in \mathbb{C} \mid \text{the orbit of the critical point of } f_c \text{ is bounded}\}.$$

This gives us an alternative definition of $\mathcal{M}$, and in fact, it is this definition that your computer uses to test whether a given $c \in \mathbb{C}$ belongs to $\mathcal{M}$.

(64) Using the theorem above, prove that $\mathcal{H} \subseteq \mathcal{M}$.

Because $\mathcal{H}$ is an open subset of $\mathbb{C}$, we actually have that $\mathcal{H} \subseteq \text{int}(\mathcal{M})$.

(65) **The BIGGEST open problem in the ENTIRE subject.** Prove or disprove:

(Density of Hyperbolicity Conjecture or DHC)

$$\mathcal{H} = \text{int}(\mathcal{M}).$$

You will win fame and fortune if you can settle this one. One way to figure this out is to settle:

(66) **The other BIGGEST open problem in the ENTIRE subject.** Prove or disprove:

(MLC) The Mandelbrot set is locally connected.

Douady and Hubbard showed that (MLC) $\implies$ (DHC), and in fact, this is why people started trying to prove (MLC); they wanted to establish (DHC). But both problems are very hard.

(67) **Noah's question, Feb 1.** Let $f : \mathbb{C} \to \mathbb{C}$ be a quadratic polynomial, and consider the quartic polynomial f^2. The polynomial f^2 has four fixed points. Two of these fixed points are fixed points of the map f, and the other two fixed points come from a periodic cycle of period 2 for the map f. Is there any way, just given the four fixed points $\{p_1, p_2, p_3, p_4\}$ of f^2, to tell which of the p_i are fixed points of f, and which are part of the 2-cycle of f?

(68) **Dylan's question, Feb 1.** When do two rational maps have the same Julia set? In the polynomial case, the answer is completely understood - have a look at the paper *The polynomials associated with a Julia set* by Schmidt and Steinmetz on the course website.

(69) **(Noah, Connor, Jacob, and Jonathon)'s question, Feb 6.** Consider $f(z) = z^2 + c$, and take U_0, a small disk around ∞, just like we did in class. We know that the basin of ∞ is hyperbolic, so the boundary of U_0 has some hyperbolic length. Let L_0 be this length. Let L_i be the hyperbolic length of the boundary of U_i, where $U_i = f^{-i}(U_0)$. What happens to the sequence $i \mapsto L_i$ as $i \to \infty$?

(70) For each of the cubic polynomials, draw the filled Julia set, and then sketch the 'Cantor pants' for each one; that is, the surface that Dylan drew on the board at the end of class on Feb 6. It might be a good idea to use FractalStream (or another program) to help with this (see Figure 1).

(a) $p(z) = z^3 - 0.6iz + i$

(b) $p(z) = z^3 - 0.6iz + 0.5i$

(c) $p(z) = z^3 - 0.6iz + 0.5i - 2$

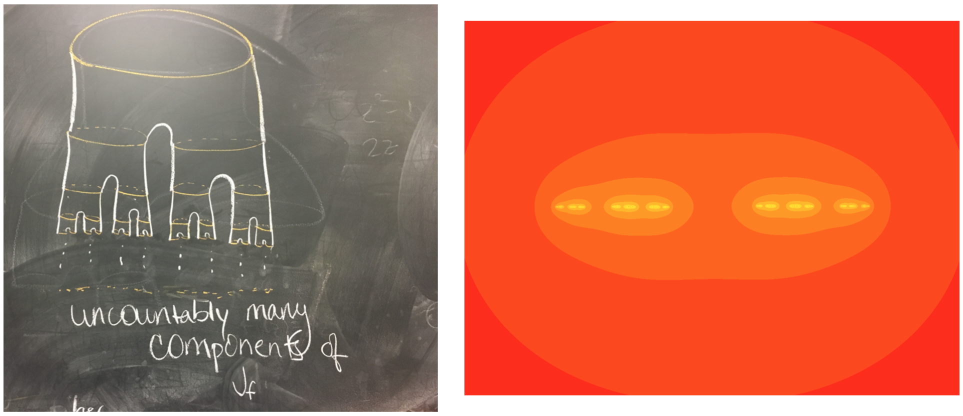


FIGURE 1. On the left, we see the Cantor pants surface from class. On the right, we see the dynamical plane for a quadratic polynomial of the form $z \mapsto z^2 + c$, where c is real and less than -2. If you project the picture on the left down onto a copy of $\mathbb{C}$, you should get the picture on the right.

(71) Consider the family of cubic polynomials $f_c(z) = z^3 - 3c^2 z + 0.4 - 0.2i$. Draw two copies of the c-parameter plane for this family. Color the first copy according to the behavior of one critical point, and color the second according to the behavior of the other critical point. You should notice something about your pictures. There is an affine transformation that will map one picture to the other. Why are these pictures related this way?

(72) Come up with a jingle to help Dylan remember the Riemann-Hurwitz formula.

(73) Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. Suppose f has a fixed point at z_0 with multiplier $\lambda = f'(z_0)$. Prove that λ does not depend on your choice of complex coordinates on $\widehat{\mathbb{C}}$. Now prove that for any polynomial $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the point at ∞ is a superattracting fixed point.

(74) Complete the argument that you do actually get a set of Cantor pants in the case where the total degree is 2 with one escaping critical point, as follows. We had set U_0 a small disk neighborhood of infinity, $U_k = f^{-k}(U_0)$, and $V_k = \overline{U_k \backslash U_{k-1}}$. Suppose that U_n is the first of these to contain the critical point.

(a) Show that V_n is a pair of pants (a sphere minus 3 disks).

(b) Show that V_{n+1} is a degree two cover of V_n. Find all topological degree two covers of a pair of pants.

(c) The boundary of V_k can be divided into an "upper" piece B_{k-1} that touches U_{k-1} and a "lower" piece B_k that touches V_{k+1}. Show that f gives a degree two covering map from B_k to B_{k-1} for all $k > 0$.

(d) Show that B_n is two circles and $f\colon B_n \to B_{n-1}$ is a homeomorphism on each component of B_n. What possibilities does this leave for V_{n+1}?

(e) Use the fact that U_{n+1} is a subset of the sphere to reduce to only one possibility for V_{n+1} and show you get an infinite tree of pairs of pants.

(75) (a) Find a cubic polynomial where one critical point escapes and the other is attracted to a periodic cycle of period 2. Draw the Julia set and describe its features.

(b) Do the same, but where one critical point is attracted to a periodic cycle of period 3. Can you find examples with pieces that look like a rabbit? An airplane?

(76) Pick a cubic polynomial example where one critical point escapes and the other is attracted to a fixed point. (For instance, you could look at one coming from previous exercises.) Some of the components of the Julia sets will have non-empty interior. Of the uncountably many components of the Julia set, only countably many can have interior, since an open subset of the plane has at most countably many connected components. Which components have non-empty interior? (You will have to figure out a scheme to describe the components of the Julia set first.)

(77) Recall the map per : $\mathcal{H} \to \mathbb{N}$. How many components are there in $\mathrm{per}^{-1}(5)$? Draw cartoons for each of them.

Sizes of Fatou components and derivatives at postcritical points.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $d \geqslant 2$. Every connected component of the interior of the filled Julia set $K(f)$ is called a *Fatou component* of the polynomial f.

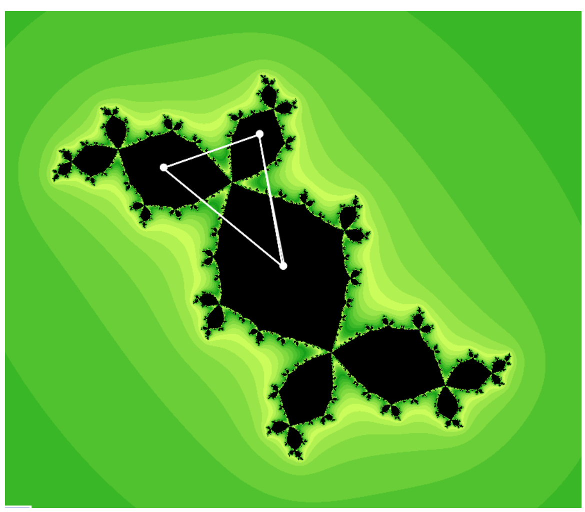


FIGURE 2. The rabbit polynomial with critical orbit marked.

Looking at the filled Julia set for the rabbit polynomial, we noticed that there is a difference in size among the three Fatou components that are periodic of period 3: namely, the body of the rabbit and his two ears. Write $p_0 = 0$, and $p_i = f^i(0)$. Label the component containing p_i as U_i. Component U_1 is smaller than U_0 because the map $f|_{U_0} : U_0 \to U_1$ is essentially $z \mapsto z^2$. But why should U_1 be smaller than U_2? As we discussed in class, to understand this phenomenon, one should investigate the derivative $f'(p_1)$.

(78) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $d \geqslant 2$. Prove that the filled Julia set of f is *full*; that is, show that $\mathbb{C} - K(f)$ is connected. *Hint:* Use the maximum modulus principle.

(79) It follows from the problem above that the connected components of the interior of $K(f)$ are homeomorphic to disks. Let U be a connected component of $\operatorname{int}(K(f))$. Show that f maps each connected component of $f^{-1}(U)$ surjectively onto U.

(80) For the rabbit polynomial f, prove that the component U_1 is smaller than U_0. And then prove that U_2 is smaller than U_1.

(81) Find a parameter $c \in \mathbb{C}$ such that for the polynomial $z \mapsto z^2 + c$, the component U_2 is smaller than U_1.

(82) Consider the family $f_\lambda(z) = -\lambda(1 - z/2)^2$. In the λ-parameter plane, draw the subset

$$\mathcal{M}' = \{\lambda \in \mathbb{C} \mid \text{the filled Julia set of } f_\lambda \text{ is connected}\}.$$

Show that $\mathcal{M}'$ is the image of the Mandelbrot set $\mathcal{M}$, scaled up by a factor of 2.

(83) **Jasmine's question, Feb 15.** Given a positive integer k, is there a quadratic polynomial $f : z \mapsto z^2 + c$ for which 0 is periodic, and in the corresponding cycle of Fatou components, U_k is the smallest? What is a systematic way to understand this in the Mandelbrot set?

(84) **Maxime's question, Feb 15.** Let f be a quadratic polynomial of the form $z \mapsto z^2 + c$, and suppose that 0 is periodic. Consider the sequence of derivatives along the postcritical points

$$2p_1, \quad 2p_2, \quad \ldots, \quad 2p_{n-1}.$$

What do these numbers measure? More specifically, are they somehow related to *Dylan's combinatorial data* (this is one of our coming attractions!)??

(85) **Sarah's question, Feb 15.** Following up on Maxime's question: what does this quantity

$$\prod_{i=1}^{n-1} \frac{1}{f'(p_i)} = \frac{1}{2^{n-1} p_1 \cdots p_{n-1}}$$

measure? *A posteriori*, this turns out to be a quantity that is geometrically meaningful, but why?

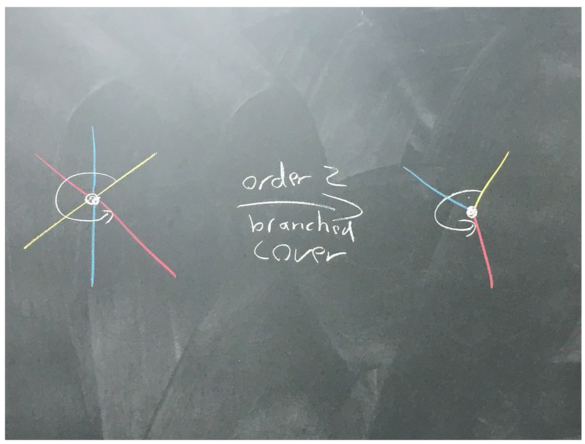
Hubbard trees, combinatorially. To describe the combinatorics of a not-necessarily-real polynomial, we generalize the kneading data by *Hubbard trees*. We consider trees as topological spaces, so, for instance, we can freely add or remove two-valent vertices, as long as they are not marked.

Definition. A *ribbon structure* on a tree (or other graph) is a choice of cyclic order on the edges incident to each vertex. (Note that there is only a choice for vertices of valence at least 3.) If a tree is embedded in $\mathbb{C}$, we give it a ribbon structure by taking the counterclockwise cyclic order at each vertex. A *ribbon graph* is a graph with a ribbon structure.

(86) Show that any ribbon tree can be embedded in $\mathbb{C}$ so that the ccw order agrees with the ribbon structure. Show that this embedding is unique up to isotopy. What is a correct statement for more general graphs?

We are interested in two special types of maps between ribbon trees. An *inclusion* $\iota\colon T_0 \hookrightarrow T_1$ between ribbon trees is a locally injective map so that the cyclic order around a vertex v of T_0 maps by ι to a subset of the cyclic order around $\iota(v)$.

Definition. A *branched covering* $f\colon T_1 \to T_0$ is a map that is a local homeomorphism (preserving the ribbon structure) at all points of T_1 except for a finite number of *branch points*. Around a branch point v of order k, the incident edges map to the incident edges around $f(v)$, repeated k times in cyclic order. Thus v is a vertex of T_1, except possibly when $k = 2$ and $f(v)$ is a leaf (1-valent vertex) of T_0.

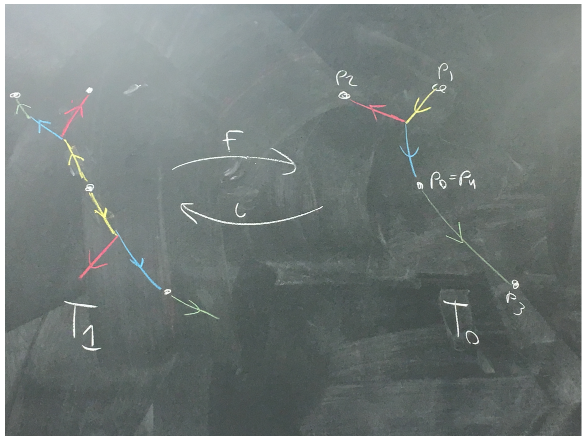


(87) Show that if $f\colon T_1 \to T_2$ is a branched cover, there is a single number n so that each edge of T_2 has n preimages in T_1.

Definition. A *combinatorial Hubbard tree* is a pair of ribbon trees T_0 and T_1, a finite marked set $P \subset T_0$, and maps $\iota\colon T_0 \to T_1$ and$f_T\colon T_1 \to T_0$ so that

- ι is an inclusion of ribbon trees and f_T is a branched covering;
- $\iota(P) \subset (f_T)^{-1}(P)$;
- P is the forward orbit of the critical values of f_T under $f_T \circ \iota$; and
- T_0 is the convex hull of P, in the sense that every point of T_0 is between two points of P. (In particular, every leaf of T_0 is a point of P.)

We showed in class that the map f_T for a combinatorial Hubbard tree can be extended, uniquely up to isotopy, to a topological branched cover $f\colon \mathbb{C} \to \mathbb{C}$, by considering the sequence of edges that you see around the boundary when you cut open the plane along the T_i.



(88) Let $f_T\colon T_1 \to T_0$ is a branched cover of total degree n. Show that if you read the edges around the boundary of T_1 in cyclic order, you see the edges around the boundary of T_0, repeated n times. (This fills in a missing step in the proof above.)

Given a combinatorial Hubbard tree, we can also consider just the self-map $f_T \circ \iota\colon T_0 \to T_0$ on the smaller tree.

(89) Suppose that you have a quadratic combinatorial Hubbard tree, with only a single branch point v of order 2. Show that the critical value $f_T(\iota(v))$ is a leaf of T_0. Conclude that in this case $f_T \circ \iota$ on T_0 determines the whole structure.

(90) Construct cubic polynomials so that the map on T_0 agrees but the whole structure (T_0, T_1, ι, f_T) is different.

(91) **Andy's question, Feb 15.** Can you find examples like above where the trees differ, even as abstract trees? That is, can you find two sets of combinatorial data $f\colon T_1 \to T_0$ and $f'\colon T_1' \to T_0'$ (plus associated ribbon data) so that $T_0 = T_0'$ and the restriction for f to T_0 agrees with the restriction of f' to T_0', but $T_1 \neq T_1'$, even as abstract trees?

Definition. Suppose that $f_1 : \mathbb{C} \to \mathbb{C}$ and $f_2 : \mathbb{C} \to \mathbb{C}$ are orientation-preserving branched covers that are postcritically finite. Let P_i be the postcritical set of f_i. We say that f_1 and f_2 are *combinatorically equivalent* if there are orientation-preserving homeomorphisms $\phi : \mathbb{C} \to \mathbb{C}$ and $\psi : \mathbb{C} \to \mathbb{C}$ such that

- the following diagram commutes

$$\begin{array}{ccc} (\mathbb{C}, P_1) & \xrightarrow{\psi} & (\mathbb{C}, P_2) \\ \Big\downarrow f_1 & & \Big\downarrow f_2 \\ (\mathbb{C}, P_1) & \xrightarrow{\phi} & (\mathbb{C}, P_2) \end{array}$$

  and

- the maps ϕ and ψ are isotopic relative to the set P_1. This means that

$$\phi|_{P_1} = \psi|_{P_1},$$

and there is a homotopy from ϕ to ψ through homeomorphisms that agree on P_1.

(92) Give an example of two distinct cubic polynomials $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ that are combinatorially equivalent.

(93) Show that if two quadratic polynomials are combinatorially equivalent, then they have the same Hubbard tree.

(94) Conversely, show that if two quadratic polynomials have the same Hubbard tree, then they are combinatorially equivalent.

(95) **Noah's question, Feb 15.** We know that the map $T_0 \to T_0$ on the smaller tree is not enough to determine the extended map on the plane uniquely in general (as Dylan showed us). Let's think about what extra information might be required - maybe something about the derivative of the map at the postcritical points as discussed above? Let's turn this into a question: can you find two distinct postcritically finite polynomials, of degree $d \geqslant 2$, with the same $T_0 \to T_0$, and with the same derivatives at corresponding points in their respective postcritical sets?

(96) Write a computer program that inputs a combinatorial Hubbard tree and outputs a polynomial that realizes the Hubbard tree (if such a polynomial exists). It might be a good idea to start with degree $d = 2$. *Hint:* consider the pullback construction that we used for the real cartoons. What happens for the $\mathbb{C}$omplex cartoons (aka Hubbard trees)?

(97) Recall the kokopelli map $f(z) = z^2 + c_K$ (see the figure below). Label the postcritical set $\{p_0, p_1, p_2, p_3\}$ as usual. Notice that the point p_3 is on kokopelli's right leg instead of her left leg. Does kokopelli have a cousin where the postcritical point p_3 is on her left leg? Why or why not?

(98) **Mandelbrot set geography.** Zoom into the component of $\text{per}^{-1}(4)$ in the Mandelbrot set where kokopelli lives, and think about the question above based on the picture you see. Click around in some of these components and study the associated Julia sets and Hubbard trees. Formulate three questions based on what you observed.

Hubbard trees, geometrically. Let's begin with a complex polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geqslant 2$ that is postcritically finite. We would like to associate a Hubbard tree to it. In class, we worked with quadratic polynomials, and we extracted a tree from the picture of the filled Julia set. To make this association precise, we require the following results from complex dynamics.

Theorem (Böttcher coordinates). Let $f(z) = z^k(1 + g(z))$ be an analytic map defined in a neighborhood $U \subseteq \mathbb{C}$ of 0, with $k \geqslant 2$. Then there exist a smaller neighborhood $V \subseteq U$ of 0 and a unique analytic map $\phi : V \to \mathbb{C}$ with $\phi'(0) = 1$ and $(\phi(z))^k = \phi(f(z))$.

Definition. The map ϕ is the *Böttcher coordinate* of the superattracting fixed point $z_0 = 0$ for f.

The map ϕ is actually injective and conjugates f to the model map $z \mapsto z^k$ in a neighborhood of 0. You can check that ϕ is injective in the following proof.

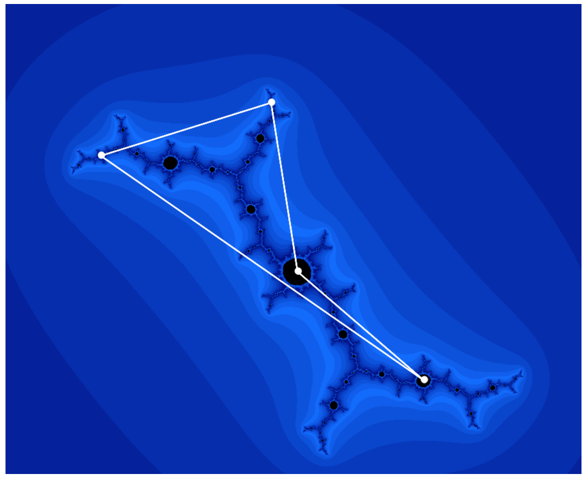


FIGURE 3. The kokopelli polynomial with critical orbit marked.

Proof. As we discussed in class, we will define a sequence of holomorphic maps $n \mapsto \phi_n$ and prove that they converge uniformly on compact subsets in a neighborhood of 0. The limit will be the map ϕ that we are looking for. Define

$$\phi_n(z) = (f^n(z))^{1/k^n}.$$

In order to make sense of this expression, we need to be careful about which root we are taking. Let's write ϕ_n as a telescoping product to think about this:

$$(f^n(z))^{1/k^n} = z \cdot \frac{(f(z))^{1/k}}{z} \cdot \frac{(f^2(z))^{1/k^2}}{(f(z))^{1/k}} \cdots \frac{(f^n(z))^{1/k^n}}{(f^{n-1}(z))^{1/k^{n-1}}}.$$

The mth term in the product is

$$\begin{aligned}\frac{(f^m(z))^{1/k^m}}{(f^{m-1}(z))^{1/k^{m-1}}} &= \frac{\left(f\left(f^{m-1}(z)\right)\right)^{1/k^m}}{\left(f^{m-1}(z)\right)^{1/k^{m-1}}} \\ &= \frac{\left(f^{m-1}(z)\right)^{k/k^m} \cdot (1+g(f^{m-1}(z)))^{1/k^m}}{\left(f^{m-1}(z)\right)^{1/k^{m-1}}} \\ &= (1+g(f^{m-1}(z)))^{1/k^m}.\end{aligned}$$

We will find $r > 0$ such that if $|z| < r$, then $|(1+g(f^{m-1}(z)))| < 1$. That way, we can get rid of the ambiguity in the choice of root by picking the principal branch of $(1+g(f^{m-1}(z)))^1/k^m$. Indeed, recall that if $|z| < 1$, then the principal root of $(1+z)^\alpha$ is defined by the binomial series

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-2)}{2} z^2 + \cdots$$

We first find $r_1 > 0$ and a constant $C > 0$ so that $|g(z)| < C|z|$ for $|z| < r_1$. Then let r_2 be the positive root of the equation

$$x^{k-1}(1+Cx) = 1,$$

and set $r := \min(r_1, r_2, 1/(2C))$. Then if $|z| < r$, we have

$$|f(z)| = |z|^k |1+g(z)| \leqslant |z| r^{k-1}(1+Cr) \leqslant z,$$

so $|f^m(z)| \leqslant r$ for all m, and

$$|g(f^{m-1}(z))| \leqslant C|f^{m-1}(z)| \leqslant \frac{C}{2C} = \frac{1}{2}.$$

Therefore, the principal branch of

$$(1 + g(f^{m-1}(z))^{1/k^m}$$

is well-defined for $|z| < r$. Now our sequence of maps $n \mapsto \phi_n(z)$ is well-defined for $|z| < r$. We will show that this sequence of maps converges for $|z| < r$. To do this, consider the associated series

$$\sum \frac{1}{k^m} \log\left(1 + g(f^{m-1}(z))\right).$$

We have

$$\left|\sum \frac{1}{k^m} \log\left(1 + g(f^{m-1}(z))\right)\right| \leqslant \sum \frac{1}{k^m} \left|\log\left(1 + g(f^{m-1}(z))\right)\right|.$$

For $|w| \leqslant 1/2$, the maximum value of $|\log(1+w)|$ is $\log 2$ which is achieved at $w = -1/2$, and so our sum is bounded above by the geometric series

$$\sum \frac{\log 2}{k^m}.$$

We have proven that $n \mapsto \phi_n(z)$ converges for $|z| < r$, and the limit is holomorphic. In fact, we can do better. We can prove that the limit is injective (see the following exercise). Let's define the open neighborhood V in the statement of the theorem to be the disk $|z| < r$.

(99) Show that the map $\phi : (V, 0) \to (V, 0)$ constructed in the theorem above is injective, and has the property that $\phi'(0) = 1$. Now show that ϕ is the unique such map with these properties.

Extending Böttcher coordinates. The previous theorem gives us a local model for the behavior of our map in a neighborhood of a superattracting fixed point. How far can we extend the map ϕ?

Let $\mathcal{B}$ be the immediate basin of $z_0 = 0$, the superattracting fixed point of f, and fix $z \in \mathcal{B}$. There is some $m > 0$ so that $f^m(z)$ is inside V. Let's try to define our Böttcher coordinate ϕ at z to be

$$\phi(z) := (f^m(z))^{1/k^m}. \tag{2}$$

This seems like a great way to extend ϕ, and it sometimes works! However, we will have trouble defining a holomorphic map this way in general because it involves choosing a root. This will be problematic if $f^m(z) = 0$, or if the basin $\mathcal{B}$ is not simply connected.

Application: polynomials. We know a bit about the circumstances under which a basin is not simply connected. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a polynomial. Then f has a superattracting fixed point at ∞, and let $\mathcal{B}$ be the basin of ∞. We know that the filled Julia set of f is connected if and only if it contains all of the critical points of f. Equivalently, $\mathcal{B}$ is simply connected if and only if it contains no critical points of f.

We can apply Böttcher's theorem to the polynomial f at the superattracting fixed point at ∞. Suppose that f has degree $d \geqslant 2$. Then Böttcher's theorem gives us a conjugacy between

f in a neighborhood of ∞ and the map $z \mapsto z^d$ (in a neighborhood of ∞). Let's write

$$\phi : (V, \infty) \to (V, \infty)$$

for our Böttcher coordinate. It turns out that we can extend ϕ, using the extension in Equation (2), until we crash into a critical point of f in the basin $\mathcal{B}$. As a consequence, if $\mathcal{B}$ contains no critical points of f, that is, if the filled Julia set $K(f)$ is connected, then we can extend our Böötcher coordinate throughout the ENTIRE basin of ∞!! Here are some really neat consequences of that. In fact, because $\phi : \mathcal{B}_{\text{rabbit}} \to \mathcal{B}_{z\mapsto z^2}$ is a conformal isomorphism, it

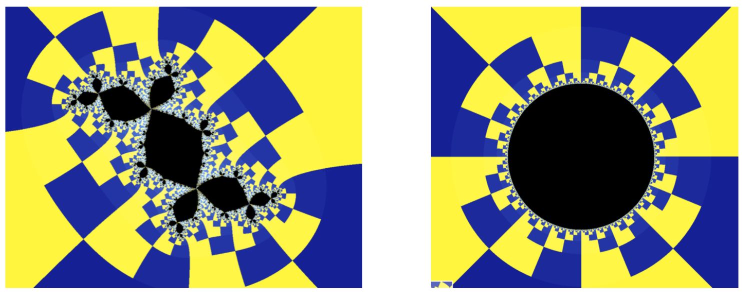


FIGURE 4. Since the filled Julia set of the rabbit polynomial is connected, the basin $\mathcal{B}_{\text{rabbit}}$ is simply connected. This means that we can extend our Böttcher coordinate through out the whole basin to a *conformal isomorphism* $\phi : \mathcal{B}_{\text{rabbit}} \to \mathcal{B}_{z\mapsto z^2}$, that conjugates the rabbit polynomial $\mathcal{B}_{\text{rabbit}} \to \mathcal{B}_{\text{rabbit}}$ to $z \mapsto z^2$ restricted to $\mathcal{B}_{z\mapsto z^2} \to \mathcal{B}_{z\mapsto z^2}$. The checkerboard pattern on the right essentially comes from polar coordinates. This structure can be pullback via ϕ to give a checkerboard pattern in the picture on the left that can be exploited to understand the rabbit polynomial asa map $\mathcal{B}_{\text{rabbit}} \to \mathcal{B}_{\text{rabbit}}$.

has a conformal inverse; $\psi : \mathcal{B}_{z\mapsto z^2} \to \mathcal{B}_{\text{rabbit}}$. Because the Julia set of the rabbit polynomial is locally connected[1], the map ψ extends continuously to the boundary of the basin to give a semi-conjugacy between the squaring map on the unit circle, and the rabbit polynomial restricted to its Julia set. Cool.

In order to define Hubbard trees in a stern & earnest way, we need to discuss internal rays.

Proposition (internal rays for quadratic polynomials). Let $p : \mathbb{C} \to \mathbb{C}$ be a quadratic polynomial whose critical point z_0 is periodic, forming the superattracting cycle $z_i = f^i(z_0)$ of period k. Let V_i be the component of the interior of the filled Julia set that contains z_i. Then

- there exists a unique homeomorphism $\eta_{V_0} : \overline{\mathbb{D}} \to \overline{V}_0$, analytic in the interior, such that
$$\eta_{V_0}(z^2) = p^k(\eta_{V_0}(z)) \quad \text{so} \quad \eta_{V_0}(0) = z_0.$$
- Let V be a connected component of the interior of the filled Julia set. Then there exists some minimal m such that $p^m : V \to V_0$ is an analytic isomorphism, so that

[1]This requires an argument

the map $\eta_V : \overline{\mathbb{D}} \to \overline{V}$ given by

$$\eta_V := (p^m|_V)^{-1} \circ \eta_{V_0}$$

is a homeomorphism, analytic in the interior $\mathbb{D} \subseteq \overline{\mathbb{D}}$.

Proof. The proof of the first bullet follows immediately from Böttcher's theorem and remarks about extending the Böttcher coordinate throughout the whole immediate basin (we can extend it until we crash into a critical point of the map).

(100) Prove the second bullet point in the proposition above.

Definition. Let p be a quadratic polynomial with periodic critical point of period k. In each connected component of the interior of the filled Julia set V, the arc

$$\eta_V(re^{2\pi i t}), 0 \leqslant r \leqslant 1$$

is called the *internal ray* of V at internal angle t. The point $\eta_V(0)$ is called the *center* of V, and the point $\eta_V(1)$ is called the *root* of V.

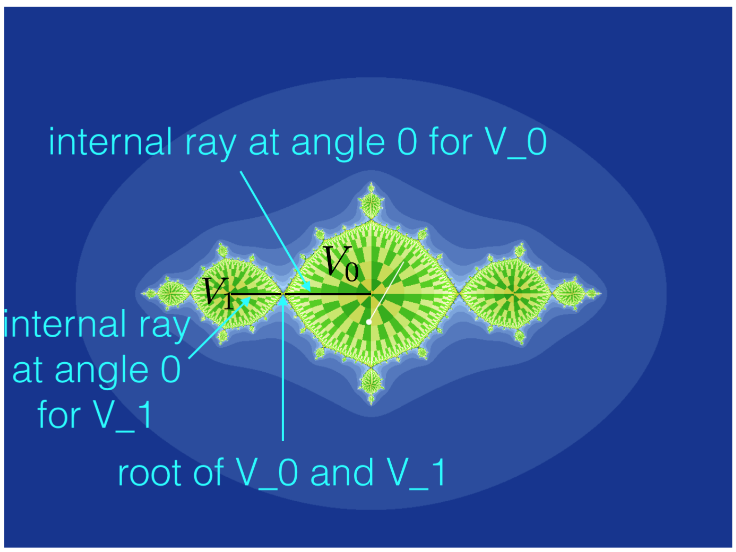


FIGURE 5. Here we can see the internal rays for the basilica. The components V_0 and V_1 are in a 2-cycle. The component V_0 contains the critical point, and V_1 contains the critical value. The second iterate of f fixes V_0, and it fixes V_1. Each restriction $f^2|_{V_i} : V_i \to V_i$ has a Böttcher coordinate that extends through all of V_i. This coordinate conjugates f^2 to the model map $z \mapsto z^2$. We use the Böttcher coordinate to transport the checkerboard pattern into V_0 and V_1. This pattern can be used to understand the polynomial f^2 on both V_0 and V_1.

Hubbard trees, finally. Let $f : \mathbb{C} \to \mathbb{C}$ be a quadratic polynomial with periodic critical point. We know then that the filled Julia set $K(f)$ is connected, and in fact, it is also locally connected (you can take this for granted, or you can see Chapter 10 in Hubbard's Teichmüller theory, vol 2 book). The filled Julia set $K(f)$ is then path connected, and any pair of points in $K(f)$ can be joined by an embedded arc in $K(f)$. We need a few properties of these arcs, established in the following exercise (Hubbard, Teich vol 2, p105):

(101) Suppose that $K \subseteq \mathbb{C}$ is compact, connected, locally connected, and full (the set K is *full* if $\mathbb{C} - K$ is connected). Prove the following statements:

- If $\Gamma \subseteq K$ is an embedded arc, then for any connected component V of the interior of K, the intersection $\overline{V} \cap \Gamma$ is connected: it is a subarc of Γ, or a single point, or it is empty.
- If $\Gamma', \Gamma'' \subseteq K$ are two embedded arcs joining the same points x and y, then:
  (a) $\Gamma' \cap \partial K = \Gamma'' \cap \partial K$
  (b) $\overline{V} \cap \Gamma' = \varnothing \iff \overline{V} \cap \Gamma'' = \varnothing$
  (c) The sets $\overline{V} \cap \Gamma'$ and $\overline{V} \cap \Gamma''$ are either both the same single point, or both arcs that enter $\overline{V}$ at the same point and exit it at the same point.

Definition. A *regulated path* in $K(f)$ is an embedded arc that intersects each component of the interior only in internal rays (hence either one ray, if the path ends at the center of the component, or two, if the path crosses the component).

Definition. The *Hubbard tree* T_f of a quadratic polynomial with a superattracting cycle is the union of all regulated paths joining pairs of points in the postcritical set.

Theorem (Douady and Hubbard, early 1980s). The Mandelbrot set is connected.

We don't have time to cover this in class. However, you can find a complete proof of this in the secret stash of course notes on the website. We are happy to talk with you more about it - in fact, if Hubbard comes to our workshop, we might ask him to explain the proof. It is one of the most beautiful applications of complex analysis I have ever seen.

Topological combinatorics of rational maps.

We will now proceed to the second half of the course, exploring the world of rational maps. For our class, we will think of a rational map as a holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Equivalently, this means that in coordinates, there are polynomials p and q so that $f(z) = p(z)/q(z)$. The degree of f is equal to the maximum of the degrees of p and q. Three examples are shown in Figure 6.

(102) Verify that these three rational maps are postcritically finite. What is the dynamics on the postcritical set?

What should a rational map be topologically? Before answering this, let's think topologically about maps between surfaces.

Definition. Let X and Y be compact, oriented topological surfaces. A continuous map $f : X \to Y$ is called a *branched covering map* provided that at every point x in the domain, there is an oriented change of coordinates in a neighborhood of x, and a change of coordinates in a neighborhood of $y = f(x)$ in which f is given by $z \mapsto z^k$ for some $k \geqslant 1$. The number k is called the *local degree* of f at x.

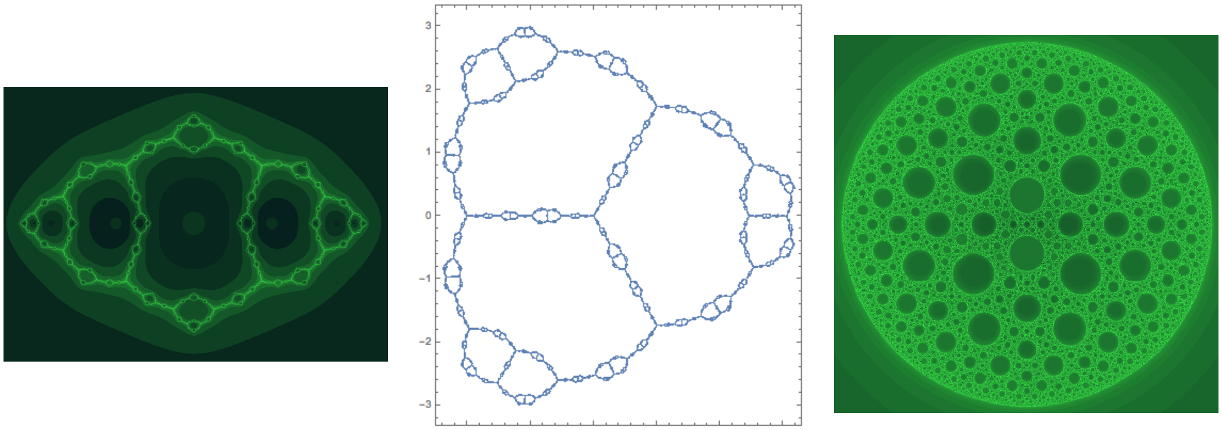


FIGURE 6. Three different Julia sets for postcritically finite rational maps, for (a) $f(z) = 1 - 1/z^2$, (b) $f(z) = z(14 + 14z^3 - z^6)/(6 + 21z^3)$, and (c) $f(z) = 4/27 \cdot (z^2 - z + 1)^3/(z(z-1))^2$.

Definition. If the local degree of f at x is greater than 1, then x is called a *critical point* of f, and $f(y)$ is called a *critical value* of f.

Definition. The degree of the branched cover $f : X \to Y$ is equal to the number of points in a generic fiber $f^{-1}(y)$.

Riemann-Hurwitz formula, topologically. The Riemann-Hurwitz formula holds in this topological setting. Suppose that $f : X \to Y$ has degree d. Then:

$$\chi(X) = d \cdot \chi(Y) - \sum_{x \in X} (\deg(f, x) - 1)$$

where $\deg(f, x)$ is the *local degree* of f at x.

Rational maps, topologically. Let S^2 denote a topological 2-sphere, equipped with an orientation. You can take the unit sphere in $\mathbb{R}^3$ if you'd like.

(103) Construct an orientation-preserving homeomorphism from S^2 to the Riemann sphere $\widehat{\mathbb{C}}$.

We will look at branched covering maps $f : S^2 \to S^2$ from the sphere to itself, of some finite degree d.

(104) Show that there are $2d - 2$ critical points of f. Show that if Σ is a closed surface other than the sphere or the torus, then there are no branched covers from Σ to Σ. Classify branched covers of the torus by itself.

To make things topological reasonable, we will restrict to the post-critically finite case.

Definition. A *W. Thurston map* or *branched self-cover* of S^2 is a branched cover $f : S^2 \to S^2$ and a finite set $P \subset S^2$ so that $f(P) \subset P$ and P contains all the critical values of f.

We use the same notion of combinatorial equivalence for polynomials, a combination of homotopy and conjugacy.

Virtual endomorphisms. Let's put this structure in a broader context.

Definition. A *virtual endomorphism* of a connected topological space X_0 is another space X_1 and a pair of maps $\pi, \phi\colon X_1 \rightrightarrows X_0$ so that π is a connected covering map of finite degree and ϕ is any continuous map.

A pcf self-cover $f\colon (S^2, P) \circlearrowleft$ gives a virtual endomorphism: set $Q = f^{-1}(P)$. The pcf property says that $P \subset Q$, so $S^2 \backslash Q \subset S^2 \backslash P$. We then have two maps $f, i\colon S^2 \backslash Q \rightrightarrows S^2 \backslash P$, where f is the restriction of the original map (now a covering map), and i is the inclusion.

The term "virtual endomorphism" comes from group theory. The theory of covering spaces tells us that finite-degree connected covers $\pi\colon X_1 \to X_0$ correspond to finite-index subgroups of $\pi_1(X_0)$. (That is, $\pi_1(X_1)$ is naturally a subgroup of $\pi_1(X_0)$.) The other map ϕ gives us a map $\phi_*\colon \pi_1(X_1) \to \pi_1(X_0)$. Putting these together, we get a partially-defined "map" $\phi_* \circ (\pi_*)^{-1}\colon \pi_1(X_0) \dashrightarrow \pi_1(X_0)$, defined on a finite-index subgroup of $\pi_1(X_0)$. We can then, for instance, start iterating this group-level map; it will be defined on smaller and smaller subgroups of $\pi_1(X_0)$ (but still of finite index).

We can also take graphical models of these virtual endomorphisms.

Definition. A *spine* for a surface Σ is a graph G embedded in Σ so that Σ deformation retracts on to G. In other words, each component of $\Sigma \backslash G$ is an annulus, with one end on G and the other end a non-compact end of Σ.

From a virtual endomorphism of surfaces $\pi_\Sigma, \phi_\Sigma\colon \Sigma_1 \rightrightarrows \Sigma_0$ (as coming from a branched self-cover), take a spine G_0 for Σ_0. Then $\pi_\Sigma^{-1}(G_0)$ is also a graph (since π_Σ is a covering map); call this graph G_1. It comes with a natural map $\pi_G\colon G_1 \to G_0$, the restriction of π_Σ. To get a proper virtual endomorphism of graphs, we need an additional map ϕ_G. This comes from the following composition:

$$\begin{array}{ccc} \Sigma_1 & \xhookrightarrow{\ \phi_\Sigma\ } & \Sigma_0 \\ i_1 \big\uparrow & & r_0 \big\downarrow \big\uparrow i_0 \\ G_1 & \xrightarrow{\ \phi_G\ } & G_0. \end{array}$$

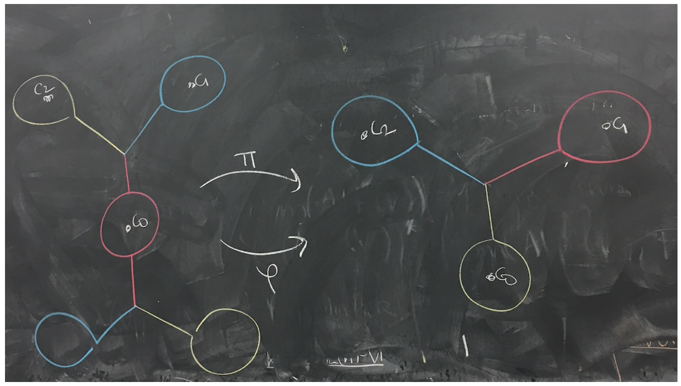
That is, we take $r_0 \circ \phi_\Sigma \circ i_1$, where

- i_1 is the inclusion of G_1 as a spine in Σ_1,
- ϕ_Σ is the inclusion of surfaces, and
- r_0 is the retraction mapping from Σ_0 to G_0, so that $r_0 \circ i_0$ is the identity and $i_0 \circ r_0$ is homotopic to the identity.

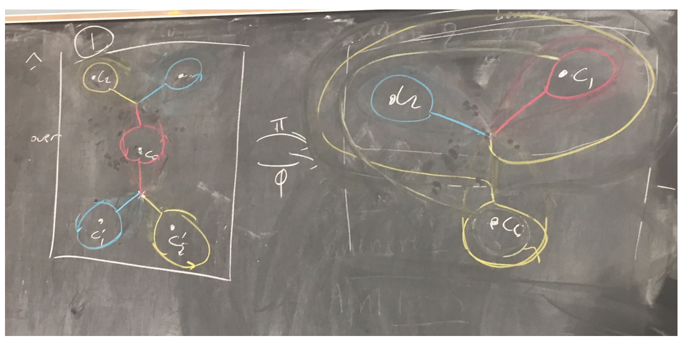
(105) Find graphical spines for one of the rational maps in Figure 6. Describe explicitly the covering map π and the map ϕ related to the inclusion. (For extra credit, do all of them.)

(106) How is the third rational map in Figure 6 related to barycentric subdivision of a triangle?

Twisting the rabbit. To see how this works, we'll do a little exercise: we will see how a twisted rabbit is equivalent to an airplane. We start with a virtual endomorphism defining the rabbit self-cover f_R:



and apply a Dehn twist to get a virtual endomorphism defining $T_\delta \circ f_R$, where T_δ is a positive (left-handed) Dehn twist along a curve δ:

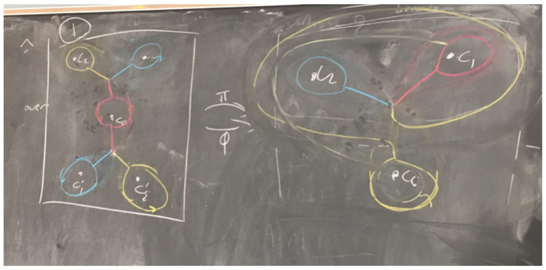


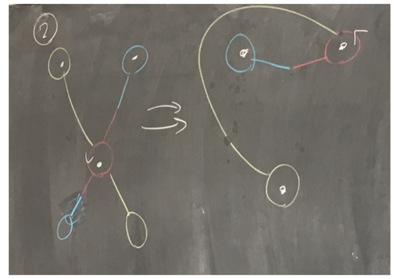
This pair of graphs G_0, G_1 in the plane can be interpreted in two ways.

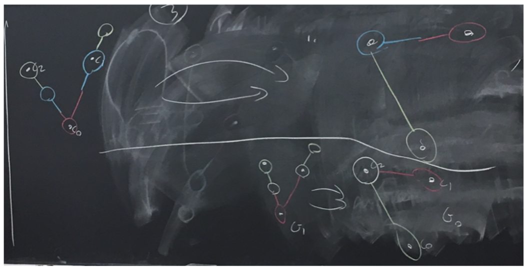
- We can think of the graphs as describing a branched self-cover directly. There is a natural covering map f_G defined on the graphs, preserving the colors and orientations. By the extension lemmas that we talked about earlier, this can be extended uniquely (up to homotopy) to a branched self-cover of the sphere.
- We can think of the graphs as describing a virtual endomorphism of graphs. The covering map is the map f_G from above. The other map ϕ_G comes from including G_1 in the plane, and then deformation retracting G_1 onto G_0.

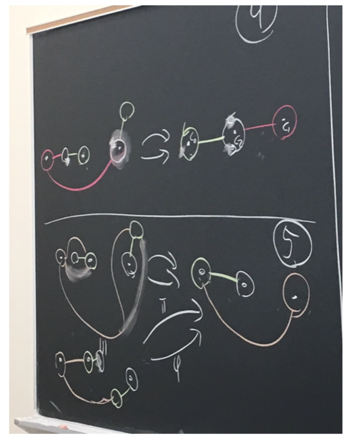
(107) Describe explicitly the map ϕ_G as a map between graphs in the twisted rabbit example above, before simplifications.

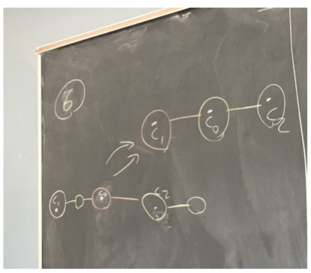
Now we want to simplify the picture, to look like one of the models we have for the rabbit, co-rabbit, or airplane. These all have the property that the graph G_0 looks like a subgraph of G_1. We will try to find this model by modifying G_0 to look more like G_1, then see how $G_1 = f^{-1}(G_0)$ changes to match. (Notice that G_0 and G_1 change simultaneously when we do this; but in some sense G_1 changes less than G_0, so we ignore the change in G_1 in deciding which change to do to G_0.)









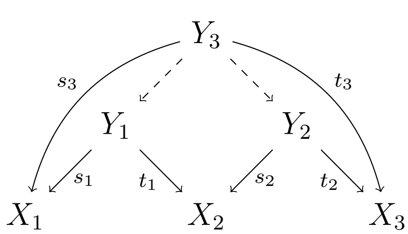


Iterating and homotopy equivalence. Just to see that we actually have dynamics, let's iterate virtual endomorphisms. First let's make an even more general setting.

Definition. A *topological correspondence* between two spaces X_1 and X_2 is a third space, Y, and maps $s\colon Y \to X_1$ and $t\colon Y \to X_2$.

You should think of a topological correspondence as giving a set of "arrows" from elements in X_1 to elements in X_2, with s giving the source of the arrow and t giving the target. For instance, a function $f\colon X_1 \to X_2$ gives a correspondence with $Y = X_1$, $s = \mathrm{id}_{X_1}$, and $t = f$. This also works if f is a partially defined function, defined on a subset $X_1' \subset X_1$; then we take $Y = X_1'$.

Definition. If we have a correspondence (Y_1, s_1, t_1) from X_1 to X_2, and a correspondence (Y_2, s_2, t_2) from X_2 to X_3, then the *composition* of the two correspondences is $Y_3 = Y_1 \times_{X_2} Y_2$, i.e., the pull-back in the diagram



with the indicated maps. Concretely, we may take

$$
\begin{aligned}
Y_3 &= \left\{ (y_1, y_2) \in Y_1 \times Y_2 \;\middle|\; t_1(y_1) = s_2(y_2) \right\} \\
s_3((y_1, y_2)) &= s_1(y_1) \\
t_3((y_1, y_2)) &= t_3(y_2).
\end{aligned}
$$

(108) Show that the definition of composition of correspondences agrees with what it should be in the case of functions or partially-defined functions.

(109) Show that "covering maps are invariant under base change": if $\pi\colon Y \to X$ is a covering map and $\phi\colon Z \to X$ is any continuous map, then the pull-back map $\phi^*(\pi)\colon Y \times_X Z \to Z$ is also a covering map. Deduce that composing two virtual endomorphisms gives a virtual endomorphism.

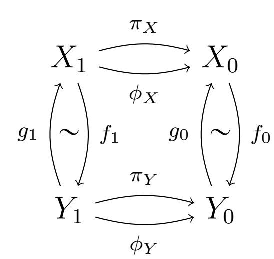
(110) Pick your favorite rational map (perhaps a polynomial, or perhaps one of the examples from Figure 6), find the corresponding virtual endomorphism of graphs, and iterate it at least twice. Compare the graphs you find with the Julia set. **Hint:** You may want to look at the poster for the course for the rational map $f(z) = (1+z^2)/(1-z^2)$.

You can also ask when two virtual endomorphisms should be considered "equivalent". (Remember, for instance, that different spines for the same rational map give different graphical models, but ultimately represent the same map.)

Definition. Let $\pi_X, \phi_X\colon X_1 \rightrightarrows X_0$ and $\pi_Y, \phi_Y\colon Y_1 \rightrightarrows Y_0$ be two virtual endomorphisms. A *morphism* between them is a pair of maps $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$ so that

$$
\begin{aligned}
f_0 \circ \pi_X &= \pi_Y \circ f_1 \\
f_0 \circ \phi_X &\sim \phi_Y \circ f_1
\end{aligned}
$$

where $\sim$ means homotopy. (Recall that ϕ_X and ϕ_Y are only defined up to homotopy.) We say that (π_x, ϕ_x) and (π_Y, ϕ_Y) are *homotopy equivalent* if there are a pair of morphisms, (f_0, f_1) from X to Y and (g_0, g_1) from Y to X, so that $f_0 \circ g_0 \sim \mathrm{id}_{Y_0}$ and $g_0 \circ f_0 \sim textrmid_{X_0}$. We can lift the homotopies to show that $f_1 \circ g_i \sim \mathrm{id}_{Y_1}$ and $g_1 \circ f_1 \sim \mathrm{id}_{X_1}$, as shown below.



(111) Show how to get a homotopy equivalence of virtual endomorphisms from each of the steps in the proof above that the twisted rabbit gives the airplane.

Multicurves. We have been using simple closed curves in the complement of a finite set P on the sphere to get interesting examples of homeomorphisms on (S^2, P), like Dehn twists. We will continue to use simple closed curves in our discussion of postcritically finite branched covers.

Definition. Let X be a compact, oriented topological surface, and let $P \subseteq X$ be a finite set. A *multicurve* on (X, P) is a collection of disjoint, simple closed curves, no two of which are homotopic relative to P. Furthermore, we require that the curves in our collection be *nonperipheral*; that is, no curve in our collection bounds a disk, and no curve in our collection bounds a once-punctured disk.

(112) Show that the maximum number of components that a multicurve on (S^2, P) can have is $|P| - 3$.

(113) Show that there are infinitely many different multicurves on (S^2, P) if $|P| \geqslant 4$.

We want to use multicurves on (S^2, P) to understand a postcritically finite branched cover $f : (S^2, P) \to (S^2, P)$ with postcritical set P. To do this, we want to study what happens to multicurves under our map. More specifically, we want to study what happens to multicurves under pullback.

(114) Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover with postcritical set P. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a multicurve on (S^2, P). Show that $f^{-1}(\Gamma)$ is a multicurve on $(S^2, f^{-1}(P))$.

(115) Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover with postcritical set P. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a multicurve on (S^2, P). Is $f(\Gamma)$ a necessarily multicurve on $(S^2, f(P))$?

Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover with postcritical set P. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a multicurve on (S^2, P). Because P contains the critical values of f, the map

$$f : (S^2, f^{-1}(P)) \to (S^2, P)$$

is an honest covering map; in particular, it is a local homeomorphism in the complement of $f^{-1}(P)$ upstairs.

The collection $f^{-1}(\Gamma)$ will be a multicurve on $(S^2, f^{-1}(P))$. In our class, we are doing *dynamics*. This means that we want to think of the multicurve $f^{-1}(\Gamma)$ on (S^2, P), and not on $(S^2, f^{-1}(P))$. Because $f(P) \subseteq P$, we have that $P \subseteq f^{-1}(P)$. We therefore have an inclusion map

$$(S^2, f^{-1}(P)) \hookrightarrow (S^2, P).$$

You can think about this map as 'erasing' the extra points in $f^{-1}(P) - P$. In doing this, we can think of the curves in $f^{-1}(\Gamma)$ as living in (S^2, P).
Paying a price: The curves in $f^{-1}(\Gamma)$ may NOT form a multicurve in (S^2, P). For example, it is possible that two of the curves in $f^{-1}(\Gamma)$ are homotopic rel P, when they were not homotopic rel $f^{-1}(P)$. Moreover, it is possible that some of the curves in $f^{-1}(P)$ are now peripheral, when they were not in the complement of $f^{-1}(P)$.

(116) Give an example of a PCF branched cover $f : (S^2, P) \to (S^2, P)$, and a multicurve Γ on (S^2, P), for which two of the curves in $f^{-1}(P)$ are homotopic rel P.

Dynamical multicurves. What should a *dynamical multicurve* be? It should somehow be a multicurve Γ on (S^2, P) for which the curves in $f^{-1}(\Gamma)$, considered in the space (S^2, P), give us the same multicurve back. Here is a definition.

Definition. Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover. The multicurve Γ in (S^2, P) is said to be *f-stable*, or *f-invariant*, if for all $\gamma \in \Gamma$, every component of $f^{-1}(\gamma)$ is either 1) peripheral (aka 'erased'), or 2) homotopic to some $\gamma' \in \Gamma$ relative to the set P.

Trivial f-stable multicurves. Dylan is not a fan of this definition. Here is one reason why: suppose we have a PCF branched cover $f : (S^2, P) \to (S^2, P)$, and a multicurve Γ that has the property that for all $\gamma \in \Gamma$, every component of $f^{-1}(\gamma)$ is erased. By the definition above, this multicurve is f-stable. We will call this a *trivial* f-stable multicurve.

Given a PCF branched cover $f : (S^2, P) \to (S^2, P)$, how do we find the f-stable multicurves?

Excellent question! It turns out that this is usually REALLY HARD. Especially given the fact that *a priori*, our map f could have *infinitely many* f-stable multicurves. Yikes!

Sarah's Conjecture, March 15. Let $f : \mathbb{C} \to \mathbb{C}$ be a PCF quadratic polynomial. Then f has NO nontrivial f-stable multicurves.

(117) Give an example of a PCF rational map f with at least two different nontrivial f-stable multicurves.

(118) (**Sarah's question, March 15.**) Does the rabbit polynomial have any nontrivial f-stable multicurves?

(119) Give an example of a PCF rational map f with a trivial f-stable multicurve.

(120) Give an example of a PCF rational map f such that EVERY multicurve in (S^2, P) is a trivial f-stable multicurve. WHAT????

(121) (**A good question for Kevin Pilgrim at our companion workshop.**) Give an example of a PCF branched cover that has *infinitely many* nontrivial f-stable mutlicurves.

(122) (**Sarah's question, March 15.**) Can a PCF rational map have infinitely many nontrivial f-stable multicurves?

(123) Prove or disprove Sarah's conjecture, stated above.

The matrix associated to an f-stable multicurve. Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be an f-stable multicurve. Consider the matrix $M_\Gamma \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ whose (i, j)-th entry is

$$\sum_{\{\alpha \in f^{-1}(\gamma_j) \,|\, \alpha \text{ is homotopic to } \gamma_i \text{ rel } P\}} \frac{1}{d_\alpha}, \qquad \text{where } d_\alpha \text{ is the degree by which } \alpha \text{ maps to } \gamma_j.$$

A few remarks are in order. First, we can see that this matrix is encoding how the nonperipheral components of $f^{-1}(\Gamma)$ map to Γ, which seems like relevant data to keep track of. We are also keeping track of degree information with d_α, but for some reason, we are inverting this number. hmm. At this point in our course, this may appear to be unmotivated - we will explain why this is the relevant information to keep track of in the next few weeks.

Perron-Frobenius Theorem, 1907. Let $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, and suppose that A has positive entries. Then A has a leading eigenvalue. That is, A has a simple eigenvalue $\lambda > 0$ with an associated eigenvector v that has positive entries. Moreover, $\lambda > |\omega|$, where ω is any other (complex) eigenvalue of A.

The leading eigenvalue of M_Γ. Our matrix M_Γ has non-negative entries (not necessarily positive entries). Since you can approximate any such matrix with a sequence of matrices with positive entries, some weaker version of the Perron-Frobenius theorem still holds in our setting. In particular, the matrix M_Γ will have a 'leading eigenvalue' $\lambda \geqslant 0$. The associated eigenvector will have non-negative entries (not necessarily positive ones). The eigenvalue λ may no longer be simple, but it will have the property that $\lambda \geqslant |\omega|$ where ω is any other (complex) eigenvalue of M_Γ.

We are just about ready to state William Thurston's Theorem, a central part of our class. We need one more definition.

Definition. Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover. The f-stable multicurve Γ is an *obstruction* if the leading eigenvalue of M_Γ is greater than or equal to 1.

William Thurston's Topological Characterization of Rational Maps, 1980s. Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover[2]. Then f is combinatorially equivalent to a rational map F if and only if f has no obstructions. If F exists, it is unique up to conjugation by Möbius transformations.

Applying this theorem in practice is notoriously difficult since it requires us to check the obstruction criterion. This means that we have find ALL f-stable multicurves and check that they are not obstructions. If you are lucky and you understand the f-stable multicurves for your branched cover f, this is an easier task. Any successful application of this theorem is usually a big deal, precisely because it is very hard to apply.

Definition. If the postcritically finite branched cover $f : (S^2, P) \to (S^2, P)$ has an obstruction, then we say that f is *obstructed*.

Twisting $z \mapsto z^2 + i$. Unlike what happens by twisting the rabbit polynomial (or any quadratic polynomial with periodic critical point), it is possible to twist a quadratic polynomial with preperiodic critical point to make it obstructed. Twisting $z \mapsto z^2 + i$ is a nice example.

Can you spot an obstruction in frame 6?

(124) Pick your favorite postcritically finite quadratic polynomial $f(z) = z^2 + c$, and let P be the postcritical set of f. Pick your favorite orientation-preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ that fixes the elements of P pointwise. Consider the composition $h \circ f$. Is it obstructed? If not, what polynomial is it equivalent to?

Matings of polynomials.
Here is a construction that leads to interesting examples of PCF branched covers.

Let S be the unit sphere in $\mathbb{C} \times \mathbb{R}$. If $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$ are two monic polynomials of the same degree $d \geqslant 2$, the formal mating of P and Q is the ramified covering $f = P \sqcup Q : S \to S$ obtained as follows.

[2] Right now, we suppose that f is not a Lattès map - this rules out a standard family of counter-examples.

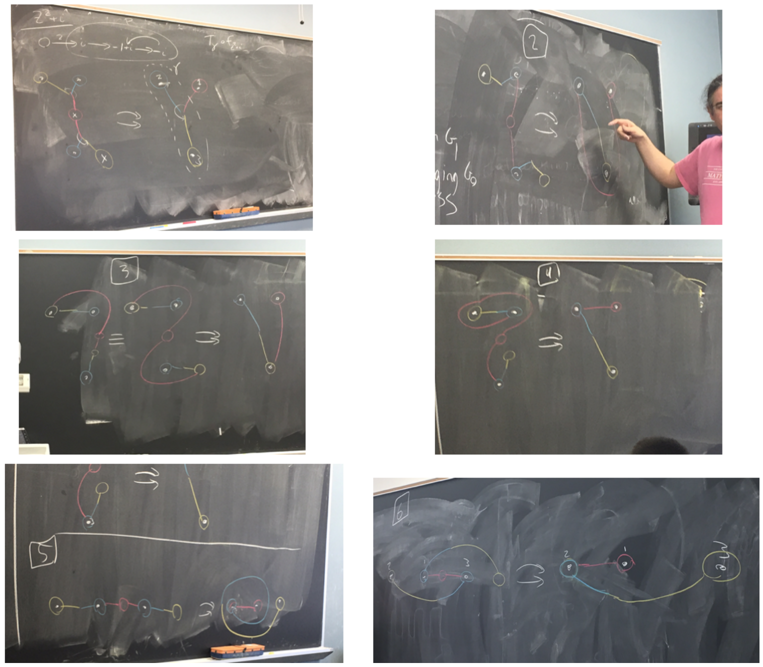


FIGURE 7. Twisting $z \mapsto z^2 + i$

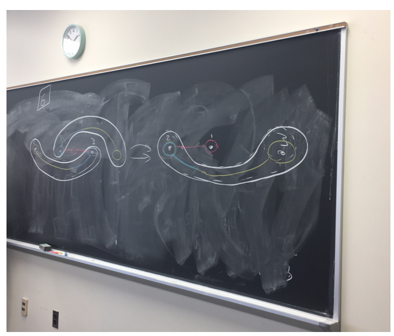


FIGURE 8. An obstruction for twisting $z \mapsto z^2 + i$.

We identify the dynamical plane of P to the upper hemisphere H^+ of S and the dynamical plane of Q to the lower hemisphere H^- of S via the gnomonic projections:

$$\nu_P : \mathbb{C} \to H^+ \quad \text{and} \quad \nu_Q : \mathbb{C} \to H^-$$

given by

$$\nu_P(z) = \frac{(z,1)}{\|(z,1)\|} = \frac{(z,1)}{\sqrt{|z|^2+1}} \quad \text{and} \quad \nu_Q(z) = \frac{(\bar{z},-1)}{\|(\bar{z},-1)\|} = \frac{(\bar{z},-1)}{\sqrt{|z|^2+1}}.$$

Since P and Q are monic polynomials of degree d, the map $\nu_P \circ P \circ \nu_P^{-1}$ defined on the upper hemisphere and $\nu_Q \circ Q \circ \nu_Q^{-1}$ defined in the lower hemisphere extend continuously to the equator of S by

$$(e^{2i\pi\theta}, 0) \mapsto (e^{2i\pi d\theta}, 0).$$

The two maps fit together so as to yield a ramified covering map $f : S \to S$, which is called the formal mating $P \sqcup Q$ of P and Q.

(125) Find an obstructing multicurve for the map f, where f is the formal mating of the basilica with itself.

Geometric mating. Let us now consider the smallest equivalence relation $\sim_{\text{ray}}$ on S such that for all $\theta \in \mathbb{R}/\mathbb{Z}$,

- points in the closure of $\nu_P\big(\mathcal{R}_P(\theta)\big)$ are in the same equivalence class, and
- points in the closure of $\nu_Q\big(\mathcal{R}_Q(\theta)\big)$ are in the same equivalence class.

In particular, for all $\theta \in \mathbb{R}/\mathbb{Z}$, $\nu_P\big(\mathcal{R}_P(\theta)\big)$ and $\nu_Q\big(\mathcal{R}_Q(-\theta)\big)$ are in the same equivalence class since the closures of these sets intersect at the point $(e^{2i\pi\theta}, 0)$ on the equator of S.

We say that a rational map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a geometric mating of P and Q if

- the quotient space $S/_{\sim_{\text{ray}}}$ is homeomorphic to S (which will have a natural orientation), and
- the formal mating $P \sqcup Q$ induces a map $S/_{\sim_{\text{ray}}} \to S/_{\sim_{\text{ray}}}$ which is topologically conjugate to $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ via an orientation preserving homeomorphism.

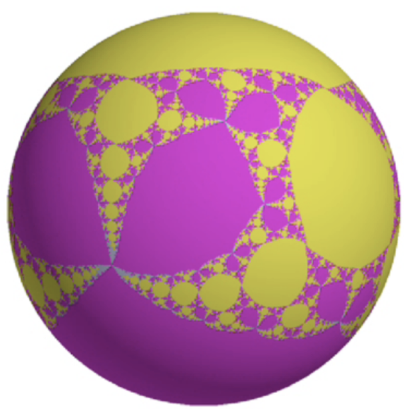


FIGURE 9. The rational map $F : z \mapsto (z^2 - e^{-2i\pi/3})/(z^2 - 1)$ is a geometric mating of the basilica polynomial and the rabbit polynomial (see Figure 1). The rational map F has a superattracting cycle of period 3 (the basin of which is colored in magenta), and a superattracting cycle of period 2 (the basin of which is colored in yellow).

Definition. Let's say the polynomial $f : \mathbb{C} \to \mathbb{C}$ is *hyperbolic* if every critical point of f is attracted to an attracting cycle of f.

(126) Prove that if f is a PCF hyperbolic polynomial, then every periodic cycle of f contained in the postcritical set is necessarily superattracting.

Theorem. (Rees) Assume $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$ are two postcritically finite hyperbolic polynomials and $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map. The formal mating $P \sqcup Q$ is combinatorially equivalent to F if and only if F is a geometric mating of P and Q.

A similar result also holds in the case P and Q are postcritically finite polynomials, not necessarily hyperbolic.

Question. Which polynomials are mateable?

Theorem. (Tan Lei, Rees, Shishikura) The polynomials $z \mapsto z^2 + c_1$ and $z \mapsto z^2 + c_2$ are mateable if and only if c_1 and c_2 do not belong to *conjugate limbs* of the Mandelbrot set.

Definition. The limbs of the Mandelbrot set are precisely the parts that are growing off of the main cardiod.

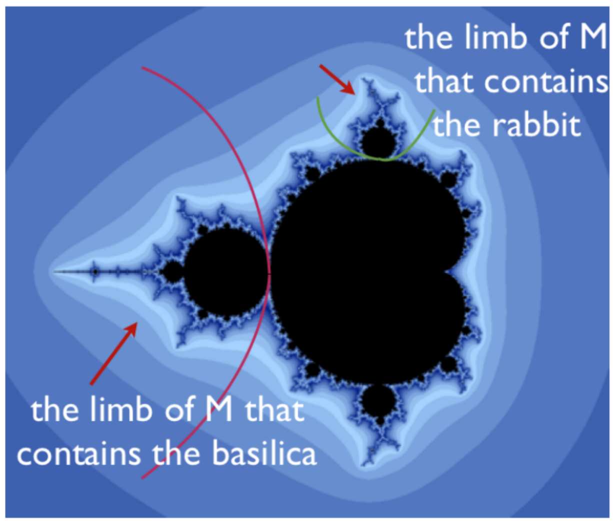


FIGURE 10. Every limb of $\mathcal{M}$ is connected to the main cardiod at a unique parabolic parameter on $\partial\mathcal{M}$. All points of $\mathcal{M}$ that can be disconnected from the main cardiod by removing this one point are in the same limb. By the theorem above, the rabbit and the basilica are mateable, but the rabbit and cokokopellli are not. The basilica and the airplane are not mateable, but the airplane and $z \mapsto z^2 + i$ are mateable.

(127) Find an example of a PCF quadratic rational function that is NOT a mating of two PCF quadratic polynomials. (*Hint:* Can you say something about the orbits of the critical points in the case of a mating?)

(128) Find an example of a *shared mating*; that is, find an example of a PCF rational map that is a mating of two polynomials in more than one way.

(129) **The definition Dylan likes:** Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover. The multicurve $\Gamma \subseteq (S^2, P)$ is f-stable if for all $\gamma \in \Gamma$, there is some $\gamma' \in \Gamma$ so

that, up to homotopy rel P, $\gamma \in f^{-1}(\gamma')$. If you use this definition, what does the corresponding matrix M_Γ look like? Give an example of a matrix that could arise as M_Γ for Γ an f-stable multicurve according to the previous definition, but could not arise if one uses this Dylan definition.

Modulus and extremal length. The uniformization theorem tells us that every Riemann surface that is homeomorphic (as a topological space) to an annulus is biholomorphic to one of the following models:

$$\left\{ \begin{array}{l} [0,\ell] \times [-\infty,\infty]/\sim \\ [0,\ell] \times [0,\infty]/\sim \\ [0,\ell] \times [0,w]/\sim \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} \mathbb{C}\backslash\{0\} \\ \mathbb{D}\backslash\{0\} \\ \{\, z \in \mathbb{C} \mid r_1 < |z| < r_2 \,\} \end{array} \right\}.$$

The left hand models are "rectangular" models, where we quotient a rectangular region by gluing vertical sides with the relation $(0,t) \sim (\ell,t)$. The right hand models are alternative "circular" models. The first two models (in either the rectangular or circular models) are (half-)infinite annuli, which we will be less concerned with.

Definition. The *modulus* of an annulus A equivalent to one of the models above is $\operatorname{Mod}(A) = w/\ell$ (rectangular model) or $\frac{1}{2\pi}\ln\left(\frac{r_2}{r_1}\right)$ (annular model). Its *extremal length* is $\operatorname{EL}(A) = 1/\operatorname{Mod}(A)$. These are defined to be ∞ or 0, respectively, for (half-)infinite annuli.

If C is a simple closed curve on a Riemann surface Σ, then the *modulus* of (the homotopy class of) C is

$$\operatorname{Mod}[C] = \sup_{A \hookrightarrow \Sigma} \operatorname{Mod}(A),$$

where the supremum runs over all conformal annuli A together with a holomorphic injection $\phi\colon A \hookrightarrow \Sigma$, so that the core curve of A maps to a curve in the homotopy class $[C]$. It is a theorem of Jenkins and/or Strebel that for non-peripheral curves C this supremum is realized, and that the supremum fills the surface: $\overline{\phi(A)} = \Sigma$. (They furthermore give more information about the supremum, relating it to quadratic differentials, which we will not get into here.)

(130) Show that $\operatorname{Mod}[C] = \infty$ if and only if C is a peripheral curve on Σ.

(131) Let Σ be a Riemann surface with a hyperbolic metric (which usually exists, by the uniformization theorem), and let C be a simple curve on Σ. Show that

$$\ell_{\text{hyp}}[C] \leqslant \pi \cdot \operatorname{EL}[C]$$

where ℓ_{hyp} means the length in the hyperbolic metric, minimized over the homotopy class.

Hint: First consider the case when Σ is an annulus A: find explicitly the unique hyperbolic structure on A from one of the models above. (It may help to think about the quotient of the upper-half plane by $z \mapsto kz$.) Then apply the Schwarz lemma.

(132) Look up the *collar lemma* from hyperbolic geometry. Use this to get an inequality the other way, an upper bound on $\operatorname{EL}[C]$ in terms of $\ell_{\text{hyp}}[C]$.

There are other definitions of extremal length/modulus.

Definition. Let $[C]$ be a homotopy class of curves on a Riemann surface. The *extremal length* of $[C]$ is

$$\mathrm{EL}[C] = \sup_{\substack{\text{conformal} \\ \text{metrics } g \text{ on } \Sigma}} \frac{\ell_g[C]^2}{\mathrm{Area}_g(\Sigma)}.$$

where:

- We take the supremum over all Riemannian metrics g on Σ, coming from the conformal class from the complex structure on Σ.
- The length $\ell_g[C]$ is the length of the curve C on the Riemannian metric g, minimized over the homotopy class.
- $\mathrm{Area}_g(\Sigma)$ is the total area of Σ with respect to the metric g.

In practice, we may restrict to metrics with finite area. (Otherwise the supremand is 0.) Note that the supremand is unchanged if we scale the metric g by a constant factor everywhere.

(133) Show that the two definitions of extremal length of an annulus agree, by showing that the Euclidean metric coming from the rectangular model of an annulus is extremal in the supremum. Do this by taking the Euclidean metric as a base metric and writing any other metric as a local rescaling of the Euclidean metric by a local factor $\rho(z)$. (You can find this written up many places, for instance in Hubbard's *Teichmüller Theory, Vol. 1*, but it is instructive to do it yourself.)

(134) Show that, for a curve C on a Riemann surface Σ, $\mathrm{EL}[C] = 0$ iff C is peripheral. (This is why we avoid peripheral curves when talking about obstructions.)

(135) Let $\Sigma = \widehat{\mathbb{C}} \setminus \{-1, 0, 1, \infty\}$, and let C be the homotopy class of a curve around 0 and 1 (in the most obvious way). What is $\mathrm{EL}[C]$? (*Hint*: First find a conformal mapping of the upper half-plane to a rectangle, taking the points $-1, 0, 1, \infty$ to the corners.) Pick a more complicated curve C_2 on the same surface Σ; can you find $\mathrm{EL}[C_2]$?

(136) More generally, let $\Sigma_\lambda = \widehat{\mathbb{C}} \setminus \{0, 1, \infty, \lambda\}$ be a sphere with four punctures, and again let C be the curve around 0 and 1. What is $\mathrm{EL}[C]$? To answer this you will need tools that we have not talked about in class, in particular the Weierstrass $\wp$ function.

This second definition of extremal length gives the Grötsch inequality: If a large annulus A contains two smaller annuli A_1 and A_2, then

$$\mathrm{Mod}(A) \geqslant \mathrm{Mod}(A_1) + \mathrm{Mod}(A_2).$$

This in turn shows that obstructions are obstructions, at least in cases that the Perron-Frobenius eigenvalue of the obstruction matrix is bigger than 1.

(137) Find an example of a PCF branched cover where (a) there is no obvious Levy cycle (so the hyperbolic geometry argument doesn't work) and (b) the leading (Perron-Frobenius) eigenvalue of the obstruction matrix is > 1.

Teichmüller theory. We are now discussing the ideas behind Bill Thurston's proof his topological characterization of rational maps. We have seen these ideas before, when we discussed real cartoons. Roughly, we will follow the same outline: given some kind of data (a kneading sequence for a real cartoon, or a PCF branched cover $f : (S^2, P) \to (S^2, P)$ in the general setting), we will define a space, and an iteration scheme on that space, so that if our iteration converges, then our initial data is realized by a rational map, and if our iteration scheme diverges, then there is a topological obstruction (which will be a collection of obstructing intervals in the case of kneading data, or an obstructing multicurve in the case of $f : (S^2, P) \to (S^2, P)$).

Let's begin with an oriented topological 2-sphere, and a finite set $B \subseteq S^2$.

Definition. The *Teichmüller space* $\mathrm{Teich}(S^2, B)$ is equal to the set

$$\mathcal{T}_B := \{\text{orientation-preserving homeomorphisms } \phi : (S^2, B) \to (\widehat{\mathbb{C}}, h(B)) \text{ such that } \phi_1 \sim_B \phi_2$$

$$\iff \exists\, \mu \in \mathrm{Aut}(\widehat{\mathbb{C}}) \text{ where } \phi_1 = \mu \circ \phi_2 \text{ on } B \text{ and } \phi_1 \text{ is isotopic to } \mu \circ \phi_2 \text{ relative to } B\}.$$

Note that an element $[\phi] \in \mathcal{T}_B$ records two pieces of data: i) information about the *points* in set B; that is $\phi|_B : B \hookrightarrow \widehat{\mathbb{C}}$, and ii) the homotopy class of $\phi : (S^2, B) \to (\widehat{\mathbb{C}}, \phi(B))$.

Fun Facts. The Teichmüller space is a complex manifold of dimension $|B| - 3$. As a complex manifold, it is isomorphic to an open bounded, simply connected subset of $\mathbb{C}^{|B|-3}$. The space $\mathcal{T}_B$ has a natural metric called the *Teichmüller metric*. In the 1970s, Royden proved that this metric coincides with the Kobayashi metric on $\mathcal{T}_B$. The upshot is that $\mathcal{T}_B$ is Kobayashi hyperbolic. (wooooo!).

Suppose that $A \subseteq B$. Thinking of marked points instead of punctures, a homeomorphism $\phi : (S^2, B) \to (\widehat{\mathbb{C}}, \phi(B))$ restricts to a homeomorphism $\phi : (S^2, A) \to (\widehat{\mathbb{C}}, \phi(A))$. Moreover, if $\phi_1 \sim_B \phi_2$, then $\phi_1 \sim_A \phi_2$ since A is contained in B. This means that there is a well-defined map $e_{B,A} : \mathcal{T}_B \to \mathcal{T}_A$ given by $e_{B,A} : [\phi]_B \mapsto [\phi]_A$. This map is a holomorphic submersion that is forgetting the points in $B - A$; we use the letter e in this notation for 'erase'.

(138) Verify the claim made above; that is, if $A \subseteq B$, show that the map $e_{B,A}$ described above is well-defined.

The pullback map. Given a PCF branched cover, we would like to define a map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$. We will do this in a few steps. Let $\phi : (S^2, P) \to (\widehat{\mathbb{C}}, \phi(P))$ represent an element of $\mathcal{T}_P$, and consider the honest covering map $f : (S^2, f^{-1}(P)) \to (S^2, P)$. Because a covering map is a local homeomorphism, we can pullback the complex structure $\phi : (S^2, P) \to (\widehat{\mathbb{C}}, \phi(P))$ by f to obtain a complex structure on $(S^2, f^{-1}(P))$. But this is just a homeomorphism

$$\psi : (S^2, f^{-1}(P)) \to \left(\widehat{\mathbb{C}}, \psi(f^{-1}(P))\right) \tag{3}$$

such that the map

$$F_{\phi,\psi} := \phi \circ f \circ \psi^{-1} : \left(\widehat{\mathbb{C}}, \psi(f^{-1}(P))\right) \to (\widehat{\mathbb{C}}, \phi(P))$$

is holomorphic (hence rational).

(139) Show that if $\psi' : (S^2, f^{-1}(P)) \to \left(\widehat{\mathbb{C}}, \psi'(f^{-1}(P))\right)$ is another homeomorphism that has the property that the map

$$F_{\phi,\psi'} := \phi \circ f \circ (\psi')^{-1} : \left(\widehat{\mathbb{C}}, \psi'(f^{-1}(P))\right) \to (\widehat{\mathbb{C}}, \phi(P))$$

is rational, then there exists a Möbius transformation μ so that $\psi = \mu \circ \psi'$, and

$$F_{\phi,\psi'} = F_{\phi,\psi} \circ \mu.$$

The previous exercise shows that the map ψ in Equation (3) is unique up to postcomposition by a Möbius transformation.

(140) Show that if $\phi_1 \sim_P \phi_2$ in $\mathcal{T}_P$, then $\psi_1 \sim_{f^{-1}(P)} \psi_2$ in $\mathcal{T}_{f^{-1}(P)}$. This essentially comes from the homotopy-lifting property for covering spaces.

As a consequence, we have a map $f^* : \mathcal{T}_P \to \mathcal{T}_{f^{-1}(P)}$ given by

$$f^* : [\phi] \mapsto [\psi].$$

Recall that $P \subseteq f^{-1}(P)$ so there is a forgetful, or erasing map

$$e_{f^{-1}(P),P} : \mathcal{T}_{f^{-1}(P)} \to \mathcal{T}_P.$$

The map we are interested is defined as

$$\sigma_f : \mathcal{T}_P \to \mathcal{T}_P, \quad \sigma_f := e_{f^{-1}(P),P} \circ f^*.$$

Definition. The map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ is called the *pullback map* associated to the PCF branched cover $f : (S^2, P) \to (S^2, P)$.

Here is one way to think about it. Let's suppose that $P = \{p_1, \ldots, p_n\}$, for $n \geqslant 3$. By postcomposing by a Möbius transformation if necessary, we may suppose that the homeomorphism $\phi : (S^2, P) \to (\widehat{\mathbb{C}}, \phi(P))$ sends p_1 to 0, sends p_2 to ∞, and sends p_3 to 1. Then:

- there is a unique homeomorphism $\psi : (S^2, P) \to (\widehat{\mathbb{C}}, \psi(P))$ so that

  $$\psi(p_1) = 0, \quad \psi(p_2) = \infty, \quad \text{and} \quad \psi(p_3) = 1,$$

  and

- there is a unique rational map $F_\phi : (\widehat{\mathbb{C}}, \psi(P)) \to (\widehat{\mathbb{C}}, \phi(P))$ so that the following diagram commutes:

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{\psi} & (\widehat{\mathbb{C}}, \psi(P)) \\ \downarrow f & & \downarrow F_\phi \\ (S^2, P) & \xrightarrow{\phi} & (\widehat{\mathbb{C}}, \phi(P)) \end{array}$$

Then the pullback map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ sends $[\phi]$ to $[\psi]$ in $\mathcal{T}_P$.

If $[\phi]$ is a fixed point of the pullback map σ_f, then there are some things we can say. Just following the definitions, this means that in the diagram above, the homeomorphisms ϕ

and ψ represent the SAME point of $\mathcal{T}_P$. This means that there is a Möbius transformation $\mu \in \mathrm{Aut}(\widehat{\mathbb{C}})$ so that

$$\phi|_P = \mu \circ \psi|_P \quad \text{and} \quad \phi \text{ is isotopic to } \mu \circ \psi \text{ rel } P.$$

But since we normalized our homeomorphisms such that

$$\phi(p_1) = \psi(p_1) = 0, \quad \phi(p_2) = \psi(p_2) = \infty, \quad \text{and} \quad \phi(p_3) = \psi(p_3) = 1,$$

the Möbius transformation μ must be the identity. As a consequence, we have

$$\phi|_P = \psi|_P.$$

(141) Show that if $\phi|_P = \psi|_P$, then the map

$$F_\phi : (\widehat{\mathbb{C}}, \psi(P)) \to (\widehat{\mathbb{C}}, \phi(P))$$

is postcritically finite, and the postcritical set of F_ϕ is equal to $\psi(P)$. Then show that the ramification portrait of F_ϕ is isomorphic (as a weighted directed graph) to the ramification portrait of our original branched cover $f : (S^2, P) \to (S^2, P)$.

So if $\phi|_P = \psi|_P$, then F_ϕ is PCF. Good. But we can do better. If $[\phi] = [\psi]$ in $\mathcal{T}_P$, then we also know that ϕ is isotopic to ψ relative to P. This means EXACTLY that the pair (ϕ, ψ) provides a combinatorial equivalence between the topological map $f : (S^2, P) \to (S^2, P)$ and the rational map $F_\phi : (\widehat{\mathbb{C}}, \psi(P)) \to (\widehat{\mathbb{C}}, \phi(P))$.

(142) Show that $f : (S^2, P) \to (S^2, P)$ is combinatorially equivalent to a rational map if and only if $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ has a fixed point.

This is the first amazing idea behind Bill Thurston's theorem: reduce the question in the theorem to a fixed point problem. This is already a good step. Now, if we only knew something about the map σ_f so that we might use some topology to draw conclusions about its fixed points...

Proposition. The map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ is a weak contraction with respect to the Teichmüller metric.

Proof. This follows from the fact that the Teichmüller metric coincides with the Kobayashi metric on $\mathcal{T}_P$ (Royden's theorem), and from the fact that the map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ is holomorphic.

Proposition. If f is not a Lattès map, the second iterate of σ_f is a strict contraction.

Corollary. Fix some $\tau \in \mathcal{T}_P$. If σ_f has a fixed point, then the sequence $n \mapsto \sigma_f^n(\tau)$ will converge to it.

Since Teichmüller space is a complete metric space with respect to the Teichmüller metric, we would really like to appeal to the Contraction Mapping Fixed Point Theorem to conclude that σ_f always has a unique fixed point. We can't do this though. The reason is that σ_f is not a uniform contraction on $\mathcal{T}_P$. The amount by which it contracts depends on where we are in $\mathcal{T}_P$. BUT if we know that the sequence of iterates $n \mapsto \sigma_f^n(\tau)$ stays in a compact subset of $\mathcal{T}_P$, then over this compact subset, we have uniform contraction, and we can conclude that the sequence $n \mapsto \sigma_f^n(\tau)$ converges, and it therefore converges to a fixed point.

Corollary. Suppose that f is not a Lattès map. Then if σ_f has a fixed point, it is unique.

Bill Thurston's theorem says that f is equivalent to a rational map if and only if f has no obstructions. Thus far in the discussion, we know that f is equivalent to a rational map if and only if the pullback map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ has a fixed point. In order to complete our discussion of the ideas behind Bill Thurston's theorem, we need to establish the following statement:

Statement. The map σ_f has no fixed point if and only if the map f has an obstruction.

We already indirectly discussed one of the implications in the Statement. In class, Dylan went over some ideas about why rational maps cannot have obstructing curves. In the Statement above, this addresses the direction: if f has an obstruction, then σ_f has no fixed point. We now address the harder direction: if σ_f has no fixed point, then f has an obstruction. One immediate question is: where the heck is the obstruction going to come from? We will have to work on a new space.

Moduli space. Given an orientation homeomorphism $\phi : (S^2, B) \to (\widehat{\mathbb{C}}, \phi(B))$, we can get rid of the homotopy information and just record $\phi|_B : B \hookrightarrow \widehat{\mathbb{C}}$. This gives us a map

$$\pi : \mathcal{T}_B \to \mathcal{M}_B$$

where $\mathcal{M}_B$ is the *moduli space* of (S^2, B). By definition, this is the space

$$\mathcal{M}_B := \{\varphi : B \hookrightarrow \widehat{\mathbb{C}} \text{ up to postcomposition by Möbius transformations}\}.$$

Let $B = \{b_1, \ldots, b_n\}$, and suppose that $n \geqslant 3$. By choosing coordinates, we may suppose that $\varphi \in \mathcal{M}_B$ satisfies $\varphi(b_1) = 0, \varphi(b_2) = \infty$, and $\varphi(b_3) = 1$. In this way, the class $[\varphi] \in \mathcal{M}_B$ is determined by the complex numbers

$$(\varphi(b_4), \ldots, \varphi(b_n)) \in \mathbb{C}^{n-3},$$

We can therefore identify $\mathcal{M}_B$ with an open subset of $\mathbb{C}^{n-3}$. The map

$$\pi : \mathcal{T}_B \to \mathcal{M}_B \quad \text{given by} \quad [\phi] \mapsto [\phi|_B]$$

is a universal covering map.

(143) Using the coordinates above prove that $\mathcal{M}_B$ is isomorphic to the complement of finitely many hyperplanes in $\mathbb{C}^{n-3}$, and be explicit.

Let's go back to our setting. We have a sequence $n \mapsto \sigma_f^n(\tau)$ in $\mathcal{T}_P$. We are assuming this sequence does not converge (if it did converge, it would converge to a fixed point). Consider the associated sequence $n \mapsto \pi(\sigma_f^n(\tau))$ in $\mathcal{M}_P$.

Lemma. If the sequence $n \mapsto \sigma_f^n(\tau)$ diverges in $\mathcal{T}_P$, then $n \mapsto \pi(\sigma_f^n(\tau))$ leaves every compact subset of $\mathcal{M}_P$.

Proof idea. The proof of this lemma is very particular to our setting. The way one proves this is by showing that the pullback map descends to a finite covering space of $\mathcal{M}_P$. More details available upon request. In my research, I have gotten A LOT of mileage out of this lemma and its cousins.

(144) Give an example of a covering map $f : X \to Y$, and a sequence $n \mapsto x_n \in X$ so that $n \mapsto x_n$ diverges but $n \mapsto f(x_n)$ converges. Can you give an example of a *finite* cover that has this property? (See the 'Proof idea' above).

The lemma above is useful because of the following theorem. For this theorem, suppose that $|P| \geqslant 3$. Then each $X \in \mathcal{M}_P$ is a Riemann sphere with $|P|$ marked points (or punctures) on it. Because $|P| \geqslant 3$, this means that X is a hyperbolic Riemann surface (of genus 0). Consider the hyperbolic metric on X. With respect to this metric, we can measure the lengths of simple closed curves. It is a fact that in each isotopy class, there is a unique simple closed curve of smallest hyperbolic length; this curve is called a *geodesic* in the given isotopy class. Given X, we can look at the geodesics in each isotopy class. Note that there are infinitely many isotopy classes of simple closed curves on X.

The Mumford Compactness theorem. Assume $|P| \geqslant 3$. Let $M_L \subseteq \mathcal{M}_P$ be the set of all $X \in \mathcal{M}_P$ whose shortest closed hyperbolic geodesic is of length $\geqslant L > 0$. Then M_L is compact.

We will apply the contrapositive of this theorem to our setting. Remember, we are assuming that the sequence $n \mapsto \sigma_f^n(\tau)$ diverges in $\mathcal{T}_P$. By the Lemma, this means that the associated sequence $n \mapsto \pi(\sigma_f^n(\tau))$ leaves every compact subset of the moduli space $\mathcal{M}_P$. This means that there are marked points on our Riemann spheres $X_n := \pi(\sigma_f^n(\tau))$ that are getting dangerously close to each other (in the hyperbolic metric). This means that as a hyperbolic surface, our Riemann spheres are developing very short curves (in the complement of the punctures, or marked points). We use these curves to find the obstruction for $f : (S^2, P) \to (S^2, P)$. Why will we get an obstruction this way? First we have to show that we get an f-stable multicurve. Then we have to show that it is an obstruction. It will be an obstruction since the curves are getting very short, there is a very large annulus that we can embed in our hyperbolic surface. This annulus will have very large modulus.

That was a sketch of the key ingredients in the proof of Bill Thurston's theorem. There are lots of details to check - it is a substantial theorem!

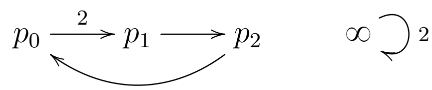
The rabbit, again. Understanding the pullback map is not easy, and part of the difficulty is that understanding points of $\mathcal{T}_P$ is not easy. The moduli space $\mathcal{M}_P$ is somewhat more user-friendly. An immediate question is: does the pullback map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ descend to yield a map $\mathcal{M}_P \to \mathcal{M}_P$ so that the following diagram commutes?

$$\begin{array}{ccc} \mathcal{T}_P & \xrightarrow{\sigma_f} & \mathcal{T}_P \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathcal{M}_P & \longrightarrow & \mathcal{M}_P \end{array}$$

This almost NEVER happens. I know of only one class of examples where σ_f descends to the moduli space, and for those examples, σ_f is actually constant (so of course it extends!). However, sometimes an *inverse* of σ_f descend, yielding a map $\mathcal{M}_P \leftarrow \mathcal{M}_P$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{T}_P & \xrightarrow{\sigma_f} & \mathcal{T}_P \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{M}_P & \longleftarrow & \mathcal{M}_P \end{array}$$

We'll call this map $g_f : \mathcal{M}_P \to \mathcal{M}_P$. Let's compute it a familiar example. Let $f : (S^2, P) \to (S^2, P)$ be a PCF branched cover with $P = \{p_0, p_1, p_2, \infty\}$, and suppose that f has the following ramification portrait.



Let $\phi : (S^2, P) \to (\widehat{\mathbb{C}}, \phi(P))$ be an orientation-preserving homeomorphism normalized so that $\phi(p_0) = 0, \phi(p_1) = 1$, and $\phi(\infty) = \infty$. For notation, set $y := \phi(p_2)$. Note that the point $[\phi|_P] \in \mathcal{M}_P$ is completely determined by the complex number $y \in \widehat{\mathbb{C}} = \{0, 1, \infty\}$. Think of y as a moduli space variable.

There is:

- a *unique* orientation-preserving homeomorphism $\psi : (S^2, P) \to (\widehat{\mathbb{C}}, \psi(P))$, so that $\psi(p_0) = 0, \psi(p_1) = 1$, and $\psi(\infty) = \infty$, and
- a *unique* rational map $F : (\widehat{\mathbb{C}}, \psi(P)) \to (\widehat{\mathbb{C}}, \phi(P))$ so that the following diagram commutes

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{\psi} & (\widehat{\mathbb{C}}, \psi(P)) \\ \downarrow f & & \downarrow F \\ (S^2, P) & \xrightarrow{\phi} & (\widehat{\mathbb{C}}, \phi(P)) \end{array}$$

For notation, set $x := \psi(p_2)$; again, think of $x \in \widehat{\mathbb{C}} - \{0, 1, \infty\}$ as a moduli space variable. Let's determine as much as we can about the rational map F:

- F is a quadratic polynomial (why?)
- F has a critical point at 0, and the corresponding critical value is 1; that is, $F(0) = 1$
- $F(0) = x$, and
- $F(1) = y$.

The previous conclusions are based on the fact that the diagram above commutes, and the coordinates we chose. The first two points above imply that a normal form for F is given by

$$F : t \mapsto At^2 + 1, \quad \text{where } A \text{ is a complex parameter.}$$

Imposing the third condition above implies that $A = -1/x^2$, so we really have

$$F : t \mapsto -\frac{t^2}{x^2} + 1.$$

Imposing the last condition above implies that $y = -1/x^2 + 1$. We just did something REMARKABLE! We found a relation between the moduli space variables x and y. In fact, we did this:

$$\begin{array}{ccc} \phi & \xrightarrow{\sigma_f} & \psi \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ y & \xleftarrow[g_f]{} & x \end{array}$$

where $g_f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is $g_f : x \mapsto y$, $y = -1/x^2 + 1$. This map g_f is rather interesting; note that it is also postcritically finite!!!! In fact, the ramification portrait for g_f is

$$0 \xrightarrow{2} \infty \xrightarrow{2} 1 \to 0$$

This is not a coincidence. The map g_f has three fixed points: ζ_r, ζ_c, and ζ_a. Note that each of these fixed points is in $\widehat{\mathbb{C}} - \{0, 1, \infty\}$, or each of the fixed points is in the *moduli space* $\mathcal{M}_P$.

(145) Prove that each of these fixed points corresponds to a quadratic polynomial of the form $t \mapsto At^2 + 1$, such that the critical point $t_0 = 0$ is in a cycle of period 3. Then draw the Julia set of g_f and find these fixed points.

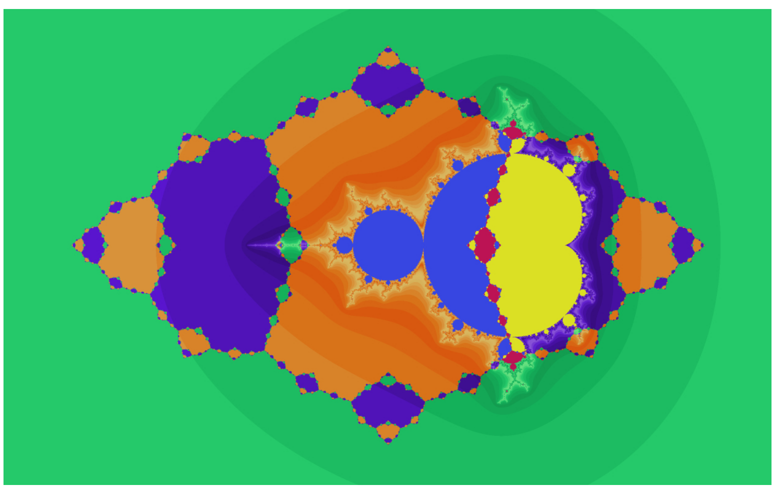


FIGURE 11. This is a superposition of two parameter spaces. One of them, the moduli space $\mathcal{M}_P \approx \widehat{\mathbb{C}} - \{0, 1, \infty\}$ is a *nondynamical parameter space*, and it has a dynamical system on it, the map g_f. The Julia set of g_f is visible in this picture. There are three repelling fixed points of the map g_f: $\zeta_r, \zeta_c, \zeta_a$. The other parameter space is a dynamical parameter space; that is, it parameterizes quadratic polynomials as dynamical systems, and it contains the Mandelbrot set M. In the coordinates chosen, the three fixed points of g_f line up EXACTLY with the corresponding parameters of M: the rabbit, corabbit, airplane. This is my absolute favorite picture.

We can lift the Julia set of g_f up to the Teichmüller space $\mathcal{T}_P$ to help understand the dynamics of the map $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$.

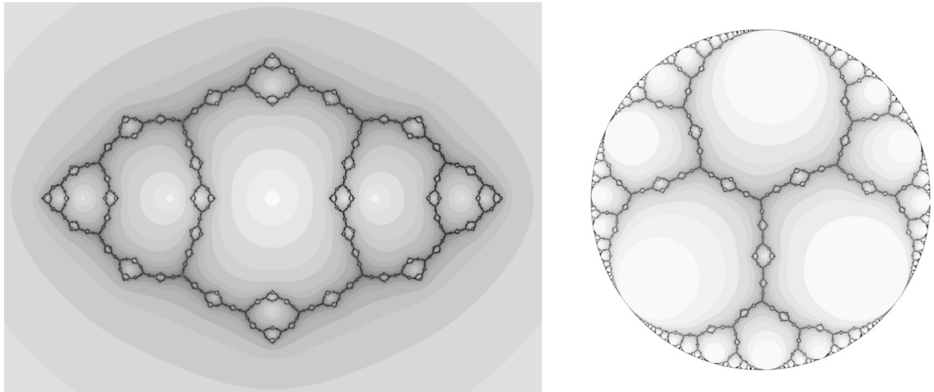


FIGURE 12. On the left is the Julia set of $g_f : x \mapsto -1/x^2 + 1$, and on the right is a picture of π^{-1}(this Julia set) $\subseteq \mathcal{T}_P$. The Teichmüller space in this example is isomorphic to the open disk, and if f is the rabbit polynomial, then σ_f has a unique fixed point $\zeta \in \mathcal{T}_P$. The disk on the right is centered at this fixed point. The dynamics of σ_f is visible in this picture. The three large components comprise one cycle of period 3. The rabbit fixed point in the center is attracting, and one can use the map g_f to compute the derivative of σ_f at the rabbit fixed point. Just compute the multiplier of g_f at the repelling fixed point ζ_r and invert it.

Note that the only thing that I used to compute the map g_f was the ramification portrait of f. I did not use the combinatorial equivalence class at all. This is another reason why the map g_f is very helpful; it can be computed from some finite combinatorial data. This is somewhat misleading; the map g_f depends on just more than the portrait. It actually depends on some nondynamical data; the covering combinatorics of f, or the *Hurwitz class* of f. It turns out that if f is a Thurston map of degree 2, then there is a unique Hurwitz class, so the portrait is enough to determine g_f.

Small wrinkle. Unfortunately, the map g_f does not always exist. What does exist is a *Hurwitz correspondence* $\mathcal{W}_f$ which is sometimes the graph of a map. Regardless of whether it is or not, it is an algebraic object that can be used to understand the dynamics of σ_f.

Elastic graphs.

We end with a *positive* characterization of rational maps, a criterion that lets us know for sure that a rational map *does* exist. This is in terms of elastic graphs.

Definition. An *elastic graph* G is a graph with a positive number $\alpha(e)$ associated to each edge e, which we will sometimes use as a metric or measure (with α giving the length of the edge). If $f\colon G \to X$ is a suitably nice map to a length space, then the *elastic energy* of f is

$$EE(f) = \int_{x \in G} |f'(x)|\, dx,$$

where the derivatives and integral are taken with respect to the length α. If the derivative is constant along each edge and is taken to a segment of length ℓ, then $|f'(x)| = \ell/\alpha$. We get an extra factor of α in the integration, so overall we get

$$EE(f) = \sum_{e \in \mathrm{Edges}(G)} \frac{\ell(f(e))^2}{\alpha(e)},$$

which looks familiar from Hooke's law for spring energy. We are also interested in the minimum of $EE(f)$ as f ranges over a homotopy class, which we will write $EE[f]$.

(146) Use calculus of variations to show that if we minimize over a homotopy class, then $E[f]$ is realized when the derivative $|f'(x)|$ is constant on each edge. What can you say about what happens at the vertices? (You may assume that X is $\mathbb{R}^n$, where some vertices are pinned at fixed locations to make the "up to homotopy" statement non-trivial.)

Definition. A map $\phi\colon G_1 \to G_2$ of elastic graphs is *loosening* if

$$\operatorname*{ess\,sup}_{y \in G_2} \sum_{x \in \phi^{-1}(y)} |\phi'(x)| < 1,$$

where ess sup is the "essential supremum", which means the supremum ignoring sets of measure 0, for instance the vertices of G_2 where the derivatives are not defined. Also define in general

$$\mathrm{Fill}(\phi) = \operatorname*{ess\,sup}_{y \in G_2} \sum_{x \in \phi^{-1}(y)} |\phi'(x)|.$$

As with the elastic energy, we will be interested in whether there is a loosening map in a homotopy class, or minimizing $\mathrm{Fill}(\phi)$ over the homotopy class.

(147) Show that if ϕ is loosening, then for any map $f\colon G_2 \to X$ to a length space, then

$$EE(f \circ \phi) < EE(f),$$

thus justifying the name "loosening". Show the same thing after minimizing in a homotopy class:

$$EE[f \circ \phi] < EE[f],$$

(You will need almost nothing about the space X, and this is not so hard.)

(148) Show that the above criterion is an if and only if. More precisely, suppose we have a map $\phi\colon G_1 \to G_2$ so that, for ANY map $f\colon G_2 \to X$ to a length space, $EE(f \circ \phi) < EE(f)$. Show that ϕ must be loosening. (This is also not so bad. You can assume that X is G_2 with another metric.)

(149) Now show that the criterion is ALSO an if and only if when you take homotopy classes. Let $\phi\colon G_1 \to G_2$ be a map so that, for any map $f\colon G_2 \to X$ to a length space, $EE[f \circ \phi] < EE[f]$. Show that there is a loosening map in $[\phi]$. (This is much harder.)

Theorem. [Positive criterion for rational maps] Suppose $f\colon (S^2, P) \circlearrowleft$ is a hyperbolic PCF branched cover (i.e., with a branch point in each cycle in P). Then the following conditions are equivalent:

- f is equivalent to a rational map.

- There is an elastic spine G for $S^2 \backslash P$ and $n > 0$ so that there is a loosening map from $f^{-n}(G)$ to G.
- For ANY elastic spine G for $S^2 \backslash P$ and ANY sufficiently large $n > N$, there is a loosening map from $f^{-n}(G)$ to G.

In the second two conditions, the map is required to be in the homotopy class coming from the deformation retraction of $S^2 \backslash P$ onto G.

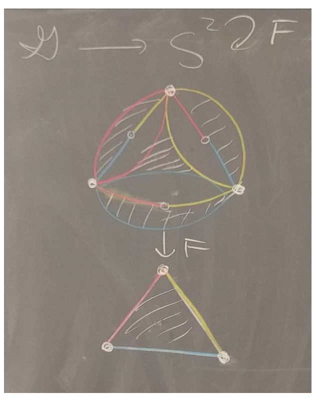
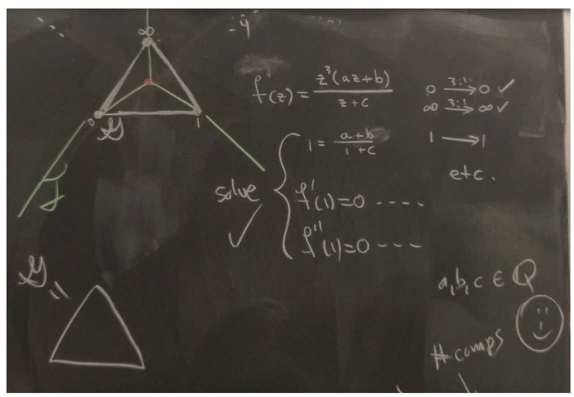
(150) We did the map $f(z) = 1 - 1/z^2$ in class, using a spine G that was topologically like a theta graph Θ. Suppose that we had chosen a different spine G', maybe one that looks topologically like a dumbbell, with two loops connected by an edge. Show that there is a loosening map from $f^{-n}(G')$ to G' for some sufficiently large G'.

(151) Consider the PCF rational map

$$f(z) = \frac{1 + z^2}{1 - z^2}.$$

Find a spine G for the post-critical set of f for which there is a loosening map from $f^{-1}(G)$ to G.

(152) Consider the critically fixed map on S^2 described in Kevin Pilgrim's talk:

Concretely, this is the map from S^2 to S^2 that, in the left picture, maps each shaded triangle on top to the shaded triangle on the bottom (preserving colors) and maps each unshaded triangle to the shaded triangle.

(a) Verify that this map has three critical points, at the corners of the triangle, each one is mapped to itself (it is "critically fixed"), and that it has degree 4.

(b) Find a spine G for $S^2 \backslash P$ and its inverse image $f^{-1}(G)$.

(c) Show that this map is equivalent to a rational map by finding an explicit loosening map from $f^{-1}(G)$ to G.

(d) Verify your work by finding the map algebraically, completing the exercise Kevin suggests in the image on the right.

(e) Can you generalize this argument?

(153) (**Milnor's question**) Prove or disprove: the map $z \mapsto z^2 - 1.5$ has an attracting periodic cycle.

(154) Name one of the filled Julia sets who doesn't already have a name.

(155) Compute the entropy of the map $z \mapsto z^2 - 2$ restricted to its filled Julia set.

(156) Prove or disprove: the Julia set of the geometric mating of the airplane and kokopelli is homeomorphic to a Sierpinski carpet.

(157) Recover the mating theorem from Dylan's machinery.

FractalStream Scripts.

(1) Draw the Mandelbrot set:

iterate $z^2 + c$ until z escapes.

(2) Draw a Julia set of a polynomial:

iterate $z^2 - 1$ until z escapes.

(3) Draw the Julia set of the second iterate of the basilica, and color the superattracting basins in different colors.

iterate $(z^2 - 1)^2 - 1$ until z escapes or z vanishes or $(z + 1)$ vanishes.

If z escapes then [blue].

If z vanishes then [green].

If $(z + 1)$ vanishes then [yellow].

(4) Enable period counter in M.

probe integer "Period Counter":

repeat 100 times:
set z to $z^2 + c$.
end.

set w to z.
set p to 0.

do
set z to $z^2 + c$.
set p to $p + 1$.
until $z = w$.

report p.
end.

iterate $z^2 + c$ until z escapes.

(5) Color M according to period.

set q to 0.
do
set z to $z^2 + c$.
set q to $q + 1$.
until $q = 100$.

set w to z.
set p to 0.

do
set z to $z^2 + c$.
set p to $p + 1$.
until $z = w$.

report p.
end.

iterate $z^2 + c$ until z escapes.