On the local Langlands conjectures for disconnected groups

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Abstract

We extend the local Langlands conjectures to a certain class of disconnected groups, allowing non-abelian component groups, and recast in this language some aspects of twisted endoscopy. We further introduce normalized twisted transfer factors and a normalized correspondence between an L-packet for a disconnected group and the set of representations of the centralizer groups of its Langlands parameter.

1 Introduction

Let F be a local field of characteristic zero. The goal of this paper is to extend the refined local Langlands conjecture to the case of disconnected groups. We recall briefly the statement of this conjecture, referring to [Kal16a] for details. Given a connected reductive F-group G' there should be a bijection between the set of (equivalence classes of) Langlands parameters $\varphi : L_F \to {}^LG'$ and the set of L-packets $\Pi_\varphi(G')$. An L-packet is a finite set of irreducible admissible representations of $G'(F)$. It is empty if and only if φ is non-relevant for G'. The L-packets are disjoint and exhaust the set of isomorphism classes of irreducible admissible representations of $G'(F)$. To enumerate the constituents of $\Pi_\varphi(G')$ one fixes an inner twisting $\xi : G \to G'$ with G quasi-split and enriches it to a rigid inner form datum (ξ, z). One further fixes a Whittaker datum $\mathfrak{w}$ for G. The inner twisting provides an identification of dual groups $\widehat{G'} = \widehat{G}$ and of L-groups ${}^LG' = {}^LG$. Let $Z \subset G$ be a finite central subgroup that is sufficiently large to realize z. Let $\bar{G} = G/Z$. The natural quotient map $G \to \bar{G}$ is an isogeny. Let $\widehat{\bar{G}} \to \widehat{G}$ be the dual isogeny and let $Z(\widehat{\bar{G}})^+$ be the preimage of $Z(\widehat{G})^\Gamma$. The element z provides a character $[z] : \pi_0(Z(\widehat{\bar{G}})^+) \to \mathbb{C}^\times$. When F is p-adic the character $[z]$ determines the equivalence class of the rigid inner twist (G', ξ, z) uniquely. When $F = \mathbb{R}$ multiple equivalence classes of rigid inner twists may lead to the same character $[z]$, and they are related by $H^1(\mathbb{R}, G'_{\text{sc}})$. Let $S_\varphi \subset \widehat{G}$ be the centralizer of the image of φ and let S_φ^+ be its preimage in $\widehat{\bar{G}}$. Let $\text{Irr}(\pi_0(S_\varphi^+), [z])$ be the set of isomorphism classes of those irreducible representations of the finite group $\pi_0(S_\varphi^+)$ whose restriction to $\pi_0(Z(\widehat{\bar{G}})^+)$ is $[z]$-isotypic. There should be a map $\Pi_\varphi(G') \to \text{Irr}(\pi_0(S_\varphi^+), [z])$. In the p-adic case it should be bijective. In the case $F = \mathbb{R}$ the map should become bijective if one replaces $\Pi_\varphi(G')$ with the disjoint union over all rigid inner twists giving rise to the same character $[z]$. In all cases, this map depends on the choice of $\mathfrak{w}$ and the rigid inner twist data. That same data provides a normalization $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi, z)]$ of the Langlands–Shelstad transfer factor for each refined endoscopic datum $\dot{\mathfrak{e}}$ for G and z-pair $\mathfrak{z}$ for it. The map $\Pi_\varphi(G') \to \text{Irr}(\pi_0(S_\varphi^+), [z])$ is expected to satisfy the endoscopic character identites with respect to this normalization of the transfer factor when the parameter φ is tempered. More precisely, if $\pi \mapsto \rho_\pi$ is the above map, then a semi-simple element $\dot{s} \in S_\varphi^+$ leads to the virtual character $\Theta_\varphi^{\dot{s}} = \sum_{\pi \in \Pi_\varphi(G')} \mathrm{tr}\rho_\pi(\dot{s})\Theta_\pi$ of $G'(F)$. At the same time the connected centralizer $\widehat{H}$ of the image $s \in \widehat{G}$ of $\dot{s}$ and the parameter φ lead to a quasi-split group H and a parameter $\varphi^{\mathfrak{z}}$ for its cover $H^{\mathfrak{z}}$ that is part of the z-pair, hence to a similar virtual character $S\Theta_{\varphi^{\mathfrak{z}}} = \sum_{\pi^{\mathfrak{z}} \in \Pi_{\varphi^{\mathfrak{z}}}(H^{\mathfrak{z}})} \dim\rho_{\pi^{\mathfrak{z}}}\Theta_{\pi^{\mathfrak{z}}}$ on $H^{\mathfrak{z}}(F)$. The transfer factor $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi, z)]$ gives rise to a correspondence of functions $f \leftrightarrow f^{\mathfrak{z}}$ between functions on $G'(F)$ and functions of $H^{\mathfrak{z}}(F)$ and the expected character identity is $\Theta_\varphi^{\dot{s}}(f) = S\Theta_{\varphi^{\mathfrak{z}}}(f^{\mathfrak{z}})$. A suitable generalization is supposed to hold in the non-tempered case once φ has been replaced by an Arthur parameter. We note here that we have absorbed the Kottwitz sign $e(G')$ into the transfer factor, rather than into the virtual character $\Theta_\varphi^{\dot{s}}$.

In this paper we extend these conjectures to certain disconnected algebraic groups whose identity component is reductive. Motivation for this comes on the one hand from the natural occurrence of disconnected groups in number theoretic contexts, most notably the orthogonal groups, and on the other hand from the natural occurrence of disconnected groups in representation theoretic contexts, for example by taking centralizers of semi-simple elements. In fact, disconnected groups appear in the classification of tempered representations of connected reductive groups. If M' is a Levi subgroup of the connected reductive group G' and σ is a square-integrable representation of $M'(F)$, the subgroup of $G'(F)$ that normalizes M' and stabilizes the isomorphism class of σ plays an important role in the decomposition into irreducible pieces of the parabolic induction of σ. In order to properly normalize the intertwining operators needed to decompose this parabolic induction one is led to study the representation theory of disconnected groups of this form. This leads to a normalized version of Arthur's local intertwining relation [Art89, §7], [Art13, §2.4]. We will present this in a forthcoming paper as an application of the results of the current paper.

The class of disconnected groups we consider in this paper are those affine algebraic F-groups $\tilde{G}'$ whose identity component G' is reductive, and for which there exists an isomorphism $\tilde{G}' \cong G' \rtimes A$ over the algebraic closure $\bar{F}$ of F, where A is a finite (possibly non-abelian) group of automorphisms of G' that preserves a $\bar{F}$-pinning. The second condition is automatically fulfilled if G' is adjoint, but in general it does restrict the class of disconnected groups we are considering. The possible forms of such groups $\tilde{G}'$ can be classified cohomologically in a manner similar to the connected case. In the connected case the classification has two steps – one first classifies quasi-split groups by means of based root data, and then inner forms in terms of Galois cohomology. In the disconnected case the classification has three steps – one first classifies quasi-split disconnected groups, then inner forms, and then (what we have called) "translation forms". A quasi-split disconnected group is of the form $G \rtimes A$, where G is a quasi-split connected reductive F-group, and A is a subgroup of its automorphism group that fixes an F-pinning. One sees easily that we may assume without loss of generality that A is a constant group scheme. Then we have $[G \rtimes A](F) = G(F) \rtimes A$. An inner form of $G \rtimes A$ is obtained by twisting via elements of $Z^1(F, G/Z(G)^A)$. A translation form is obtained by twisting via elements of $Z^1(F, Z^1(A, Z(G)))$. These two twisting steps can be performed in either order. While in the quasi-split case the split exact sequence

$$1 \to G \to G \rtimes A \to A \to 1$$

remains exact on F-points and retains a canonical splitting, after inner twisting or translational twisting neither of these statements is true in general. More precisely, given $\bar{z} \in Z^1(F, G/Z(G)^A)$ there is a natural subgroup $A^{[\bar{z}]} \subset A$ so that if $\tilde{G}_{\bar{z}}$ is the corresponding inner form of the quasi-split group $\tilde{G} = G \rtimes A$, then the sequence

$$1 \to G_{\bar{z}}(F) \to \tilde{G}_{\bar{z}}(F) \to A^{[\bar{z}]} \to 1$$

is exact, but it is not equipped with a natural splitting even if it is split. A similar remark applies to translation forms. In this paper we extend the formulation of the refined local Langlands correspondence to inner forms of quasi-split disconnected groups, leaving the treatment of translation forms, as well as the removal of the condition $(\tilde{G}')_{\bar{F}} \cong (G' \rtimes A)_{\bar{F}}$, to a future paper.

There are multiple questions one must answer when attempting to extend the Langlands conjectures to the disconnected setting: What will be the dual group, or the L-group, of a disconnected group? What will be the concept of a Langlands parameter and of its centralizer? What are endoscopic groups? What are transfer factors and how does one normalize them?

We hasten to say that we do not perform any non-trivial harmonic analysis in this paper. Instead, we use the already established framework of twisted endoscopy and the fundamental results of Langlands, Shelstad, Kottwitz, Arthur, Waldspurger, and Ngo. Part of this paper consists of introducing a slightly different language for this theory. We hope that this language will be beneficial for some applications. One advantage it provides is that the statements of the conjectures for disconnected groups become formally very similar to the statements for connected groups. Another advantage it provides is in organizing multiple automorphisms of a connected reductive group. We note further that our language does not encompass the full generality of twisted endoscopy, even if we restirct attention to a cyclic component group. Indeed, we do not consider a character $\omega : G(F) \to \mathbb{C}^\times$, and the automorphisms of $G_{\bar{z}}$ we obtain from elements of $\tilde{G}_{\bar{z}}(F)$ are not as general as the theory of twisted endoscopy allows.

With this in mind, the answers we give to the above questions are the following. Consider a quasi-split disconnected reductive group $\tilde{G} = G \rtimes A$ and an inner form $\tilde{G}_{\bar{z}}$ corresponding to an element $\bar{z} \in Z^1(F, G/Z(G)^A)$. The groups $\tilde{G}$ and $\tilde{G}_{\bar{z}}$ will have the same set of Langlands parameters, and this is the set of Langlands parameters for the connected group G (and its inner form $G_{\bar{z}}$). Thus, the notion of Langlands parameters remains unchanged when we pass from the connected group G to the disconnected group $\tilde{G}$. What changes is the notion of equivalence. Two parameters for $G_{\bar{z}}$ are considered equivalent if they are conjugate under $\widehat{G}$. We declare two parameters for $\tilde{G}_{\bar{z}}$ to be equivalent if they are conjugate under $\widehat{G} \rtimes A^{[\bar{z}]}$. Note that the notion of equivalence depends on the inner form $\bar{z}$ being considered, in contrast to the case of connected groups.

The new notion of equivalence leads to a new notion of the centralizer group S_φ of a Langlands parameter $\varphi : L_F \to {}^LG$. Indeed, S_φ can be viewed as the group of self-equivalences of φ. In the disconnected case we now obtain $\tilde{S}_\varphi^{[\bar{z}]}$ as the group of self-equivalences of φ in the new sense of equivalence. In other words, $\tilde{S}_\varphi^{[\bar{z}]}$ is the centralizer of φ in the group $\widehat{G} \rtimes A^{[\bar{z}]}$. We obtain the exact sequence

$$1 \to S_\varphi \to \tilde{S}_\varphi^{[\bar{z}]} \to A^{[\varphi],[\bar{z}]} \to 1,$$

where $A^{[\varphi],[\bar{z}]} = A^{[\varphi]} \cap A^{[\bar{z}]}$ is the stabilizer in $A^{[\bar{z}]}$ of the $\widehat{G}$-conjugacy class of φ.

Let $\Pi_\varphi(\tilde{G}_{\bar{z}})$ denote the set of irreducible admissible representations of $\tilde{G}_{\bar{z}}(F)$ whose restriction to $G_{\bar{z}}(F)$ intersects $\Pi_\varphi(G_{\bar{z}})$. We think of $\Pi_\varphi(\tilde{G}_{\bar{z}})$ as the L-packet for the disconnected group $\tilde{G}_{\bar{z}}(F)$ associated to the parameter φ. To enumerate its members, choose a lift $z \in Z^1(u \to W, Z \to G)$ of $\bar{z}$, where $Z \subset Z(G)^A$ is a sufficiently large finite subgroup, thereby realizing $\tilde{G}_{\bar{z}}$ as a rigid inner form of $\tilde{G}$. The stabilizer $A^{[z]}$ of the cohomology class of z for the action of A equals $A^{[\bar{z}]}$. Choose an A-invariant Whittaker datum for G. As above we obtain from z a character $[z] : \pi_0(Z(\widehat{\bar{G}})^+) \to \mathbb{C}^\times$. Let $\tilde{S}_\varphi^{+,[z]}$ be the preimage of $\tilde{S}_\varphi^{[z]}$ in $\widehat{\bar{G}} \rtimes A^{[z]}$. This group surjects onto $A^{[\varphi],[z]}$ and we have the exact sequence

$$1 \to S_\varphi^+ \to \tilde{S}_\varphi^{+,[z]} \to A^{[\varphi],[z]} \to 1.$$

Then there should be a map

$$\Pi_\varphi(\tilde{G}_{\bar{z}}) \to \mathrm{Irr}(\pi_0(\tilde{S}_\varphi^{+,[z]}), [z]),$$

which is again bijective in the p-adic case, and becomes bijective in the real case once its target has been replaced by a suitable disjoint union. As in the connected case, this map should lead to character identities with respect to a normalized transfer factor $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi, z)]$. More precisely, a semi-simple element $\dot{\tilde{s}} \in \tilde{S}_\varphi^+$ leads to the virtual character $\Theta_\varphi^{\dot{\tilde{s}}} = \sum_{\tilde{\pi} \in \Pi_\varphi(\tilde{G}_{\bar{z}})} \mathrm{tr}\rho_{\tilde{\pi}}(\dot{\tilde{s}})\Theta_{\tilde{\pi}}$ of $\tilde{G}_{\bar{z}}(F)$. The connected centralizer $\widehat{H}$ of the image $\tilde{s} \in \widehat{G} \rtimes A$ of $\dot{\tilde{s}}$ and the parameter φ lead to a quasi-split connected reductive group H and a parameter $\varphi^{\mathfrak{z}}$ for its cover $H^{\mathfrak{z}}$, hence to a similar virtual character $S\Theta_{\varphi^{\mathfrak{z}}} = \sum_{\pi^{\mathfrak{z}} \in \Pi_{\varphi^{\mathfrak{z}}}(H^{\mathfrak{z}})} \dim\rho_{\pi^{\mathfrak{z}}}\Theta_{\pi^{\mathfrak{z}}}$ on $H^{\mathfrak{z}}(F)$. The transfer factor $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi, z)]$ gives rise to a correspondence of functions $f \leftrightarrow f^{\mathfrak{z}}$ between functions on $\tilde{G}_{\bar{z}}(F)$ and functions of $H^{\mathfrak{z}}(F)$ and the expected character identity is $\Theta_\varphi^{\dot{\tilde{s}}}(f) = S\Theta_{\varphi^{\mathfrak{z}}}(f^{\mathfrak{z}})$. Note the strong similarity with the connected case.

The normalization of the transfer factor is one of the main tasks of this paper. The relative transfer factor for twisted endoscopy was introduced in [KS99] and some adjustments were later made in [KS]. It is a function that assigns a complex number to two pairs of elements $(\gamma, \tilde{\delta})$, where γ is a sufficiently regular semi-simple element of $H^{\mathfrak{z}}(F)$, and $\tilde{\delta}$ is a strongly regular semi-simple element of $\tilde{G}_{\bar{z}}(F)$ that lies in a fixed coset determined by the image of $\tilde{s}$ in A. If one fixes arbitrarily a pair $(\gamma, \tilde{\delta})$, then one obtains from this a function of just one such pair, called an absolute transfer factor. But the arbitrary choice means that this function is well-defined only up to multiplication by a non-zero complex scalar. A specific normalization useful for applications was given in [KS99, §5.3] for quasi-split twisted groups. In this paper we provide a normalization for all (rigid) inner forms of quasi-split twisted groups. We call this factor Δ_{KS}. By a simple averaging procedure we obtain from it the transfer factor $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi, z)]$ used in the above paragraph, which may now be supported on multiple cosets of $G_{\bar{z}}(F)$ in $\tilde{G}_{\bar{z}}(F)$.

The normalization of Δ_{KS} involves two ingredients. The first is a definition of an absolute term $\Delta_{III}^{\text{new}}$ that replaces the relative term Δ_{III} constructed in [KS99, §4.4]. The construction we offer here is shorter and simpler than the one of loc. cit. for two reasons. First, our setting ensures that the class $\mathbf{z}$ of [KS99, Lemma 3.1.A(3)] is trivial. This implies that the transfer of twisted classes between the twisted group and its quasi-split form is defined over F, and that the rational structure of the endoscopic group H does not need a shift. Second, we define an absolute invariant $\text{inv}(\gamma^{\mathfrak{z}}, \tilde{\delta})$ that measures the relative position of a related pair $(\gamma^{\mathfrak{z}}, \tilde{\delta})$, thus avoiding the complications caused by dealing with two related pairs simultaneously. The construction of the invariant involves a blend of the techniques from [KS99] and [Kal16b], and an interpretation of conjugacy classes in inner form of disconnected groups in terms of a certain non-abelian cohomology set $H^1(F, G \rightrightarrows G)$. The second ingredient of Δ_{KS} is a generalization of the Kottwitz sign $e(G')$ defined in [Kot83] to the case of inner forms of twisted groups, or equivalently to cosets of inner forms of quasi-split disconnected reductive groups.

Besides stating the conjectures, we prove a number of reduction results in this paper. We show how the conjectures for disconnected groups can be reduced to the conjectures for connected groups, a conjecture on the compatibility of the conjectures for connected groups with automorphisms, and a certain amplification of the endoscopic character identity conjecture in twisted endoscopy. We also discuss various functorial constructions, such as restriction and induction of component groups.

Finally, we prove our conjectures in the special case when the identity component is a torus. In this case, the representation theory of the identity component essentially disappears and one can clearly see the additional information present in the consideration of disconnectedness. The core of the proof consists of showing that two group extensions, produced from the same data but on in terms of G and the other in terms of $\widehat{G}$, are canonically isomorphic. To illustrate the point let us discuss the case of pure inner forms. Consider a torus T and a finite group of F-automorphisms A. Then $\tilde{T} = T \rtimes A$ is a quasi-split disconnected group in our sense. Let $z \in Z^1(F, T)$ and let $\tilde{T}_z$ be the corresponding pure inner form. Of course the identity components of $\tilde{T}$ and $\tilde{T}_z$ are canonically identified, but the disconnected groups $\tilde{T}(F) = T(F) \rtimes A$ and $\tilde{T}_z(F)$ are not. Let $\varphi : W_F \to {}^LT$ be a Langlands parameter and let $\theta : T(F) \to \mathbb{C}^\times$ be the corresponding character. Let $A^{[\varphi],[z]}$ be the subgroup of A that fixes both the $\widehat{T}$-conjugacy class of φ and the cohomology class of z. Simple arguments reduce the problem to the case $A = A^{[\varphi],[z]}$. The L-packet $\Pi_\varphi(\tilde{T}_z)$ consists of the irreducible representations of (the usually non-abelian) group $\tilde{T}_z(F)$ whose restriction to $T(F)$ is θ-isotypic. The set $\mathrm{Irr}(\tilde{S}_\varphi, [z])$ consists of the irreducible representations of $\tilde{S}_\varphi$ whose restriction to $\widehat{T}^\Gamma$ is $[z]$-isotypic (note we do not need the covers $\tilde{S}_\varphi^+$ and $[\widehat{\widehat{T}}]^+$ since we are using a pure inner form). Therefore we are led to consider the following two push-out diagrams

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T(F) & \longrightarrow & \tilde{T}_z(F) & \longrightarrow & A & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \theta} & & & & & & \\ & & \mathbb{C}^\times & & & & & & \end{array}$$

and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{T}^\Gamma & \longrightarrow & \tilde{S}_\varphi & \longrightarrow & A & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle [z]} & & & & & & \\ & & \mathbb{C}^\times & & & & & & \end{array}$$

Both push-outs are central extensions of A by $\mathbb{C}^\times$. The id-isotypic irreducible representations of the first extension are in canonical bijection with $\Pi_\varphi(\tilde{T}_z)$, while those of the second extension are in canonical bijection with $\mathrm{Irr}(\tilde{S}_\varphi, [z])$. The conjecture about the internal structure of L-packets requires us to show that these two extensions are canonically isomorphic. There appears to be no a-priori reason why these extensions should even be isomorphic, let alone canonically. But we are able to produce a canonical isomorphism. We then continue to show that this isomorphism satisfies the endoscopic character identities with respect to the normalized transfer factor.

We now describe the contents of the paper. In §3 we discuss basic results about disconnected groups, such as the classification of their forms in §3.1, focusing on inner forms in §3.2. In §3.3 we recall facts about twisted conjugacy classes and norms from [KS99] and adapt them to our present language. In §3.4 we discuss Whittaker data invariant under A. In §3.5 we extend the definition of the Kottwitz sign $e(G')$ of a connected reductive group to inner forms of quasi-split twisted groups.

The next two sections – §4 and §5 – contain the statments of the refined local Langlands conjecture in the settings of pure respectively rigid inner forms. We have decided to present these cases separately, rather than only dealing with the general case of rigid inner forms, because we feel that the setting of pure inner forms illustrates more clearly the ideas behind conjugacy classes, relative positions, and invaraints, as well as the structure of the conjecture. The more general case of rigid inner forms follows the same structure and ideas, but combines them with a technical cohomological discussion. In §4 we first discuss the concept of rational conjugacy classes across pure inner forms, their norms, and the associated invariants. These are based on the non-abelian cohomology set $H^1(F, G \rightrightarrows G)$. The constructions are ultimately the same as those of [KS99], but our language is slightly different and our situation is more specialized, which makes the arguments simpler and shorter. For this reason we have given them in full detail in the hope that this would be helpful to the reader. In §4.6 we state the first part of the refined local Langlands conjecture – the correspondence between parameters and packets and the internal structure of packets. We then recall the notion of twisted endoscopic data from [KS99]. Our definitions are in fact slightly different, both for data and for their isomorphisms. This difference is very mild; it ensures that absolute transfer factors are invariant under isomorphisms. We then turn to the normalization of transfer factors in the setting of pure inner forms. In §4.9 we explain how the factor $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi, z)]$ is related to the twisted factor Δ_{KS}. The normalization of Δ_{KS} is done in §4.10 and §4.11. We have again split the exposition in the hope that this will make the construction most transparent, by first treating the less technical set-up when a z-pair is not needed, and then the more general case when it is. In §4.12 we summarize the fundamental results of Langlands, Shelstad, Kottwitz, Arthur, Waldspurger, and Ngo, on endoscopic transfer of functions. These results allow us to state the second part of the refined local Langlands conjecture – the character identities – in §4.13.

The treatment of rigid inner forms in §5 requires the blending of the hypercohomology techniques of [KS99, A.3] and the Galois gerbes of [Kal16b]. This is done in the first two subsections. We the rest of the section consists of slight generalizations of material of §4, and we allow ourselves to be more brief. The refined local Langlands conjecture for rigid inner forms is stated in §5.6.

In §6 we discuss how the parameterization of the internal structure of L-packets depends on the choice of Whittaker datum. This is the disconnected analog of the corresponding results from [Kal13].

In §4.4 we discuss how the conjectures change when we change the component group of the disconnected group. The simplest possible change is passing to a subgroup of the component group. It is discussed in §7.1. In §7.2 we discuss how the conjectures for disconnected groups can be related to those for connected groups and twisted endoscopy. This clarifies the information that the disconnected case caries beyond the connected case and the twisted case. There is an operation dual to restriction, which may be called induction. If G is a connected reductive group on which a finite group of automorphisms A operates, and B is a group containing A, one can form the connected reductive group $H = \mathrm{Ind}_A^B G$, on which B operates. We discuss in §7.5 how the conjectures for inner forms of $G \rtimes A$ imply those for inner forms of $H \rtimes B$. The discussion here is elementary, but unfortunately rather long and technical. We have included it because the process of induction appears quite often in applications.

In §8 we prove the conjectures made here in the special case of tori. We close the paper is three short appendices. In §A we formulate a conjecture about the compatibility of the refined local Langlands correspondence for connected groups with automorphisms. This conjecture is undoubtedly well-known to experts, but we have not been able to locate a reference. In §B we discuss automorphisms of reductive groups that arise via Weil-restriction. In §C we review orthogonality relations for irreducible projective representations of finite groups.

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2 Notation

Throughout the paper, F will denote a local field of characteristic zero, Γ the absolute Galois group with respect to a fixed algebraic closure $\bar{F}$ of F, and W_F the Weil group. We will write $Z^1(\Gamma, G)$ for the set of continuous 1-cocycles of Γ valued in the discrete group $G(\bar{F})$, $H^1(\Gamma, G)$ for the set of cohomology classes of such cocycles, $\tilde{Z}^1(\Gamma, G)$ for the set of continuous sections $\tilde{z} : \Gamma \to G(\bar{F}) \rtimes \Gamma$ of the natural projection, and $\tilde{H}^1(\Gamma, G)$ for the set of $G(\bar{F})$-conjugacy classes of such sections. The assignment $z \mapsto \tilde{z}$ defined by $\tilde{z}(\sigma) = z(\sigma) \rtimes \sigma$ is a bijection $Z^1(\Gamma, G) \to \tilde{Z}^1(\Gamma, G)$ that descends to a bijection $H^1(\Gamma, G) \to \tilde{H}^1(\Gamma, G)$. We will switch freely in the notation between $\tilde{z}$ and z.

Given an automorphism a of G and $\delta \in G$ we have the element $\tilde{\delta} = \delta \rtimes a \in G \rtimes a \subset G \rtimes \langle a \rangle$. The assignment $\tilde{\delta} \mapsto \delta$ translates the action of G on $G \rtimes a$ by conjugation to the action of G on itself by a-twisted conjugation. When a is understood from the context, we will switch freely between $\tilde{\delta}$ and δ.

3 DISCONNECTED GROUPS

{sec:disc_grps}

3.1 Split disconnected groups and their forms

{sub:disc_forms}

Let F be a local field of characteristic zero. We denote by W_F the Weil group of F and by Γ the absolute Galois $\mathrm{Gal}(\overline{F}/F)$. In this paper we will study affine algebraic groups defined over F whose connected component is reductive. We will call such groups "disconnected reductive" for short. We will however restrict our attention to those disconnected reductive groups $\tilde{G}$ that satisfy the following condition:

{cnd:main}

Condition 3.1. *There exists an isomorphism defined over $\overline{F}$*

$$\tilde{G} \to G \rtimes A$$

where G is a connected reductive group, A is a finite group, and A acts on G by automorphisms which preserve a fixed $\bar{F}$-pinning.

Not all disconnected reductive groups satisfy this condition. The most basic counterexample is the normalizer of the torus in SL_2. On the other hand, this condition is satisfied by many naturally occurring disconnected reductive groups, including the orthogonal groups as well as the groups involved in the classification of tempered representations of connected reductive groups. The latter are among the main motivations for our study.

Just as in the connected case, one can classify the possible $\tilde{G}$ that satisfy the above condition in terms of root data and Galois cohomology. First, one can consider a split connected reductive group G defined over F and a finite group A, interpreted as a constant groups scheme over F, and let A act on G and preserve a fixed F-pinning. Then $G \rtimes A$ is a special case of a disconnected reductive group defined over F and we will call it "split". This adjective carries for us a double meaning – not only is the connected component G split, but the extension $G \rtimes A$ is also split. It is clear that the split disconnected reductive group $G \rtimes A$ is classified by the root datum of G and the action of A on this root datum.

Now fix an isomorphism $\iota : \tilde{G} \to G \rtimes A$ as in Condition 3.1. We may assume without loss of generality that G is split and that A preserves an F-pinning. Then

$$\Gamma \to \mathrm{Aut}(G \rtimes A), \quad \sigma \mapsto \iota^{-1}\sigma(\iota)$$

is a 1-cocycle and the isomorphism class of $\tilde{G}$ is determined by the split form $G \rtimes A$ and the cohomology class of this 1-cocycle.

In order to understand this cohomology better, we look more closely at $\mathrm{Aut}(G \rtimes A)$. The action of G on $G \rtimes A$ by conjugation factors through $G/Z(G)^A$, where $Z(G)^A$ is the group of fixed points for the action of A on the center of G. Thus $G/Z(G)^A$ is a subgroup of $\mathrm{Aut}(G \rtimes A)$, and in fact this subgroup is normal, because G is a characteristic subgroup of $G \rtimes A$ (being the neutral connected component).

Consider now the group $Z^1(A, Z(G))$ of 1-cocycles of A valued in $Z(G)$. The map sending $z \in Z^1(A, Z(G))$ to the automorphism $g \rtimes a \mapsto gz(a) \rtimes a$ of $G \rtimes A$ embeds $Z^1(A, Z(G))$ as a normal subgroup of $\mathrm{Aut}(G \rtimes A)$. The two normal subgroups $G/Z(G)^A$ and $Z^1(A, Z(G))$ of $\mathrm{Aut}(G \rtimes A)$ commute. Their intersection can be described as the subgroup $Z(G)/Z(G)^A$ of $G/Z(G)^A$, or equivalently its isomorphic image $B^1(A, Z(G)) \subset Z^1(A, Z(G))$ under the differential $z \mapsto z \cdot a(z)^{-1}$.

We thus have the normal subgroup $G/Z(G)^A \cdot Z^1(A, Z(G))$ of $\mathrm{Aut}(G \rtimes A)$. It has a complement. In order to specify it, we use the pinning of G defined over F and preserved by the action of A and let $\mathrm{Aut}_{\mathrm{pin}}(G \rtimes A)$ be those automorphisms of $G \rtimes A$ whose restriction to G preserves the pinning *and* which preserve the subgroup $1 \rtimes A$ of $G \rtimes A$. We conclude

$$\mathrm{Aut}(G \rtimes A) = (G/Z(G)^A \cdot Z^1(A, Z(G))) \rtimes \mathrm{Aut}_{\mathrm{pin}}(G \rtimes A).$$

This means that any form of $G \rtimes A$ can be obtained by a 3-step process: First, using an element of $Z^1(\Gamma, \mathrm{Aut}_{\mathrm{pin}}(G \rtimes A))$ one twists the rational structure of $G \rtimes A$. The result is again a group of the form $G \rtimes A$, where now G is a quasi-split connected reductive group, A is a (not necessarily constant) finite group scheme over F, and A acts on G again by automorphisms that preserve a fixed F-pinning. We shall call such disconnected reductive groups "quasi-split". Second, using an element $z \in Z^1(\Gamma, G/Z(G)^A)$ we twist the quasi-split group $\tilde{G} = G \rtimes A$ and obtain an "inner form" $\tilde{G}_z$ of it. Finally, we twist $\tilde{G}_z$ by an element of $Z^1(\Gamma, Z^1(A, Z(G)))$ to obtain a "translation form" of $\tilde{G}_z$.

3.2 Inner forms

{sub:inner}

Let G be a connected reductive group, defined and quasi-split over F. Let $(T, B, \{X_\alpha\})$ be an F-pinning of G and let A be a finite group that acts on G by pinned automorphisms. Assume given an action of Γ on A so that for $\sigma \in \Gamma$ we have $\sigma(a(g)) = \sigma(a)(\sigma(g))$. Thus $\tilde{G} = G \rtimes A$ is a quasi-split disconnected group in the sense of the previous subsection.

We have an exact sequence of algebraic groups

$$1 \to G \to \tilde{G} \to A \to 1$$

which leads to an exact sequence of topological groups

$$1 \to G(F) \to \tilde{G}(F) \to A^\Gamma \to 1.$$

Both of these extensions are split and come equipped with a splitting.

The group G acts on $\tilde{G}$ by conjugation and this action preserves the decomposition $\tilde{G} = \bigsqcup_{a \in A} G \rtimes a$ of $\tilde{G}$ into left G-cosets. In addition, the group $\tilde{G}$ acts on itself by conjugation, and this action preserves the group G.

Writing $\bar{G} = G/Z(G)^A$, the group $\bar{G} \rtimes \Gamma$ acts on the group $\tilde{G}$, with $\bar{g} \rtimes \sigma$ acting as the automorphism $\mathrm{Ad}(\bar{g}) \circ \sigma$. Given $\bar{z} \in Z^1(\Gamma, \bar{G})$, we denote by $\tilde{G}_{\bar{z}}$

the algebraic group defined over F which satisfies $\tilde{G}_{\bar{z}}(\overline{F}) = \tilde{G}(\overline{F})$ and where Γ acts on $\tilde{G}_{\bar{z}}(\overline{F})$ via the homomorphism $\tilde{\bar{z}} : \Gamma \to \bar{G} \rtimes \Gamma$ and the action of $\bar{G} \rtimes \Gamma$ on $\tilde{G}(\overline{F})$. We call $\tilde{G}_{\bar{z}}$ the *inner form* of $\tilde{G}$ corresponding to $\bar{z}$.

We still have the exact sequence of algebraic groups

$$1 \to G_{\bar{z}} \to \tilde{G}_{\bar{z}} \to A \to 1$$

but the sequence of F-points

$$1 \to G_{\bar{z}}(F) \to \tilde{G}_{\bar{z}}(F) \to A,$$

need not be exact. The image of the last map lies in A^{Γ} and we denote it by $A^{[\bar{z}]} \subset A^{\Gamma}$. We obtain an extension

$$1 \to G_{\bar{z}}(F) \to \tilde{G}_{\bar{z}}(F) \to A^{[\bar{z}]} \to 1.$$

Unlike the case of the quasi-split group $\tilde{G}$, this extension, even when it is split, does not come equipped with a distinguished splitting.

The action of $G_{\bar{z}}(F)$ on $\tilde{G}_{\bar{z}}$ preserves the subset $\tilde{G}_{\bar{z}}(F)$. The action of $\tilde{G}_{\bar{z}}(F)$ on $G_{\bar{z}}$ is realized by automorphisms defined over F and in particular preserves the subset $G_{\bar{z}}(F)$.

If we replace A by A^{Γ} the group $\tilde{G}_{\bar{z}}(F)$ remains unchanged. Since we shall ultimately be interested in the topological group $\tilde{G}_{\bar{z}}(F)$ and its representations, we will assume from now on that the action of Γ on A is trivial. In other words, we will treat the group A as a constant group scheme.

3.3 Strongly regular semi-simple elements and norms

{sub:norms}

We recall some material from [KS99]. An automorphism θ of G is called quasi-semi-simple if it preserves a Borel pair. A maximal torus that is part of a θ-stable Borel pair is called θ-admissible. The automorphism θ is furthermore called strongly regular if Cent(θ, G) is abelian. For such an automorphism θ, there is a unique θ-admissible maximal torus of G, namely Cent(Cent$(\theta, G), G)$. If $S \subset G$ is a θ-invariant maximal torus we will write $S_\theta = S/(1-\theta)S$ for the quotient of θ-coinvariants.

We shall call an element of $G \rtimes A$ (strongly regular) semi-simple, if the automorphism of G it induces by conjugation is (strongly regular) quasi-semi-simple. Clearly these notions are invariant under conjugacy by $G \rtimes A$.

{lem:c1}

Lemma 3.2. 1. *Let $\tilde{\delta} = \delta \rtimes a \in \tilde{G}(\bar{F})$ be semi-simple. Given an a-admissible maximal torus $S \subset G$ there exists $g \in G(\bar{F})$ such that $\tilde{\delta}^* = g^{-1}\tilde{\delta}g$ belongs to $S(\bar{F}) \rtimes a$.*

2. *Write $\tilde{\delta}^* = \delta^* \rtimes a$, so that $\delta^* \in S(\bar{F})$. Write $\gamma \in S_a(\bar{F})$ for the image of δ^* in the torus $S_a = S/(1-a)S$ of a-coinvariants. The set of pairs (S, γ) obtained in this way for a fixed $\tilde{\delta}$ and varying g forms a single $G^{a,\circ}(\bar{F})$-conjugacy class.*

3. *If (S_1,γ_1) and (S_2,γ_2) are two such pairs, then all $g \in G^{a,\circ}(\bar F)$ such that $Ad(g)(S_1,\gamma_1)=(S_2,\gamma_2)$ induce the same isomorphism $Ad(g): S_1 \to S_2$.*

4. *Given $\gamma \in S_a(\bar F)$, the set of $\tilde\delta \in \tilde G(\bar F)$ corresponding to the $G^{a,\circ}(\bar F)$-conjugacy class of (S,γ) is a single $G(\bar F)$-conjugacy class.*

Proof. This is essentially [KS99, Lemma 3.2.A]. Let $(T_{\tilde\delta}, B_{\tilde\delta})$ be a Borel pair normalized by $\tilde\delta$ and let C be a Borel subgroup containing S and normalized by a. Let $g \in G(\bar F)$ be such that $\mathrm{Ad}(g)(S,C)=(T_{\tilde\delta},B_{\tilde\delta})$. Set $\tilde\delta^*=g^{-1}\tilde\delta g$. Then (S,C) is normalized by both $\tilde\delta^*$ and a, so also by δ^*, hence $\delta^* \in S(\bar F)$.

For the second point, we fix for $i=1,2$ elements $g_i \in G(\bar F)$ such that $\tilde\delta_i^* = g_i^{-1}\tilde\delta g_i \in S_i(\bar F) \rtimes a$ and the image of δ_i^* is γ_i. Choose a-stable Borel subgroups C_i of G defined over $\bar F$ and containing S_i. Since any two a-stable Borel pairs are conjugate under $G^{a,\circ}(\bar F)$, we may modify g_2 to assume $S_1=S_2=S$ and $C_1=C_2=C$. Thus $\tilde\delta_1^*$ and $\tilde\delta_2^*$ belong to $S \rtimes a$ and are conjugate by $g := g_2^{-1}g_1$. It follows that $S^{a,\circ}$ and $\mathrm{Ad}(g^{-1})S^{a,\circ}$ are maximal tori of $\mathrm{Cent}(\tilde\delta_1^*,G)^\circ$. Modifying g_1 on the right we may assume $\mathrm{Ad}(g^{-1})$ normalizes S^a. Then it normalizes S and its image in $\Omega(S,G)$ is a-fixed. It is thus representable by an element of $G^{a,\circ}(\bar F)$.

For the third point, let $\delta_1^* \in S_1(\bar F)$ and $\delta_2^* \in S_2(\bar F)$ be elements mapping to γ_1 and γ_2 and such that $\tilde\delta_1^*$ and $\tilde\delta_2^*$ are $G(\bar F)$-conjugate to $\tilde\delta$. A given $g \in G^{a,\circ}(\bar F)$ with $\mathrm{Ad}(g)(S_1,\gamma_1)=(S_2,\gamma_2)$ can only be modified to hg for $h \in G^{a,\circ}(\bar F)$ normalizing S_2 and fixing γ_2. Thus there exists $s \in S_2(\bar F)$ such that $\mathrm{Ad}(sh) \in \mathrm{Cent}(\tilde\delta_2^*,G)=S_2^a$. It follows that the isomorphism $\mathrm{Ad}(g): S_1 \to S_2$ carrying γ_1 to γ_2 does not depend on the choice of g.

The final point follows immediately from the fact that the set of $\tilde\delta^*=\delta^* \rtimes a$ such that δ^* maps to γ forms a single $S(\bar F)$-conjugacy class. □

{dfn:norm}

Definition 3.3. Let $\tilde\delta = \delta \rtimes a \in \tilde G_{\bar z}(F)$ be strongly regular semi-simple. A *norm* of $\tilde\delta$ is a pair (S,γ) consisting of a maximal torus $S \subset G$ defined over F and a-admissible, and an element $\gamma \in S_a(F)$, such that there exists $g \in G(\bar F)$ the property that $\tilde\delta^* = g^{-1}\tilde\delta g \in S(\bar F) \rtimes a$ and the image of $\delta^* \in S(\bar F)$ in $S_a(\bar F)$ equals γ.

{lem:c2}

Lemma 3.4. *Let $\tilde\delta = \delta \rtimes a \in \tilde G_{\bar z}(F)$ be strongly regular semi-simple.*

1. *There exists a norm (S,γ) of $\tilde\delta$.*

2. *For any two norms (S_1,γ_1) and (S_2,γ_2) of $\tilde\delta$ the canonical isomorphism $Ad(g): S_1 \to S_2$, $g \in G^{a,\circ}(\bar F)$, carrying γ_1 to γ_2 is defined over F.*

Proof. The arguments for the first point are contained in the proofs of [KS99, Lemmas 3.3.B,3.3.C]. By Lemma 3.2 we may find $h \in G(\bar F)$ such that $\tilde\delta^0 := \mathrm{Ad}(h)^{-1}\tilde\delta \in T(\bar F) \rtimes A$, where we recall that T is the maximal torus that is part of the A-invariant F-pinning of G. Since $\tilde\delta$ is fixed by $\mathrm{Ad}(z_\sigma) \rtimes \sigma$ for all $\sigma \in \Gamma$, its G-conjugacy class is fixed by σ, and part (4) of Lemma 3.2 implies that the $\Omega(T,G)^a$-orbit of the image $\gamma^0 \in T_a(\bar F)$ of δ^0 is Γ-invariant. Thus for every $\sigma \in \Gamma$ there exists $w_\sigma \in \Omega(T,G)^a$ such that $w_\sigma\sigma(\gamma^0) = \gamma^0$. Since $\tilde{\delta}$ is strongly regular, no element of $\Omega(T,G)^a$ fixes γ^0, and hence w_σ is determined by σ. The map $\sigma \mapsto w_\sigma$ is a 1-cocycle. Since $\Omega(T,G)^a = \Omega(T_{\text{sc}}^a, G_{\text{sc}}^a)$, Steinberg's theorem implies the existence of $g \in G_{\text{sc}}^a$ such that $g^{-1}\sigma(g)$ normalizes T_{sc}^a and induces w_σ. Set $\tilde{\delta}^* = \text{Ad}(gh^{-1})\tilde{\delta}$. Then $S = \text{Ad}(g)T$ is the unique $\tilde{\delta}^*$-admissible maximal torus and we have $\delta^* \in S(\bar{F})$. The image $\gamma \in S_a(\bar{F})$ of $\tilde{\delta}^*$ under the projection $S \to S_a$ coincides with the image of γ^0 under $\text{Ad}(g) : T_a \to S_a$ and is Γ-fixed.

The second point follows from part (3) of Lemma 3.2, since both $\text{Ad}(g)$ and $\text{Ad}(\sigma(g))$ map (S_1,γ_1) to (S_2,γ_2). □

3.4 A-special Whittaker data

{sub:awhit}

We continue with a connected reductive group G defined and quasi-split over F, and A a finite group of automorphisms that leaves invariant an F-pinning of G.

Let $(T, B, \{X_\alpha\})$ be an F-pinning and let $\psi_F : F \to \mathbb{C}^\times$ be a non-trivial character. Recall from [KS99, §5.3] that one obtains a generic character $\psi : U(F) \to \mathbb{C}^\times$, where $U \subset B$ is the unipotent radical, via the following procedure: The fixed pinning induces an isomorphism from the abelianization $U^{\text{ab}} = U/[U,U]$ to $\prod_{\alpha\in\Delta} \mathbb{G}_a$. This isomorphism is defined over F if we let Γ act on the product in a way compatible with the action of Γ on Δ. The summation map $\prod_{\alpha\in\Delta} \mathbb{G}_a \to \mathbb{G}_a$ is then defined over F. The generic character ψ is given by the composition

$$U(F) \to U^{\text{ab}}(F) \to \Big(\prod_{\alpha\in\Delta} \mathbb{G}_a\Big)(F) \to F \to \mathbb{C}^\times.$$

Definition 3.5. An *A-special Whittaker datum* is a Whittaker datum obtained from an A-invariant F-pinning and a non-trivial character of F via the above procedure.

It is clear that an A-special Whittaker datum is A-invariant.

Fact 3.6. *Any two A-special Whittaker data are conjugate by $G_{ad}(F)^A$.*

Proof. We may realize the two A-special Whittaker data using two A-invariant F-pinnings and the same character $\psi : F \to \mathbb{C}^\times$. The result follows from the fact that the set of F-pinnings is a torsor for $G_{\text{ad}}(F)$. □

{fct:gasurj}

Fact 3.7. *The map $G^A \to G_{ad}^A$ is surjective.*

Proof. Let $(T, B, \{X_\alpha\})$ be an A-invariant F-pinning of G. Applying the Bruhat decomposition it is enough to consider a single cell $B_{\text{ad}}wB_{\text{ad}}$ for $w \in \Omega(T,G)^A$. Let $n \in N(T,G)$ be the Tits lift of w with respect to the pinning. Then $n \in N(T,G)^A$. The cell for G is then the set-wise direct product $T \times U \times \{n\} \times (U \cap w^{-1}\bar{U}w)$, where $\bar{U}$ is the unipotent radical of the Borel subgroup T-opposite to B. The restriction to U of the map $G \to G_{\text{ad}}$ is an isomorphism, and the cell for

G_{ad} is the set-wise direct product $T_{\text{ad}} \times U \times \{n\} \times (U \cap w^{-1}\bar{U}w)$. By directness of the product and the fact that n is A-fixed it is enough to prove that $T^A \to T_{\text{ad}}^A$ is surjective. But since A fixes a basis of $X^*(T_{\text{ad}})$ the group T_{ad}^A is connected and the result follows. □

{cor:asw}

Corollary 3.8. *Let $\mathfrak{w}$ be an A-special Whittaker datum. The set of $G(F)^A$-conjugacy classes of A-special Whittaker data is in 1-1 correspondence with*

$$im\Big(G_{ad}^A(F) \to H^1(F, Z(G)^A)\Big).$$

3.5 The twisted Kottwitz sign

{sub:sgn_twisted}

Let G be a quasi-split connected reductive F-group G, let a be an automorphism preserving an F-pinning, and let $\bar{z} \in Z^1(\Gamma, G/Z(G)^a)$. We have the inner form $[G \rtimes a]_{\bar{z}}$ of the coset $G \rtimes a$. We assume that $[G \rtimes a]_{\bar{z}}$ has F-point. Under this assumption we are going to define a sign $e([G \rtimes a]_{\bar{z}}) \in \{\pm 1\}$ generalizing the definition of the sign $e(G_{\bar{z}})$ due to Kottwitz [Kot83] in the sense that $e([G \rtimes \text{id}]_{\bar{z}}) = e(G_{\bar{z}})$.

By definition we will have $e([G \rtimes a]_{\bar{z}}) = e([G_{\text{ad}} \rtimes a]_{\bar{z}})$. Therefore, to lighten the notation, we may assume that G is adjoint. We then have $z \in Z^1(\Gamma, G)$, where we have dropped the bar from the notation. The existence of an F-point in $[G \rtimes a]_z$ is equivalent to the class of z being fixed by a. Let Z be the center of G_{sc} and let $\xi \in H^2(\Gamma, Z)^a$ be the image of the class of z under the connecting homomorphism for the exact sequence $1 \to Z \to G_{\text{sc}} \to G \to 1$.

We will now construct an element $\lambda \in H^0(\Gamma, X^*(Z))_a$, whose cup-product with ξ will be an element of order 2 in $H^2(\Gamma, \mathbb{G}_m)$. Its invariant will be the desired sign $e([G \rtimes a]_z) \in \{\pm 1\}$. Let (T, B) be a Borel pair in G_{sc} invariant under Γ and a. Let $\Omega \subset X^*(T)$ be the set of fundamental weights. This set receives an action of $\Gamma \times \langle a \rangle$. Let

$$\lambda_T = \sum_O \sum_{\chi \in O} \chi,$$

where O runs over a set of representatives for the action of $\langle a \rangle$ on the set of Γ-orbits in Ω. It is clear that $\lambda \in H^0(\Gamma, X^*(T))$ and that its image in the group $H^0(\Gamma, X^*(T))_a$ of a-coinvariants is independent of the choice of representatives. Via restriction we obtain the desired element $\lambda \in H^0(\Gamma, X^*(Z))_a$. Since any two Borel pairs of G that are invariant under both Γ and a are conjugate under $G_{\text{sc}}^a(F)$, we see that λ does not depend on the choice of a Borel pair (T, B). The definition of $e([G \rtimes a]_z) \in \bar{F}^\times$ is thus complete. We will see momentarily (Corollary 3.13) that $2\lambda = 0$ and hence $e([G \rtimes a]_z) \in \{\pm 1\}$.

{fct:sgn_kott}

Fact 3.9. *If $a = 1$ then λ_T is one half the sum of the positive roots and hence $e([G \rtimes a]_z) = e(G_z)$.*

{fct:sgn_prod}

Fact 3.10. *The sign $e([G \rtimes a]_z)$ is multiplicative: given (G_i, a_i, z_i) for $i = 1, 2$ we have $e([(G_1 \times G_2) \rtimes (a_1, a_2)]_{(z_1, z_2)}) = e([G_1 \rtimes a_1]_{z_1}) \cdot e([G_2 \rtimes a_2]_{z_2})$.*

Lemma 3.11. *Let $H = G \times G \cdots \times G$ and let b be the automorphism of H given by $b(g_0, \dots, g_{n-1}) = (g_1, \dots, g_{n-1}, a(g_0))$. Then* {lem:sgn_ind}

$$e([H \rtimes b]_z) = e([G \rtimes a]_z),$$

where we have used the obvious identification $H^1(\Gamma, H)^b = H^1(\Gamma, G)^a$.

Proof. If (T, B) is a Borel pair of G invariant under Γ and a, then $T_H = T \times \cdots \times T$ and $B_H = B \times \cdots \times B$ is a Borel pair of H invariant under Γ and b. The element $\lambda_{T_H} \in X^*(T_H) = X^*(T) \oplus \cdots \oplus X^*(T)$ is equal to $(\lambda_T, 0, \dots, 0)$, while the diagonal embedding $G \to H$ realizes the identification $H^1(\Gamma, H)^b = H^1(\Gamma, G)^a$. The claim follows. □

These observations reduce the study of $e([G \rtimes a]_z)$ to quasi-split adjoint groups of the form $G = \mathrm{Res}_{E/F} H$, where H is an absolutely simple quasi-split adjoint group defined over a finite extension E/F. Via the Shapiro isomorphism $H^1(F, G) = H^1(E, H)$, which on the level of cocycles is given by restriction followed by evaluation at 1, we obtain from z an element $z' \in Z^1(E, H)$. On the other hand, Lemma B.1 shows that the pinned automorphism a of G is related to a pinned isomorphism $a' : H \to H^{\sigma_0}$, for some $\sigma_0 \in \Gamma$ that normalizes E. What we mean by a' being a pinned isomorphism is this. We have by construction $H \times_E \bar{E} = H^{\sigma_0} \times_E \bar{E}$. The F-pinning of G arises from an E-pinning of H, which via this identification gives a pinning of $H^{\sigma_0} \times_E \bar{E}$ which is immediately seen to be Galois-invariant, i.e. an E-pinning.

We would like to relate the sign $e([G \rtimes a]_z)$ to a sign $e([H \rtimes a']_{z'})$, but for this we need to generalize the definition to allow for the more general situation that now a' is not an automorphism of the E-group H, but rather an isomorphism $H \to H^{\sigma_0}$. This is however very easy. Indeed, fixing a Borel pair (T, B) of H that is preserved by a' in the sense just explained, the set $\Omega \subset X^*(T)$ of fundamental weights receives an action of $\Gamma_E \rtimes \langle a' \rangle$, where $\Gamma_E = \mathrm{Gal}(\bar{F}/E)$ and a' acts on Γ_E via conjugation by σ_0. We still have an action of $\langle a' \rangle$ on the set of Γ_E-orbits in Ω and the formula for λ_T still makes sense. At the same time, for any Γ_E-module M with isomorphism $a' : M \to M^{\sigma_0}$ we have an action of a' on $H^i(\Gamma_E, M)$, given by conjugation by σ_0 on Γ_E, and the action of a' on M. One checks that $[z'] \in H^2(\Gamma_E, Z(H_{\mathrm{sc}}))^{a'}$ and $\lambda \in H^0(\Gamma_E, X^*(Z(H_{\mathrm{sc}})))_{a'}$. Pairing these gives the sign $e([H \rtimes a']_{z'})$.

Lemma 3.12. *We have $e([G \rtimes a]_z) = e([H \rtimes a']_{z'})$.* {lem:sgn_weil}

Corollary 3.13. *We have $e([G \rtimes a]_z)^2 = 1$.* {cor:sgn_sgn}

Proof. By the previous reduction steps it is enough to consider the case when G is absolutely simple and adjoint and a is a pinned isomorphism $G \to G^{\sigma_0}$ for some finite order automorphism σ_0 of F. If a is trivial then $e([G \rtimes a]_z)$ is the Kottwitz sign $e(G_z)$, so assume that a is non-trivial. Then G is of type A_n, D_n, or E_6. Consider the action of a on $X^*(Z(G_{\mathrm{sc}}))$. In type A_n this is the action of negation on $\mathbb{Z}/n\mathbb{Z}$, whose group of coinvariants is of order 1 or 2. In type

D_n the automorphism a has non-trivial coinvariants in $X^*(Z(G_{\text{sc}}))$ only if it is of order 2, in which case these coinvariants are again of order 2. In type E_6 the automorphism a acts by negation on $X^*(Z(G_{\text{sc}})) = \mathbb{Z}/3\mathbb{Z}$ and hence has a trivial group of coinvariants. In all cases we see that $H^0(\Gamma, X^*(Z(G_{\text{sc}})))^a$ is of order at most 2, hence the claim. □

Consider a parabolic pair (M_z, P_z) of G_z whose $G(\bar{F})$-conjugacy class is a-invariant. For example, this is the case for any minimal parabolic pair. Choose an a-invariant Borel pair (T_0, B_0) of G. There exists $g \in G(\bar{F})$ and a unique standard parabolic pair (M, P) such that $\text{Ad}(g)(M, P) = (M_z, P_z)$. Since $\text{Ad}(z_\sigma) \circ \sigma$ preserves (M_z, P_z) for each $\sigma \in \Gamma$, the standard pair (M, P) is Γ-invariant. Replacing z_σ by $g^{-1} z_\sigma \sigma(g)$ we may assume that $(M, P) = (M_z, P_z)$, i.e. (M, P) is Γ-invariant both as a parabolic pair of G as well as of G_z. This implies $z_\sigma \in Z^1(\Gamma, M)$ for all $\sigma \in \Gamma$.

Consider now $g_z \rtimes a \in [G \rtimes a]_z(F)$. By assumption $\text{Ad}(g_z \rtimes a)(M, P)$ is $G(\bar{F})$-conjugate to (M, P), hence also $G_z(F)$-conjugate. Thus, upon multiplying $g_z \rtimes a$ by an element of $G_z(F)$ on the left we may achive that it preserves (M, P). This means again that the $G(\bar{F})$-conjugacy class of (M, P) is a-invariant, hence (M, P) is itself a-invariant, which in turn implies $g_z \in M(\bar{F})$. We conclude $g_z \rtimes a \in [M \rtimes a](F)$. These preparations allow us to state the following.

Lemma 3.14. *$e([G \rtimes a]_z) = e([M \rtimes a]_z)$.* {lem:sgn_levi}

Proof. We maintain the notation of the preceding two paragraphs. We have $[z] \in H^1(\Gamma, M)^a$ and its image under $H^1(\Gamma, M)^a \to H^1(\Gamma, G)^a \to H^2(\Gamma, Z_{G_{\text{sc}}})^a$ is used in the construction of $e([G \rtimes a]_z)$; let us call this image h_G. On the other hand the image under $H^1(\Gamma, M)^a \to H^1(\Gamma, M_{\text{ad}})^a \to H^2(\Gamma, Z_{M_{\text{sc}}})^a$ is used in the construction of $e([M \rtimes a]_z)$; let us call this image h_2. Let $M^\dagger$ be the preimage of M in G_{sc}. This is a Levi subgroup of G_{sc} and its derived subgroup is the simply connected group M_{sc}. Therefore $Z_{G_{\text{sc}}} \subset Z_{M^\dagger} \supset Z_{M_{\text{sc}}}$. A look at the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z_{G_{\text{sc}}} & \longrightarrow & G_{\text{sc}} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & Z_{G_{\text{sc}}} & \longrightarrow & M^\dagger & \longrightarrow & M & \longrightarrow & 1 \\
 & & & & \uparrow & & \downarrow & & \\
1 & \longrightarrow & Z_{M_{\text{sc}}} & \longrightarrow & M_{\text{sc}} & \longrightarrow & M_{\text{ad}} & \longrightarrow & 1
\end{array}$$

shows that the images of h_1 and h_2 in $H^2(\Gamma, Z_{M^\dagger})^a$ agree. On the other hand we may consider the element $\lambda_{T_{G_{\text{sc}}}} \in H^0(\Gamma, X^*(T_{G_{\text{sc}}}))_a$ computed in terms of the Borel pair $(T_{G_{\text{sc}}}, B_{G_{\text{sc}}})$ of G_{sc} that is the preimage of (T_0, B_0). Let $B_{M^\dagger} = B_{G_{\text{sc}}} \cap M^\dagger$, $T_{M_{\text{sc}}} = T_{G_{\text{sc}}} \cap M_{\text{sc}}$, $B_{M_{\text{sc}}} = B_{G_{\text{sc}}} \cap M_{\text{sc}}$. Then $(T_{G_{\text{sc}}}, B_{M^\dagger})$ is a Borel pair for $M^\dagger$ and $(T_{M_{\text{sc}}}, B_{M_{\text{sc}}})$ is a Borel pair for M_{sc}. The set $\Delta_G^\vee$ of simple coroots for $(T_{G_{\text{sc}}}, B_{G_{\text{sc}}})$ is a basis for $X_*(T_{G_{\text{sc}}})$. It contains the set $\Delta_M^\vee$ of simple coroots for $(T_{M_{sc}}, B_{M_{sc}})$, which in turn is a basis for $X_*(T_{M_{sc}}) \subset X_*(T_{G_{sc}})$. Let $\Omega_M \subset \Omega$ be the set of those fundamental weights that pair non-trivially with an element of $\Delta_M^\vee$. The image of Ω_M under $X^*(T_{G_{sc}}) \to X^*(T_{M_{sc}})$ is a basis for $X^*(T_{M_{sc}})$, while the image of $\Omega \smallsetminus \Omega_M$ is zero. The map $X^*(T_{G_{sc}}) \to X^*(T_{M_{sc}})$ is equivariant both under Γ and a and the subset Ω_M of Ω is stable under both Γ and a. Therefore the image of under $X^*(T_{G_{sc}}) \to X^*(T_{M_{sc}})$ under $\lambda_{T_{G_{sc}}}$ equals the element $\lambda_{T_{M_{sc}}}$ computed in terms of $(T_{M_{sc}}, B_{M_{sc}})$. This shows that the image of $\lambda_{T_{G_{sc}}}$ under $X^*(\lambda_{T_{G_{sc}}}) \to X^*(Z(M^\dagger))$ has the property of having the same restriction to $X^*(Z_{G_{sc}})$ as $\lambda_{T_{G_{sc}}}$ and the same restriction to $X^*(Z_{M_{sc}})$ as $\lambda_{T_{M_{sc}}}$. □

We will now give alternative expressions for the sign $e([G \rtimes a]_z)$ in the two cases when F is assumed real or p-adic, beginning with the p-adic case. For a moment consider a connected reductive F-group J and an automorphism b of it. We are not assuming that J is quasi-split and we make no assumptions on b. Let (M_0, P_0) be a minimal F-parabolic pair for J. There exists $g \in J(F)$ such that $\mathrm{Ad}(g) \circ b$ preserves (M_0, P_0) and g is unique up to multiplication on the left by elements of $M_0(F)$. Therefore $\mathrm{Ad}(g) \circ b$ induces an action on the maximal split central torus A_0^J of M_0 and this action depends only on the image of b in the group $\mathrm{Out}(G)(F)$. We will write $(A_0^J)^b$ for the group of fixed points for the action of $\mathrm{Ad}(g) \circ b$ on A_0^J. We can apply this construction to the quasi-split adjoint p-adic group G and its inner form G_z, both of which have an F-rational automorphism a. In the case of G_z, that automorphism is well-defined only up to multiplication by an inner automorphism, but this is enough. Let A_0 and A_z be maximal split tori in G_0 and G_z.

{lem:sgn_padic}

Lemma 3.15. $e([G \rtimes a]_z) = (-1)^{\dim(A_0^a) - \dim(A_z^a)}$.

Proof. Define $e'([G \rtimes a]_z) = (-1)^{\dim(A_0^a) - \dim(A_z^a)}$, so that we want to show $e([G \rtimes a]_z) = e'([G \rtimes a]_z)$. The sign $e'([G \rtimes a]_z)$ does not change if we replace G by its adjoint group, because $\dim(A_0^a) = \dim((A_0/A_G)^a) + \dim(A_G^a)$ and analogously $\dim(A_z^a) = \dim((A_z/A_G)^a) + \dim(A_G^a)$. Note that $(A_0/A_G)^a$ and $(A_z/A_G)^a$ are tori, since in the notation of the group J above the action of $\mathrm{Ad}(g) \circ b$ on A_0^J preserves the set of simple relative roots, and those form a basis of $X^*(A_0^J/A_J)$.

One checks that e' satisfies the analogs of Fact 3.10 and Lemmas 3.11 and 3.12, and 3.14. This reduces to the case that G is adjoint and absolutely simple and G_z is anisotropic. Thus $G = \mathrm{PGL}_n$ and G' corresponds to a division algebra of degree n and invariant r/n for some r coprime to n. The class of G' in $H^1(\Gamma, G) = \mathbb{Z}/n\mathbb{Z}$ must be fixed by a, which acts by multiplication by -1. This forces $a = 1$ and the claim follows from Fact 3.9 and [Kot83]. □

4 The conjecture for pure inner forms

{sec:pure}

4.1 Pure inner forms

{sub:pure}

Let $z \in Z^1(\Gamma, G)$ and let $\tilde{z} : \Gamma \to G \rtimes \Gamma$ be the corresponding section. We have the inner form $\tilde{G}_z$ as in Subsection 3.2. We will call such inner forms pure, in analogy with the case of connected groups. In the exact sequence

$$1 \to G_z(F) \to \tilde{G}_z(F) \to A^{[z]} \to 1,$$

the group $A^{[z]}$ is the stabilizer in A^Γ of the cohomology class $[z] \in H^1(\Gamma, G)$.

4.2 Rational conjugacy classes

{sub:pure_rat}

For a given $a \in A$ we want to describe those $\tilde{\delta} = \delta \rtimes a \in G \rtimes A$ that are rational for $\tilde{G}_z$, i.e. $\tilde{\delta} \in \tilde{G}_z(F)$. This is by definition equivalent to the commutativity of $\tilde{z}(\sigma) = z(\sigma) \rtimes \sigma$ and $\tilde{\delta}$ for all $\sigma \in \Gamma$. Following Vogan's suggestion [Vog93] in the case of connected reductive groups, we shall consider all pure inner forms together, and are thus lead to consider the set of pairs $(\tilde{z}, \tilde{\delta})$, where $\tilde{z} \in \tilde{Z}^1(\Gamma, G)$ and $\tilde{\delta} \in G(\bar{F}) \rtimes A$ commute. This is the set of rational elements of all pure inner forms of $\tilde{G}$. The group $G(\bar{F})$ acts on this set by conjugation. Two elements $(\tilde{z}, \tilde{\delta}_1)$ and $(\tilde{z}, \tilde{\delta}_2)$ with the same first component lie in the same conjugacy class if and only if $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are conjugate under $G_z(F)$. Thus the set of $G(\bar{F})$-orbits of commuting pairs $(\tilde{z}, \tilde{\delta})$ can be seen as the set of rational conjugacy classes of rational elements of pure inner forms of $\tilde{G}$.

The set of rational elements, and its quotient under rational conjugacy, have the following cohomological interpretation. Given a pair $(\tilde{z}, \tilde{\delta})$, with $\tilde{z}(\sigma) = z(\sigma) \rtimes \sigma$ and $\tilde{\delta} = \delta \rtimes a$, the commutativity of $\tilde{z}$ and $\tilde{\delta}$ is equivalent to the equation $a(z(\sigma)) = \delta^{-1} z(\sigma) \sigma(\delta)$ for all $\sigma \in \Gamma$. This equation says that δ is a coboundary between the 1-cocycles z and $a(z)$. This leads us to consider the set

$$Z^1(\Gamma, G \underset{a}{\overset{1}{\rightrightarrows}} G)$$

consisting of pairs (z, δ), where $z \in Z^1(\Gamma, G)$ and $\delta \in G$ satisfy the above equation. Slightly more generally one could consider for two group homomorphisms $(b, a) : G \rightrightarrows G$ the set of pairs (z, δ) consisting of $z \in Z^1(\Gamma, G)$ and $\delta \in G$ such that $a(z(\sigma)) = \delta^{-1} b(z(\sigma)) \sigma(\delta)$. For our purposes the case $b = \mathrm{id}$ will be sufficient. In order to ease typesetting, we shall use the notation $Z^1_a(\Gamma, G \rightrightarrows G)$ instead. As just discussed, the set $Z^1_a(\Gamma, G \rightrightarrows G)$ is identified with the disjoint union $\bigsqcup_{z \in Z^1(\Gamma, G)} [G \rtimes a]_z(F)$. Taking the union over $a \in A$ we obtain an identification between the disjoint union $\bigsqcup_{z \in Z^1(\Gamma, G)} \tilde{G}_z(F)$ and the disjoint union $\bigsqcup_{a \in A} Z^1_a(\Gamma, G \rightrightarrows G)$.

Fix $a \in A$. The action of $G(\bar{F})$ by conjugation on the set of pairs $(\tilde{z}, \tilde{\delta})$ is translated to the action of $G(\bar{F})$ on $(z, \delta) \in Z^1_a(\Gamma, G \rightrightarrows G)$ by $g(z, \delta) = (gz(\sigma)\sigma(g^{-1}), g\delta a(g)^{-1})$. We let $H^1_a(\Gamma, G \rightrightarrows G)$ be the set of orbits of that action and thus obtain an identification of

$$\bigsqcup_{a \in A} H^1_a(\Gamma, G \rightrightarrows G)$$

with the set of rational conjugacy classes of rational elements of pure inner forms of $\tilde{G}$.

Keeping in line with our notation, we shall write $\tilde{Z}^1_a(\Gamma, G \rightrightarrows G)$ for the set of commuting $\tilde{z} \in \tilde{Z}^1(\Gamma, G)$ and $\tilde{\delta} \in G \rtimes a$, and $\tilde{H}^1_a(\Gamma, G \rightrightarrows G)$ for their G-conjugacy classes, and will freely use the bijections $Z^1_a(\Gamma, G \rightrightarrows G) \to \tilde{Z}^1_a(\Gamma, G \rightrightarrows G)$ and $H^1_a(\Gamma, G \rightrightarrows G) \to \tilde{H}^1_a(\Gamma, G \rightrightarrows G)$ given by $(z, \delta) \mapsto (\tilde{z}, \tilde{\delta})$.

4.3 The invariant

{sub:pure_inv}

We are particularly interested in the G-conjugacy classes of pairs $(\tilde{z}, \tilde{\delta})$ for which $\tilde{\delta}$ is semi-simple and strongly regular. According to Lemma 3.4 such a conjugacy class has a norm (S, γ), well-defined up to $G^{a,\circ}(\bar{F})$-conjugacy. We shall now define an element $\text{inv}(\gamma, (z, \delta)) \in H^1_a(\Gamma, S \rightrightarrows S)$.

{lem:inv}

Lemma 4.1. *1. If (z^*, δ^*) is a representative of the equivalence class of (z, δ) such that $\delta^* \in S$ and the image of δ^* in S_a is γ, then $z^*(\sigma) \in S$ and hence $(z^*, \delta^*) \in Z^1_a(\Gamma, S \rightrightarrows S)$. The class $inv(\gamma, (z, \delta)) \in H^1_a(\Gamma, S \rightrightarrows S)$ is independent of the choice of (z^*, δ^*).*

2. If (S', γ') is another norm of the same equivalence class, and (z', δ') the corresponding representative, the unique isomorphism $Ad(g) : S \to S'$ mapping γ to γ' induces an isomorphism $H^1_a(\Gamma, S \rightrightarrows S) \to H^1_a(\Gamma, S' \rightrightarrows S')$ identifying the class $inv(\gamma, (z, \delta))$ of (z^*, δ^*) with the class $inv(\gamma', (z, \delta))$ of (z', δ').

Proof. Since the element γ is Γ-fixed, the $S(\bar{F})$-conjugacy class of $\tilde{\delta}^*$ is Γ-fixed (in fact the two statements are equivalent). For $\sigma \in \Gamma$ let $s(\sigma) \in S$ be such that $\sigma(\tilde{\delta}^*) = \text{Ad}(s(\sigma))\tilde{\delta}^*$. Since $\tilde{z}^*(\sigma) = z^*(\sigma) \rtimes \sigma$ commutes with $\tilde{\delta}^*$ we see that $z^*(\sigma)s(\sigma)$ commutes with $\tilde{\delta}^*$, thus $z^*(\sigma)s(\sigma) \in S^a$, thus $z^*(\sigma) \in S$.

Let us show that the class of $(\tilde{z}^*, \tilde{\delta}^*)$ in $\tilde{H}^1_a(\Gamma, S \rightrightarrows S)$ is independent of the choice of $(\tilde{z}^*, \tilde{\delta}^*)$. Another such choice is of the form $\text{Ad}(h)(\tilde{z}^*, \tilde{\delta}^*)$ for some $h \in G(\bar{F})$. By assumption $h\delta^* a(h^{-1})$ maps to γ, so there exists $s \in S(\bar{F})$ such that $sh\delta^* a(sh)^{-1} = \delta^*$, i.e. $sh \in \text{Cent}(\tilde{\delta}, G) = S^a$, thus $h \in S$, as claimed.

Now let (S', γ') be another norm and choose by Lemma 3.2 an element $g \in G^{a,\circ}(\bar{F})$ such that $\text{Ad}(g) : S \to S'$ carries γ to γ'. Then $\text{Ad}(g)(\tilde{z}^*, \tilde{\delta}^*)$ is a representative of the conjugacy class of $(\tilde{z}, \tilde{\delta})$ and lies in $\tilde{Z}^1(\Gamma, S' \rightrightarrows S')$. □

4.4 Comparison with [KS99]

{sec:comp}

It would be informative to compare the notions of norms and invariants given here with those in [KS99]. In short, the notions of norms are the same apart from cosmetics, while the notion of invariant introduced here is the twisted analog of the untwisted absolute invariant introduced in [Kal11, §2.1], and thus refines the relative invariant introduced in [KS99, §4.4], and generalizes the absolute invariant introduced in the setting of quasi-split groups in [KS99, §5.3].

More precisely, let $(\tilde{z}, \tilde{\delta}) \in \tilde{Z}^1_a(\Gamma, G \rightrightarrows G)$. Then $\theta = \text{Ad}(\tilde{\delta})$ is an automorphism of G_z defined over F. The element $\tilde{\delta}$ is semi-simple and strongly-regular if and only if $1 \in G_z(F)$ is θ-semi-simple and θ-strongly regular. Let $\theta^* = a$.

The map $m : Cl(G_z, \theta) \to Cl(G, \theta^*)$ of [KS99, §3.1] is given by $h \mapsto h \cdot \tilde{\delta}$ and in particular sends 1 to $\tilde{\delta}$. The 1-cochain defined in [KS99, Lemma 3.1.A] and denote by z_σ there (beware that this is not the same as our z_σ here) is identically equal to 1.

Now let $(\tilde{z}^*, \tilde{\delta}^*)$ and $\gamma \in S_a(F)$ be be as in Lemma 4.1. Then γ is a norm of 1 in the sense of [KS99, §3.3]. Moreover, if $g \in G$ is such that $g^{-1}(\tilde{z}^*, \tilde{\delta}^*)g = (\tilde{z}, \tilde{\delta})$, then the 1-cocycle $v(\sigma) = gu(\sigma)\sigma(g)^{-1}$ considered in [KS99, Lemma 4.4.A], which takes values in S_{sc}, when composed with the natural map $S_{\text{sc}} \to S$, becomes equal to z^*.

We have the the isomorphism

$$Z^1_a(\Gamma, S \rightrightarrows S) \to Z^1(\Gamma, S \xrightarrow{1-a} S), \qquad (z, \delta) \mapsto (z^{-1}, \delta) \tag{4.1}$$ {eq:ksiso}

and it allows us to view $\text{inv}(\gamma, (z, \delta))$ as an element of $H^1(\Gamma, S \xrightarrow{1-a} S)$. This element is then an absolute version of the relative invariant $\text{inv}(\gamma, \delta; \bar{\gamma}, \bar{\delta})$ introduced in [KS99, §4.4], and a generalization of the absolute invariant $\text{inv}(\gamma, \delta)$ that was introduced in [KS99, §5.3].

4.5 The dual group

{sub:dual}

Let $\widehat{G}$ be a complex dual group for G. We rigidify it by fixing a Γ-invariant pinning $(\widehat{T}, \widehat{B}, \{\widehat{X}_\alpha\})$ and requiring it to be dual to the fixed pinning of G. That is, we assume given an identification $X^*(\widehat{T}) = X_*(T)$ under which the B-positive coroots are identified with the $\widehat{B}$-positive roots. We define the L-group of G as ${}^LG = \widehat{G} \rtimes W_F$, where W_F acts on $\widehat{G}$ by fixing the pinning. We also let the group A act on $\widehat{G}$ by fixing the pinning. More precisely, given $a \in A$, we have the automorphism a_* of $X_*(T)$ given by $(a_*\lambda)(x) = a(\lambda(x))$ for $x \in \mathbb{G}_m$, and we let a act on $\widehat{T} = \text{Hom}(X_*(T), \mathbb{C}^\times)$ by $[at](\lambda) = t(a_*^{-1}\lambda)$ for $t \in \widehat{T}$ and $\lambda \in X_*(T)$. Note that the automorphism a of $\widehat{G}$ obtained in this way is related to the automorphism $\widehat{\theta}$ of $\widehat{G}$ obtained from $\theta^* = a$ as in [KS99, §1.2] by $a = \widehat{\theta}^{-1}$. This will later have the effect of $H^1(\Gamma, S \xrightarrow{1-a} S)$ being paired with $H^1(W_F, \widehat{S} \xrightarrow{1-a^{-1}} \widehat{S})$.

4.6 The local correspondence

{sub:pure_llc}

Given an irreducible admissible representation of the locally profinite group $\tilde{G}_z(F)$, its restriction to $G_z(F)$ is a finite length semi-simple admissible representation. We shall say that a representation of $\tilde{G}_z(F)$ is G-tempered respectively G-discrete, if its restriction to $G_z(F)$ contains (equivalently, consists of) a tempered respectively discrete representations.

We will now begin formulating the refined local Langlands conjecture for the disconnected groups $\tilde{G}_z$. The irreducible admissible G-tempered representations of $\tilde{G}_z$ will again be parameterized by pairs (ϕ, ρ). The first part of the pair, the Langlands parameter ϕ, will remain unchanged. That is, we will use the same tempered Langlands parameters $\phi : L_F \to {}^LG$ as for the connected group G. However, we will change what we mean by equivalence of parameters. Two parameter will be seen as $\tilde G$-equivalent if they are conjugate under the group $\widehat{G} \rtimes A$. Given a parameter ϕ, its group of $\tilde G$-self-equivalences is then $\tilde S_\phi = \text{Cent}(\phi, \widehat{G} \rtimes A)$. This group contains the group $S_\phi = \text{Cent}(\phi, \widehat{G})$ of G-self-equivalences of ϕ. We have the exact sequence

$$1 \to S_\phi \to \tilde S_\phi \to A^{[\phi]} \to 1,$$

where $A^{[\phi]}$ is the stabilizer in A of the G-equivalence class of ϕ. This exact sequence leads to the exact sequence

$$1 \to \pi_0(S_\phi) \to \pi_0(\tilde S_\phi) \to A^{[\phi]} \to 1.$$

Recall from [Kot86] that the cohomology class $[z]$ gives a character $\pi_0(Z(\widehat{G})^\Gamma) \to \mathbb{C}^\times$, which we will also denote by $[z]$. The stabilizer of this character in A is equal to the stabilizer of the cohomology class of z – this is immediate if F is p-adic, and can be checked if $F = \mathbb{R}$. Let $A^{[\phi],[z]} = A^{[\phi]} \cap A^{[z]}$. If we pull back the above extension to $A^{[\phi],[z]}$ we obtain the extension

$$1 \to \pi_0(S_\phi) \to \pi_0(\tilde S_\phi^{[z]}) \to A^{[\phi],[z]} \to 1,$$

where $\tilde S_\phi^{[z]} = \text{Cent}(\phi, \widehat{G} \rtimes A^{[z]})$. The pull-back of an irreducible representation of $\pi_0(\tilde S_\phi^{[z]})$ to $\pi_0(Z(\widehat{G})^\Gamma)$ is either $[z]$-isotypic, or it does not contain $[z]$. We write $\text{Irr}(\pi_0(\tilde S_\phi^{[z]}), [z])$ for the set of irreducible representations of $\pi_0(\tilde S_\phi^{[z]})$ whose pull-back to $\pi_0(Z(\widehat{G})^\Gamma)$ is $[z]$-isotypic.

Let us remark at this point that we could alternatively consider the set $\text{Irr}(\pi_0(\tilde S_\phi), [z])$ of those irreducible representations whose restriction to $\pi_0(Z(\widehat{G})^\Gamma)$ contains the character $[z]$. Since $Z(\widehat{G})^\Gamma$ is not central in $\tilde S_\phi$, this restriction will contain other characters as well. According to Clifford theory induction from $\tilde S_\phi^{[z]}$ to $\tilde S_\phi$ gives a bijection between $\text{Irr}(\pi_0(\tilde S_\phi^{[z]}), [z])$ and $\text{Irr}(\pi_0(\tilde S_\phi), [z])$. Indeed, any element of $\tilde S_\phi$ that normalizes $\tilde S_\phi^{[z]}$ and stabilizes $\rho \in \text{Irr}(\pi_0(\tilde S_\phi^{[z]}), [z])$ also stabilizes $[z]$ and thus belongs to $\tilde S_\phi^{[z]}$, so $\text{Ind}\rho$ is irreducible. However, working with $\text{Irr}(\pi_0(\tilde S_\phi^{[z]}), [z])$ will be more convenient for us.

Consider for a moment the special case $z = 1$. Choose an A-special Whittaker datum $\mathfrak{w}$ for G. Any $\mathfrak{w}$-generic representation π of $G(F)$ has a canonical extension $\tilde\pi$ to $\tilde G(F) = G(F) \rtimes A^\Gamma$, obtained by setting $\tilde\pi(a)$ to be the unique $G(F)$-map $\pi \circ a^{-1} \to \pi$ that preserves one (hence any) $\mathfrak{w}$-Whittaker functional. We shall say that these $\tilde\pi$ are $\mathfrak{w}$-generic representations of $\tilde G(F)$. We can now state the first part of the local Langlands conjecture for the groups $\tilde G_z$. In the case $F = \mathbb{R}$ set ${}^K\tilde G_z$ to be the associated K-group, i.e. the disjoint union of $\tilde G_{z'}$ for all z' in the image of $H^1(\mathbb{R}, G_{z,\text{sc}}) \to H^1(\mathbb{R}, G_z) \to H^1(\mathbb{R}, G)$.

{cnj:llc_pure_is}

Conjecture 4.2. *The choice of an A-special Whittaker datum $\mathfrak{w}$ on G determines a bijection between the set of irreducible admissible G-tempered representations of $\tilde{G}_z(F)$ when $F/\mathbb{Q}_p$, or any member of ${}^K\tilde{G}_z(F)$ when $F=\mathbb{R}$, and the set of $\widehat{G}\rtimes A$-conjugacy classes of pairs $(\phi,\tilde{\rho})$, where $\phi : L_F \to {}^LG$ is a tempered Langlands parameter, and $\tilde{\rho}\in Irr(\pi_0(\tilde{S}_\phi^{[z]}),[z])$. When $z=1$ the representation corresponding to $(\phi,\tilde{\rho})$ is $\mathfrak{w}$-generic if and only if $\tilde{\rho}=1$.*

Let us write $\tilde{\Pi}_{\phi,z}$ for the finite set of representations of $\tilde{G}_z(F)$ corresponding to pairs $(\phi,\tilde{\rho})$ for a fixed ϕ and varying $\tilde{\rho}$. These can be called L-packets for the disconnected group $\tilde{G}_z(F)$. In the §4.13 we will add another piece of the conjecture, which will in particular determine uniquely the sets $\tilde{\Pi}_{\phi,z}$ in terms of the L-packets of the connected group G_z. The new information in Conjecture 4.2 is thus contained in the bijection between $\tilde{\Pi}_{\phi,z}$ and $\mathrm{Irr}(\pi_0(\tilde{S}_\phi^{[z]}),[z])$.

4.7 Endoscopic data

{sub:endo}

We shall use essentially the same notion of endoscopic data as in [KS99, §2.1], with one minor but important difference that affects both the definition of datum as well as of an isomorphism of data. More precisely, an endoscopic datum will be a tuple $\mathfrak{e}=(G^\mathfrak{e},\mathcal{G}^\mathfrak{e},\tilde{s}^\mathfrak{e},\xi^\mathfrak{e})$ consisting of

(4.7.1) a quasi-split connected reductive group $G^\mathfrak{e}$ defined over F;

(4.7.2) a split extension $\mathcal{G}^\mathfrak{e}$ of W_F by $\widehat{G}^\mathfrak{e}$ (but without the choice of splitting);

(4.7.3) a semi-simple element $\tilde{s}^\mathfrak{e}\in\widehat{G}\rtimes A$;

(4.7.4) a homomorphism $\xi^\mathfrak{e} : \mathcal{G}^\mathfrak{e}\to {}^LG$ of extensions;

and satsfying

(4.7.5) the homomorphism $W_F \to \mathrm{Out}(\widehat{G}^\mathfrak{e})$ arising from the extension $\mathcal{G}^\mathfrak{e}$ is transported under the canonical identification $\mathrm{Out}(\widehat{G}^\mathfrak{e})=\mathrm{Out}(G^\mathfrak{e})$ to the one given by the rational structure of $G^\mathfrak{e}$;

(4.7.6) $\xi^\mathfrak{e}$ induces an isomorphism $\widehat{G}^\mathfrak{e}\to\mathrm{Cent}(\tilde{s}^\mathfrak{e},\widehat{G})^\circ$; {item:e0}

(4.7.7) $\tilde{s}^\mathfrak{e}$ commutes with the image of $\xi^\mathfrak{e}$. {item:e1}

This completes the description of the tuple $\mathfrak{e}$. An isomorphism $\mathfrak{e}\to\mathfrak{e}'$ is an element $g\in\widehat{G}$ satisfying

(4.7.8) $\xi^{\mathfrak{e}'}=\mathrm{Ad}(g)\circ\xi^\mathfrak{e}$;

(4.7.9) $\tilde{s}^{\mathfrak{e}'}=\mathrm{Ad}(g)\tilde{s}^\mathfrak{e}$ modulo $Z(\widehat{G})^\circ$. {item:e2}

The difference between these definitions and those in [KS99] is the following. First, we are only considering here the case $\omega = 1$ and hence $\mathbf{a} = 1$. Second, our requirement (4.7.7) is stricter than [KS99, (2.1.4a)]. The definition [KS99, (2.1.6)] of isomorphism however implies that every isomorphism class of endoscopic data in the sense of [KS99] contains a representative that satisfies (4.7.7). Third, our requirement (4.7.9) is stricter than [KS99, (2.1.6)]. This implies that a single isomorphism class in the sense of [KS99] can consist of multiple isomorphism classes in the sense of our definition.

4.8 Two constructions of endoscopic data

{sub:endocnst}

We now review two constructions of endoscopic data, one geometric and one spectral. In the case of connected groups, they are summarized in [She83, §4.2]. In the twisted case the geometric appears in the proof of [KS99, Lemma 7.2] and the spectral one appears at the end of [KS99, §2].

We begin with the spectral construction, which is a little easier to describe. Let $\phi : L_F \to {}^LG$ be an L-parameter and $\tilde{s} \in \tilde{S}_\phi$ a semi-simple element. The pair $(\phi, \tilde{s})$ leads to an endoscopic datum as follows. Set $\widehat{G}^\mathfrak{e} = \text{Cent}(\tilde{s}, \widehat{G})^\circ$, $\mathcal{G}^\mathfrak{e} = \widehat{G}^\mathfrak{e} \cdot \phi(W_F)$, and let $\xi^\mathfrak{e}$ be the natural inclusion. Let $G^\mathfrak{e}$ be the quasi-split group defined over F that is dual to $\widehat{G}^\mathfrak{e}$ and whose rational structure is determined by $\Gamma \to \text{Out}(\widehat{G}^\mathfrak{e}) = \text{Out}(G^\mathfrak{e})$, the first map coming from the extension $\mathcal{G}^\mathfrak{e}$. Then $(G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, \tilde{s}, \xi^\mathfrak{e})$ is an endoscopic datum. Note that ϕ factors through $\xi^\mathfrak{e}$ and thus becomes a parameter for $\mathcal{G}^\mathfrak{e}$ (in order to relate it to $G^\mathfrak{e}$, one needs to further choose a z-pair).

For the geometric construction, let $\tilde{G}_{\bar{z}}$ be an inner form of $\tilde{G}$ and let $\tilde{\delta} \in \tilde{G}_{\bar{z}}(F)$ be strongly regular semi-simple. Let $S' \subset G$ be the maximal torus $\text{Cent}(\text{Cent}(\tilde{\delta}, G), G)$. As a maximal torus of $G_{\bar{z}}$ it is defined over F, and $\text{Ad}(\tilde{\delta}) : S' \to S'$ is an automorphism defined over F. Let $\kappa \in H^1(W_F, (1-\tilde{\delta}) : \widehat{S}' \to \widehat{S}')$. The pair $(\tilde{\delta}, \kappa)$ leads to an endoscopic datum $(G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, \tilde{s}^\mathfrak{e}, \xi^\mathfrak{e})$ and a stable conjugacy class of elements $\gamma^\mathfrak{e} \in G^\mathfrak{e}(F)$ as follows.

Choose a norm (S, γ) of $\tilde{\delta}$ and $g \in G(\bar{F})$ such that $g^{-1}\tilde{\delta}g = \tilde{\delta}^* = \delta^* \rtimes a \in S(\bar{F}) \rtimes a$ with $\delta^* \mapsto \gamma$. Then $\text{Ad}(g)$ provides an isomorphism $H^1(W_F, (1-\tilde{\delta}) : \widehat{S}' \to \widehat{S}') \to H^1(W_F, (1-a) : \widehat{S} \to \widehat{S})$. Choose an a-invariant Borel pair $(\widehat{T}, \widehat{B})$ of $\widehat{G}$ and an a-invariant Borel subgroup C containing S. These lead to an equivariant isomorphism $\widehat{S} \to \widehat{T}$ under which a 1-hypercocycle representing κ is transported to a pair (t_w^{-1}, s) satisfying, for every $w \in W_F$, the relation $s^{-1}\sigma_S(s) = t_w^{-1}a(t_w)$, where $\sigma \in \Gamma$ is the image of w and σ_S is the transport of the action of σ on $\widehat{S}$ to $\widehat{T}$. This transport is given by $\omega_\sigma \rtimes \sigma$, for a uniquely determined $\omega_\sigma \in \Omega(\widehat{T}, \widehat{G})^a$. The map $\sigma \mapsto \omega_\sigma$ belongs to $Z^1(\Gamma, \Omega(\widehat{T}, \widehat{G})^a)$. In the long exact sequence of Γ-cohomology associated to the short exact sequence

$$1 \to \widehat{T}^{a,\circ} \to N(\widehat{T}^{a,\circ}, \widehat{G}^{a,\circ}) \to \Omega(\widehat{T}, \widehat{G})^a \to 1$$

the image of the class of ω_σ is an element of $H^2(\Gamma, \widehat{S})$, whose restriction to $H^2(W_F, \widehat{S})$ vanishes according to [Lan79, Lemma 4]. It follows that there exist lifts $n_\sigma \in N(\widehat{T}^{a,\circ}, \widehat{G}^{a,\circ})$ of ω_σ so that $w \mapsto n_w \rtimes w$ is a homomorphism $W_F \to N(\widehat{T}^{a,\circ}, \widehat{G}^{a,\circ}) \rtimes W_F$. Then $\eta : w \mapsto t_w n_w \rtimes w$ is a group homomorphism $W_F \to N(\widehat{T}, \widehat{G})$ whose image commutes with $s \rtimes a$. Define $\tilde{s}^\mathfrak{e} = s \rtimes a$, $\widehat{G}^\mathfrak{e} = \mathrm{Cent}(\tilde{s}^\mathfrak{e}, \widehat{G})^\circ$, $\mathcal{G}^\mathfrak{e} = \widehat{G}^\mathfrak{e} \cdot \eta(W_F)$, and let $\xi^\mathfrak{e}$ be the natural inclusion. Let $G^\mathfrak{e}$ be the quasi-split group defined over F, dual to $\widehat{G}^\mathfrak{e}$, and with rational structure determined by $\Gamma \to \mathrm{Out}(\widehat{G}^\mathfrak{e}) = \mathrm{Out}(G^\mathfrak{e})$, where the first map comes from the extension $\mathcal{G}^\mathfrak{e}$. It is immediately checked that $(G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, \tilde{s}^\mathfrak{e}, \xi^\mathfrak{e})$ is an endoscopic datum.

The a-equivariant isomorphism $\widehat{S} \to \widehat{T}$ and the inclusion $\widehat{T}^{a,\circ} \to \widehat{G}^\mathfrak{e}$ give a caonical $G^\mathfrak{e}(\bar{F})$-conjugacy class of embeddings $S_a \to G^\mathfrak{e}$. Thus the element γ gives a canonical $G^\mathfrak{e}(\bar{F})$-conjugacy class of strongly regular semi-simple elements of $G^\mathfrak{e}(\bar{F})$. This class is Γ-invariant, so by [Kot82, Corollary 2.2] gives a stable class of elements $\gamma^\mathfrak{e} \in G^\mathfrak{e}(F)$. This completes the geometric construction.

4.9 Normalized transfer factor invariant under $\tilde{G}_z(F)$

{sub:pure_tf}

Fix an A-special Whittaker datum as in §3.4. Let $\mathfrak{e} = (G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, \tilde{s}^\mathfrak{e}, \xi^\mathfrak{e})$ be an endoscopic datum. There may or may not exist an isomorphism $\mathcal{G}^\mathfrak{e} \to {}^LG^\mathfrak{e}$ of extensions of W_F by $\widehat{G}^\mathfrak{e}$. If it does we choose one such, denote it by $\xi^\mathfrak{z}$ and write $G^\mathfrak{z} = G^\mathfrak{e}$. If it does not, we can choose a z-extension $G^\mathfrak{z} \to G^\mathfrak{e}$ and apply [KS99, Lemma 2.2.A] which guarantees that the inclusion $\widehat{G}^\mathfrak{e} \to \widehat{G}^\mathfrak{z}$ always extends to an L-embedding $\xi^\mathfrak{z} : \mathcal{G}^\mathfrak{e} \to {}^LG^\mathfrak{z}$. We denote by $\mathfrak{z}$ the pair $(G^\mathfrak{z}, \xi^\mathfrak{z})$.

We will define a normalized absolute transfer factor $\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}]$ as a function that assigns complex values to pairs $(\gamma^\mathfrak{z}, \tilde{\delta})$ of $G^\mathfrak{z}(F) \times \tilde{G}_z(F)$, where both $\gamma^\mathfrak{z}$ and $\tilde{\delta}$ are strongly regular semi-simple. As a function of $\tilde{\delta}$ the transfer factor $\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}]$ will be conjugation-invariant under the full group $\tilde{G}_z(F)$. The definition is by the formula

$$\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^\mathfrak{z}, \tilde{\delta}) = \sum_{c \in \tilde{G}_z(F)/G_z(F)} \Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^\mathfrak{z}, c\tilde{\delta}c^{-1}), \qquad (4.2)$$ {eq:pure_tf}

which in turn uses a normalized absolute Kottwitz-Shelstad transfer factor $\Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}]$ that we will define below. The latter is a function that assigns complex values to pairs $(\gamma^\mathfrak{z}, \tilde{\delta})$ of $G^\mathfrak{z}(F) \times [G \rtimes b^{-1}]_z(F)$, where $b \in A$ is the image of $\tilde{s}^\mathfrak{e}$ and both $\gamma^\mathfrak{z}$ and $\tilde{\delta}$ are strongly regular semi-simple. In the variable $\tilde{\delta}$ this function is only $G_z(F)$-conjugation invariant.

Following [KS, §5.5], the factor $\Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}]$ is defined by

$$\Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}] := e([G \rtimes b^{-1}]_{\bar{z}}) \epsilon_L(V, \psi) (\Delta_I^{\mathrm{new}})^{-1} \Delta_{II} (\Delta_{III}^{\mathrm{new}})^{-1} \Delta_{IV}. \qquad (4.3)$$ {eq:pure_tf1}

The terms $\epsilon_L(V, \psi)$, Δ_I^{new}, Δ_{II}, and Δ_{IV} have already been defined, in [KS99, §5.3], [KS, §3.4], [KS99, §4.3], and [KS99, §4.5], respectively, but we will recall them for the convenience of the reader below. They are absolute, i.e. they depend on a single pair of elements $(\gamma^\mathfrak{z}, \tilde{\delta})$. The term $\Delta_{III}^{\mathrm{new}}$ will be defined in this paper. It is also absolute. A relative version of it, i.e. one depending on two pairs of related elements, was defined in [KS99, §4.4]; in the quasi-split case $z = 1$ an absolute version was defined in [KS99, §5.3]; in the untwisted case an absolute version was defined in [Kal11] for pure inner forms of p-adic groups, and generalized in [Kal16b] to arbitrary inner forms of connected groups over local fields. In this paper we will define an absolute version for arbitrary z in the twisted setting. The term $e([G \rtimes b^{-1}]_{\bar{z}})$ was defined in §3.5. When $\bar{z} = 1$ it is equal to 1, while the term $\Delta_{III}^{\text{new}}$ coincides with the absolute term Δ_{III} defined in [KS99, §5.3]. The factor Δ_{KS} is an absolute transfer factor whose relative version is the factor Δ' of [KS, §5.4]. When $z = 1$ then Δ_{KS} coincides with the absolute factor [KS, (5.5.2)]. When $b = 1$ the factor Δ_{KS} differs from the factor [Kal16b, (5.10)] by the term $e(G_z)$, which in the notation of loc. cit. would be $e(G')$. This change is made for convenience of exposition and will be reflected in the absence of the term $e(G_z)$ in the character identities (4.4) as compared to [Kal16b, (5.9),(5.11)].

Before we come to $\Delta_{III}^{\text{new}}$ we briefly recall the definition of the other factors. These factors depend on auxiliary data that we describe first. Let $(T, B, \{X_\alpha\})$ be an F-pinning of G invariant under b. Let $\psi : F \to \mathbb{C}^\times$ be a non-trivial character. It is assumed that the Whittaker datum arising from the pinning and the character is the given datum $\mathfrak{w}$. Let $R_{\text{res}}(S, G)$ be the set of restrictions to S_{sc}^b of the absolute roots of S in G. Fix a-data and χ-data for $R_{\text{res}}(S, G)$.

Let $\gamma^{\mathfrak{e}} \in G^{\mathfrak{e}}$ be the image of $\gamma^{\mathfrak{z}}$ under the natural map $G^{\mathfrak{z}} \to G^{\mathfrak{e}}$. The complex number $\Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^{\mathfrak{z}}, \tilde{\delta})$ is zero unless $\gamma^{\mathfrak{e}}$ transfers to a norm of $\tilde{\delta}$. More precisely, let (S, γ) be a norm of $\tilde{\delta}$ in the sense of Definition 3.3. It exists and is unique up to $G^b(\bar{F})$-conjugacy according to Lemma 3.4. In order for $\Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^{\mathfrak{z}}, \tilde{\delta})$ to not be zero, there must exists an admissible isomorphism $S^{\mathfrak{e}} \to S_b$ carrying $\gamma^{\mathfrak{e}}$ to γ, where $S^{\mathfrak{e}}$ is the centralizer of $\gamma^{\mathfrak{e}}$. We now assume such an isomorphism exists. It is then uniquely determined by the pair $(\gamma^{\mathfrak{e}}, \gamma)$ and we call it $\varphi_{\gamma^{\mathfrak{e}},\gamma}$. Via this isomorphism we obtain an embedding $R(S^{\mathfrak{e}}, G^{\mathfrak{e}}) \to R_{\text{res}}(S, G)$. We can transport the chosen a-data and χ-data to $R(S^{\mathfrak{e}}, G^{\mathfrak{e}})$. Recall that S is a b-admissible maximal torus of G, $\gamma \in S_b(F)$, and there exists $g \in G(\bar{F})$ such that $g^{-1}\tilde{\delta}g = \tilde{\delta}^* = \delta^* \rtimes b^{-1}$ with $\delta^* \in S(\bar{F})$ mapping to γ.

The term $\epsilon_L(V, \psi)$ is the root number of the virtual Γ-representation $V = X^*(T)_{\mathbb{C}}^b - X^*(T^{\mathfrak{e}})_{\mathbb{C}}$, where $T^{\mathfrak{e}}$ is the (unique up to conjugation) minimal Levi subgroup of $G^{\mathfrak{e}}$.

The term Δ_{II} is a fraction. Its numerator is a product over Γ-orbits of $\alpha_{\text{res}} \in R_{\text{res}}(S, G)$, where the factor corresponding to α_{res} is $\chi_{\alpha_{\text{res}}}((N\alpha(\delta^*) - 1)/a_{\alpha_{\text{res}}})$ when α_{res} is of type R1 or R2, and $\chi_{\alpha_{\text{res}}}(N\alpha(\delta^*) + 1)$ if α_{res} is of type R3. Here $\alpha \in R(S, G)$ is any preimage of α_{res} and $N\alpha$ is the sum of the members of the b-orbit of α, which we recall is uniquely determined by α_{res}. The denominator is a product over Γ-orbits of $\alpha_{\mathfrak{e}} \in R(S^{\mathfrak{e}}, G^{\mathfrak{e}}) \subset R_{\text{res}}(S, G)$, where the factor corresponding to $\alpha_{\mathfrak{e}}$ is $\chi_{\alpha_{\mathfrak{e}}}((\alpha_{\mathfrak{e}}(\gamma^{\mathfrak{e}}) - 1)/a_{\alpha_{\mathfrak{e}}})$.

The term Δ_{IV} is again a fraction. Its numerator is a product over $\alpha_{\text{res}} \in R_{\text{res}}(S, G)$, where the factor corresponding to α_{res} is $|N\alpha(\delta^*) - 1|^{\frac{1}{2}}$ when α_{res} is of type R1 or R2, and $|N\alpha(\delta^*) + 1|^{\frac{1}{2}}$ if α_{res} is of type R3. The denominator is a product over $\alpha_{\mathfrak{e}} \in R(S^{\mathfrak{e}}, G^{\mathfrak{e}}) \subset R_{\text{res}}(S, G)$, where the factor corresponding to

$\alpha_{\mathfrak{e}}$ is $|\alpha_{\mathfrak{e}}(\gamma^{\mathfrak{e}})-1|^{\frac{1}{2}}$.

The term Δ_I is obtained by taking the Tate-Nakayama pairing of an element $t\in H^1(\Gamma,S^b_{\text{sc}})$ with an element $s_S\in\pi_0([\widehat{S}_{\text{ad}}]^\Gamma_b)$, or equivalently of the image of t in $H^1(\Gamma,S^{b,\circ})$ with an element $s_S\in\pi_0([\widehat{S}]^\Gamma_b)$. The element t is the twisted splitting invariant of S, obtained as follows. Let $C\subset G$ be a Borel subgroup defined over $\bar{F}$ containing S and invariant under b; its existence is the definition of b-admissibility of S. Choose $h\in G^b_{\text{sc}}$ such that $h(T,B)h^{-1}=(S,C)$. Let $w(\sigma)\in N(T^b_{\text{sc}},G^b_{\text{sc}})/T^b_{\text{sc}}$ be the image of $h^{-1}\sigma(h)$ and let $n(\sigma)\in N(T^b_{\text{sc}},G^b_{\text{sc}})$ be the Tits lift of $w(\sigma)$ with respect to the chosen pinning of G. Thus $n(\sigma)(h^{-1}\sigma(h))^{-1}\in T^b$. Let $y(\sigma)=\prod\alpha^\vee(a_{\alpha_{\text{res}}})\in S^b_{\text{sc}}$, where the product runs over those $\alpha\in R(S,C)$ that satisfy $-\sigma(\alpha)\in R(S,C)$. Then t is the class of $y(\sigma)\cdot hn(\sigma)\sigma(h)^{-1}$. To obtain the element $s_S\in\pi_0([\widehat{S}]^\Gamma_b)$, choose a member of the canonical $\widehat{G}^{\mathfrak{e}}$-conjugacy class of embeddings $\widehat{S}^{\mathfrak{e}}\to\widehat{G}^{\mathfrak{e}}$ and compose it with $\xi^{\mathfrak{e}}$ and $\widehat{\varphi}_{\gamma^{\mathfrak{e}},\gamma}$ to obtain an embedding $\widehat{S}^{b,\circ}\to\widehat{G}$. The image of this embedding commutes with $\tilde{s}^{\mathfrak{e}}$. It extends uniquely to an admissible embedding $\widehat{S}\to\widehat{G}$, see Lemma 4.3 below. Replacing $\mathfrak{e}$ by an isomorphic datum if necessary we may arrange that the image of $\widehat{S}$ is b-invariant. Writing $\tilde{s}^{\mathfrak{e}}=s^{\mathfrak{e}}\rtimes b$ we see that $s^{\mathfrak{e}}$ commutes with the image of $\widehat{S}^{b,\circ}$, hence also with the image of $\widehat{S}$, and hence lies in that image. We transport $s^{\mathfrak{e}}$ to $\widehat{S}$ and project to $[\widehat{S}]_b$ and obtain s_S.

4.10 Normalized factor Δ_{KS} without z-pair

{sub:pure_tf1}

We turn to the construction of $\Delta^{\text{new}}_{III}$. For simplicity we shall first assume that there exists an L-isomorphism $\xi^{\mathfrak{z}}:\mathcal{G}^{\mathfrak{e}}\to{}^LG^{\mathfrak{e}}$, so that $G^{\mathfrak{z}}=G^{\mathfrak{e}}$ and $\gamma^{\mathfrak{z}}=\gamma^{\mathfrak{e}}$.

In Lemma 4.1 we defined an element $\text{inv}(\gamma,(z,\delta))\in H^1_{b^{-1}}(\Gamma,S\rightrightarrows S)$ which we transport via the isomorphism (4.1) to $H^1(\Gamma,(1-b^{-1}):S\to S)$. On the other hand, the constructions of [KS99, §4.4] provide an element A_0 of $H^1(W_F,(1-b):\widehat{S}\to\widehat{S})$. We recall them here, as they take a particularly simple form in our set-up. Namely, transport the choosen χ-data for $R_{\text{res}}(S_b,G)$ via the admissible isomorphism $S^{\mathfrak{e}}\to S_b$ to $R_{\text{res}}(S^{\mathfrak{e}},G^{\mathfrak{e}})$. From these χ-data one obtains L-embeddings $\xi^{\mathfrak{e}}_S:{}^LS^{\mathfrak{e}}\to{}^LG^{\mathfrak{e}}$ and $\xi^1_S:{}^LS_b\to{}^LG^1$, where G^1 is the principal endoscopic group of $G\rtimes b^{-1}$, i.e. the quasi-split connected reductive group with L-group $\widehat{G}^1\rtimes W_F$, where $\widehat{G}^1=\widehat{G}^{b,\circ}$. We have the natural embedding ${}^LG^1\to{}^LG$ and the L-isomorphism ${}^L\varphi_{\gamma^{\mathfrak{e}},\gamma}:{}^LS_b\to{}^LS^{\mathfrak{e}}$ corresponding to $\varphi_{\gamma^{\mathfrak{e}},\gamma}$. The two L-embeddings ${}^LS_b\to{}^LG$ given by ξ^1_S and $\xi^{\mathfrak{z}}\circ\xi^{\mathfrak{e}}_S\circ{}^L\varphi_{\gamma^{\mathfrak{e}},\gamma}$ are defined up to $\widehat{G}$-conjugacy and we arrange them so that their restrictions to $\widehat{S}_b$ are equal. Let $\xi_S:{}^LS\to{}^LG$ be the unique extension of ξ^1_S of Lemma 4.3. By Corollary 4.5 there exists a (uniquely determined) 1-cocycle $a_S:W_F\to\widehat{S}$ such that $\xi^{\mathfrak{e}}\circ\xi^{\mathfrak{e}}_S\circ{}^L\varphi_{\gamma^{\mathfrak{e}},\gamma}(t\rtimes w)=\xi_S(ta_S(w)\rtimes w)$ for all $t\rtimes w\in\widehat{S}^b\rtimes W_F$. The property $\text{Ad}(\tilde{s}^{\mathfrak{e}})\xi^{\mathfrak{e}}=\xi^{\mathfrak{e}}$ and the above equation immediately imply that $\tilde{s}^{\mathfrak{e}}$ commutes both with $\xi^1_S(\widehat{S}_b)$ and with $w\mapsto\xi_S(a_S(w)\rtimes w)$. Commuting with $\xi^1_S(\widehat{S}^b)$ implies $\tilde{s}^{\mathfrak{e}}=s\rtimes b$ with $s\in\xi_S(\widehat{S})$. Let $s_S\in\widehat{S}$ be the preimage of s under

ξ_S. The commuting of $\tilde{s}$ with $\xi_S(a_S(w) \rtimes w)$ is then equivalent to $(1-b)a_S(w) = s_S \cdot (\sigma_S(w)s_S)^{-1}$, which says that $(a_S^{-1}, s_S) \in Z^1(W_F, (1-b) : \widehat{S} \to \widehat{S})$. We let A_0 be the class of (a_S^{-1}, s_S).

In [KS99, §A.3] a pairing was defined between the cohomology groups $H^1(W_F, (1-b) : \widehat{S} \to \widehat{S})$ and $H^1(\Gamma, (1-b^{-1}) : S \to S)$. We define $\Delta_{III}^{\text{new}}(\gamma^{\mathfrak{z}}, \tilde{\delta}) = \langle \text{inv}(\gamma, (z, \delta)), A_0 \rangle$.

{lem:lembex}

Lemma 4.3. *There exists a unique L-embedding $\xi_S : {}^L S \to {}^L G$ extending ξ_S^1.*

Proof. If ξ_S were given, for $s \rtimes w \in \widehat{S} \rtimes W_F = {}^L S$ the equality $\xi_S(s \rtimes w) = \xi_S(s) \cdot \xi_S^1(w)$ would hold. Two different extensions of ξ_S^1 would thus differ by an element of $\Omega(S, G)(F)$ that induces a trivial action on S_b, equivalently on S^b, but this only holds for $1 \in \Omega(S, G)(F)$. This shows uniqueness. For existence we fix Γ-invariant Borel pairs $(\widehat{T}, \widehat{B})$ of $\widehat{G}$ and (T, B) of G. Then $(\widehat{T}^b, \widehat{B}^b)$ is a Γ-invariant Borel pair of $\widehat{G}^{b,\circ}$. Fix a b-invariant Borel subgroup C of G defined over $\bar{F}$ and containing S, and let $g \in G^{b,\circ}$ conjugate (T, B) to (S, C). Composing the dual of $\text{Ad}(g) : T \to S$ with the natural identification of the dual of T with $\widehat{T}$ given by B and $\widehat{B}$ gives an admissible isomorphism $\xi_S : \widehat{S} \to \widehat{T}$. Its restriction $\widehat{S}^{b,\circ} \to \widehat{T}^{b,\circ}$ is also an admissible isomorphism, and after conjugating ξ_S^1 within $\widehat{G}^{b,\circ}$ we can arrange that this latter isomorphism coincides with ξ_S^1. Let $\omega_\sigma \in \Omega(T, G)^b = \Omega(\widehat{T}, \widehat{G})^b$ be the image of $g^{-1}\sigma(g) \in N(S^b, G^{b,\circ})$. Then the transport via ξ_S of the action of $w \in W_F$ on $\widehat{S}$ is given by $\omega_{\sigma_w} \rtimes w$ on $\widehat{T}$. The same is true for ξ_S^1 and we see that $\xi_S^1(1 \rtimes w) \in N(\widehat{T}^{b,\circ}, \widehat{G}^{b,\circ}) \rtimes W_F$ lifts $\omega_{\sigma_w} \rtimes w$. It follows that $\text{Ad}(\xi_S^1(1 \rtimes w))\xi_S(s) = \xi_S(wsw^{-1})$, hence $s \rtimes w \mapsto \xi_S(s) \cdot \xi_S^1(1 \rtimes w)$ is an L-embedding extending ξ_S^1. ☐

{fct:lembimg}

Fact 4.4. *Let $\mathcal{G}$ be an extension of W_F by $\widehat{G}$ and let $\xi : {}^L S \to \mathcal{G}$ be an L-embedding.*

1. *The image of ξ is the subgroup of $\mathcal{G}$ defined by*

$$\mathcal{S} = \{x \in \mathcal{G} | \forall s \in \widehat{S} : x\xi(s \rtimes 1)x^{-1} = \xi(\sigma_x(s) \rtimes 1), \}$$

   where $\sigma_x \in \Gamma$ is the image of x.

2. *In particular, the image of ξ depends only on the restriction of ξ to $\widehat{S}$.*

3. *ξ is a homeomorphism onto its image.*

Proof. Certainly the image of ξ is contained in $\mathcal{S}$. Given $x \in \mathcal{S}$ let w be its image in W_F and consider $x' = \xi(1 \rtimes w) \in \mathcal{S}$. Then $x'x^{-1} \in \widehat{G}$ commutes with $\xi(\widehat{S})$ and thus belong to $\xi(\widehat{S})$, i.e. $x = \xi(s \rtimes w)$. The second point is immediate from the first, and the third follows from the open mapping theorem and the fact that ${}^L S$ is locally compact, Hausdorff, and σ-compact. ☐

{cor:lembcomp}

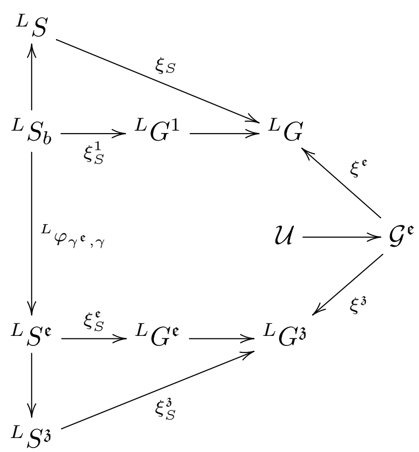
Corollary 4.5. *There exists a 1-cocycle $a_S : W_F \to \widehat{S}$ such that the L-embeddings $\xi^{\mathfrak{e}} \circ \xi_S^{\mathfrak{e}} \circ {}^L\varphi_{\gamma^{\mathfrak{e}}, \gamma}$ and $\xi_S \circ \tilde{a}_S$ are $\widehat{G}$-conjugate.*

Proof. Let $\xi_S : {}^LS \to {}^LG$ and $\xi'_S : {}^LS \to {}^LG$ be the unique extensions of ξ^1_S and $\xi^{\mathfrak{e}} \circ \xi^{\mathfrak{e}}_S \circ {}^L\varphi_{\gamma^{\mathfrak{e}},\gamma}$ given by Lemma 4.3. Their restrictions to $\widehat{S}$ are $\widehat{G}$-conjugate, so we assume they are equal. It follows from Fact 4.4 that ξ_S and ξ'_S have the same image and are homeomorphisms onto it, so we may form $\xi'_S \circ \xi_S^{-1}$. This is an automorphism of the topological group $\widehat{S} \rtimes W_F$ restricting to the identity on both $\widehat{S}$ and W_F. Thus it is given by multiplication by $a_S \in Z^1(W_F, \widehat{S})$. □

4.11 Normalized factor Δ_{KS} with z-pair {sub:pure_tf2}

We now drop the assumption that there exists an L-isomorphism ${}^LG^{\mathfrak{e}} \cong \mathcal{G}^{\mathfrak{e}}$ and instead choose a z-pair $\mathfrak{z} = (G^{\mathfrak{z}}, \xi^{\mathfrak{z}})$. We denote by $S^{\mathfrak{z}}$ the centralizer of $\gamma^{\mathfrak{z}}$. Let $S^{\mathfrak{z}}_1$ be the fiber product of $S \to S_b \cong S^{\mathfrak{e}} \leftarrow S^{\mathfrak{z}}$. The automorphism $b \times \mathrm{id}$ of $S \times S^{\mathfrak{z}}$ preserves $S^{\mathfrak{z}}_1$ and we denote the automorphism it induces by b_1. It fixes the kernel of $S^{\mathfrak{z}}_1 \to S$ pointwise. Hence the endomorphism $(1 - b_1^{-1})$ of $S^{\mathfrak{z}}_1$ induces a homomorphism $(1 - b_1^{-1}) : S \to S^{\mathfrak{z}}_1$. We are going to refine the invariant $\mathrm{inv}(\gamma, (z, \delta)) \in H^1_{b^{-1}}(\Gamma, S \rightrightarrows S) \cong H^1(\Gamma, (1 - b^{-1}) : S \to S)$ constructed above to an element $\mathrm{inv}(\gamma^{\mathfrak{z}}, (z, \delta)) \in H^1(\Gamma, (1 - b_1^{-1}) : S \to S^{\mathfrak{z}}_1)$. If $(\tilde{z}^*, \tilde{\delta}^*)$ is a representative of the G-conjugacy class of $(\tilde{z}, \tilde{\delta})$ as in Lemma 4.1, then $\delta^{\mathfrak{z}} = (\delta^*, \gamma^{\mathfrak{z}})$ belongs to $S^{\mathfrak{z}}_1(\bar{F})$ and satisfies $(b_1^{-1} - 1)z^*(\sigma) = (\delta^{\mathfrak{z}})^{-1}\sigma(\delta^{\mathfrak{z}})$, so $(z^{*,-1}, \delta^{\mathfrak{z}}) \in Z^1(\Gamma, (1 - b_1^{-1}) : S \to S^{\mathfrak{z}}_1))$ and its class is the invariant $\mathrm{inv}(\gamma^{\mathfrak{z}}, (z, \delta))$ we want.

This invariant will be paired with an element $A_0 \in H^1(W_F, (1 - b) : \widehat{S}^{\mathfrak{z}}_1 \to \widehat{S})$, whose construction is essentially the one given in [KS99, §4.4]. We have as above the L-embeddings $\xi^1_S : {}^LS_b \to {}^LG^1$ and $\xi^{\mathfrak{e}}_S : {}^LS^{\mathfrak{e}} \to {}^LG^{\mathfrak{e}}$. They will become part of a diagram as follows



All arrows are L-embeddings. The unnamed arrows ${}^LG^1 \to {}^LG$, ${}^LG^{\mathfrak{e}} \to {}^LG^{\mathfrak{z}}$, ${}^LS_b \to {}^LS$, and ${}^LS^{\mathfrak{e}} \to {}^LS^{\mathfrak{z}}$ are the canonical ones. The arrows ξ_S and $\xi^{\mathfrak{z}}_S$ are the unique ones extending ξ^1_S and $\xi^{\mathfrak{e}}_S$ by Lemma 4.3. We would like to apply Corollary 4.5, but unfortunately have no embedding ${}^LG^{\mathfrak{e}} \to {}^LG$. Instead, following Fact 4.4 we define

$$\mathcal{U} = \{x \in \mathcal{G}^{\mathfrak{e}} | \forall s \in \widehat{S} : x\xi_S^{\mathfrak{e}}(s \rtimes 1)x^{-1} = \xi_S^{\mathfrak{e}}(\sigma_x(s) \rtimes 1)\},$$

which would be the image of $\xi_S^{\mathfrak{e}}$ if we had an identification ${}^LG^{\mathfrak{e}} \cong \mathcal{G}^{\mathfrak{e}}$, which we do not. It is still an extension of W_F by $\widehat{S}^{\mathfrak{e}}$ and Fact 4.4 implies $\xi^{\mathfrak{z}}(\mathcal{U}) \subset \xi_S^{\mathfrak{z}}({}^LS^{\mathfrak{z}})$ and $\xi^{\mathfrak{e}}(\mathcal{U}) \subset \xi_S({}^LS)$. Applying again the open mapping theorem we obtain L-embeddings $\alpha_0 : \mathcal{U} \to {}^LS^{\mathfrak{z}}$ and $\beta : \mathcal{U} \to {}^LS$. Compose α_0 with the L-automorphism of ${}^LS^{\mathfrak{z}}$ given by inversion on $\widehat{S}^{\mathfrak{z}}$ to obtain $\alpha : \mathcal{U} \to {}^LS^{\mathfrak{z}}$, and consider $\alpha \times \beta : \mathcal{U} \to {}^L(S \times S^{\mathfrak{z}})$. Its composition with ${}^L(S \times S^{\mathfrak{z}}) \to {}^LS_1^{\mathfrak{z}}$ kills $\widehat{S}^{\mathfrak{e}} \subset \mathcal{U}$, thus descends to an L-homomorphism $\tilde{a}_S : W_F \to {}^LS_1^{\mathfrak{z}}$, i.e. a 1-cocycle $a_S : W_F \to \widehat{S}_1^{\mathfrak{z}}$. As before one checks that $\tilde{s}^{\mathfrak{e}} = \xi_S(s_S) \rtimes b$ and $(a_S^{-1}, s_S) \in Z^1(W_F, (1-b_1) : \widehat{S}_1^{\mathfrak{z}} \to \widehat{S})$, and we define A_0 to be the class of this element.

As in Subsection 4.10 we define $\Delta_{III}^{\text{new}}(\gamma^{\mathfrak{z}}, \tilde{\delta})$ to be the pairing of $\text{inv}(\gamma^{\mathfrak{z}}, (z, \delta))$ and A_0.

4.12 Transfer of functions

{sub:trans}

In §4.10 and §4.11 we defined a factor $\Delta_{III}^{\text{new}}$, which leads to the factor $\Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}]$ via (4.3), which in turn leads to the factor $\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}]$ via (4.2).

Let $f \in \mathcal{C}_c^\infty(\tilde{G}_z(F))$. For any $\tilde{\delta} \in \tilde{G}_z(F)$ we can form the integral of f over the $\tilde{G}_z(F)$-conjugacy class of $\tilde{\delta}$, after fixing an invariant measure on this conjugacy class. We will call this integral $O_{\tilde{\delta}}(f)$. Since the $\tilde{G}_z(F)$-conjugacy class of δ decomposes as a disjoint union of finitely many $G_z(F)$-conjugacy classes, $O_{\tilde{\delta}}(f)$ is a sum of finitely many twisted orbital integrals.

{lem:trans}

Lemma 4.6. *For any function $f \in \mathcal{C}_c^\infty(\tilde{G}_z(F))$ there exists a function $f^{\mathfrak{z}} \in \mathcal{H}(G^{\mathfrak{z}})$ such that for all strongly regular $\gamma^{\mathfrak{z}} \in G^{\mathfrak{z}}(F)$ we have*

$$SO_{\gamma^{\mathfrak{z}}}(f^{\mathfrak{z}}) = \sum_{\tilde{\delta}} \Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^{\mathfrak{z}}, \tilde{\delta}) O_{\tilde{\delta}}(f)$$

where the sum runs over the set of strongly regular $\tilde{G}_z(F)$-conjugacy classes in $\tilde{G}_z(F)$. More precisely, if $f^{\mathfrak{z},KS}$ is the function that satisfies

$$SO_{\gamma^{\mathfrak{z}}}(f^{\mathfrak{z},KS}) = \sum_{\tilde{\delta}} \Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^{\mathfrak{z}}, \tilde{\delta}) O_{\tilde{\delta}}(f),$$

where now $\tilde{\delta}$ runs over the strongly regular semi-simple elements in $[G \rtimes a]_z(F)$ modulo $G_z(F)$-conjugacy and $a^{-1} \in A$ is the image of $\tilde{s}^{\mathfrak{e}}$, then $f^{\mathfrak{z}} = f_0^{\mathfrak{z},KS}$, where $f_0(\tilde{\delta}) = \sum_{c \in \tilde{G}_z(F)/G_z(F)} f(c^{-1}\tilde{\delta}c)$.

Proof. This follows immediately from the deep results on geometric transfer in twisted endoscopy due to Shelstad [She12] in the archimedean case and Ngo [Ngô10] and Waldspurger [Wal97], [Wal08] in the non-archimedean case. Indeed, in $\sum_{\tilde{\delta}} \Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^{\mathfrak{z}}, \tilde{\delta}) O_{\tilde{\delta}}(f)$ we are summing over $\tilde{G}_z(F)$-conjugacy classes in $\tilde{G}_z(F)$, and then integrating over each such class. We may equally well sum over $G_z(F)$-conjugacy classes in $\tilde{G}_z(F)$, and then integrate over each such class. After this reparameterization, we plug in (4.2) and use the fact that Δ_{KS} is invariant under $G_z(F)$-conjugation in the variable $\tilde{\delta}$ to switch the sums over c and $\tilde{\delta}$. This brings the right hand side to

$$\sum_{c\in\tilde{G}_z(F)/G_z(F)}\ \sum_{\tilde{\delta}\in\tilde{G}_z(F)/G_z(F)-conj}\Delta_{KS}[\mathfrak{w},\mathfrak{e},\mathfrak{z}](\gamma^{\mathfrak{z}},c\tilde{\delta}c^{-1})\int_{x\in G_z(F)/G_z(F)_{\tilde{\delta}}}f(x\tilde{\delta}x^{-1})dx.$$

Changing variables to replace $\tilde{\delta}$ and x by $c^{-1}\tilde{\delta}c$ and cxc^{-1} and moving the sum over c to the right we obtain

$$\sum_{\tilde{\delta}\in\tilde{G}_z(F)/G_z(F)-conj}\Delta_{KS}[\mathfrak{w},\mathfrak{e},\mathfrak{z}](\gamma^{\mathfrak{z}},\tilde{\delta})\int_{x\in G_z(F)/G_z(F)_{\tilde{\delta}}}\ \sum_{c\in\tilde{G}_z(F)/G_z(F)}f(c^{-1}x\tilde{\delta}x^{-1}c)dx.$$

Let $a^{-1}\in A$ be the image of $\tilde{s}^{\mathfrak{e}}$. Then $\Delta_{KS}=0$ unless $\tilde{\delta}\in[G\rtimes a]_z(F)$. Fix $\tilde{\delta}_0\in[G\rtimes a]_z(F)$ and write $\theta=\mathrm{Ad}(\tilde{\delta}_0)$. Let $f_0(\delta)=\sum_{c\in\tilde{G}_z(F)/G_z(F)}f(c^{-1}\delta\tilde{\delta}_0c)$. Then we obtain

$$\sum_{\delta\in G_z(F)/\theta-conj}\Delta_{KS}[\mathfrak{w},\mathfrak{e},\mathfrak{z}](\gamma^{\mathfrak{z}},\delta\tilde{\delta}_0)\int_{x\in G_z(F)/G_z(F)_{\delta\theta}}f_0(x\delta\theta(x^{-1})).$$

By construction $\Delta_{KS}[\mathfrak{w},\mathfrak{e},\mathfrak{z}](\gamma^{\mathfrak{z}},\delta\tilde{\delta}_0)$ is a normalization of the Kottwitz-Shelstad transfer factor for the twisted group (G_z,θ) and its twisted endoscopic datum $\mathfrak{e}$, evaluated at $(\gamma^{\mathfrak{z}},\delta)$. The results of Shelstad, Ngo, and Waldspurger now imply the existence of a function $f^{\mathfrak{z}}$ so that the above formula becomes equal to $SO_{\gamma^{\mathfrak{z}}}(f^{\mathfrak{z}})$. □

4.13 Character identities

{sub:pure_charid}

Consider a parameter $\phi: L_F\to{}^LG$ and a semi-simple element $\tilde{s}\in\tilde{S}_\phi^{[z]}$. The pair $(\phi,\tilde{s})$ leads to an endoscopic datum $\mathfrak{e}=(G^{\mathfrak{e}},\mathcal{G}^{\mathfrak{e}},\tilde{s},\xi^{\mathfrak{e}})$ by the spectral construction described in Subsection 4.8. Choose a z-pair $(G^{\mathfrak{z}},\xi^{\mathfrak{z}})$ and let $\phi^{\mathfrak{z}}=\xi^{\mathfrak{z}}\circ\phi$, a tempered parameter for $G^{\mathfrak{z}}$. We assume the existence of an L-packet $\Pi_{\phi^{\mathfrak{z}}}$ on $G^{\mathfrak{z}}(F)$ and of its stable character $S\Theta_{\phi^{\mathfrak{z}}}$. Let us write the bijection $\mathrm{Irr}(\pi_0(\tilde{S}_\phi^{[z]}),[z])\to\tilde{\Pi}_{\phi,z}$ from Conjecture 4.2 as $\tilde{\rho}\mapsto\tilde{\pi}_{\tilde{\rho}}$.

{cnj:llc_pure_ci}

Conjecture 4.7. *For any pair of functions f and $f^{\mathfrak{z}}$ as in Lemma 4.6 we have*

$$S\Theta_{\phi^{\mathfrak{z}}}(f^{\mathfrak{z}})=\sum_{\tilde{\rho}}tr\tilde{\rho}(\tilde{s})\cdot\Theta_{\tilde{\pi}_{\tilde{\rho}}}(f),\tag{4.4}$$ {eq:charid}

where $\tilde{\rho}$ runs over $Irr(\pi_0(\tilde{S}_\phi^{[z]}),[z])$.

As we have already remarked in §4.9, equation (4.4) applied to the connected case $\tilde{G}=G$ differs from equation [Kal16b, (5.9),(5.11)] because it is missing the factor $e(G')$. This factor has now been built into the definition of the transfer factor (4.3), because in the disconnected case this is notationally more convenient.

We will refer to Conjectures 4.2 and 4.7 together as the *refined local Langlands conjecture for pure inner forms of quasi-split disconnected groups*.

Remark 4.8. Let $a \in A$ be the image of $\tilde{s}$. If the function f is supported away from the $G_z(F)$-cosets in $\tilde{G}_z(F)$ which are A-conjugate to a^{-1}, then $f^{\mathfrak{s}} = 0$. Thus the conjecture contains the statement that the right hand side is also zero in this case.

Remark 4.9. Let $\chi : A \to \mathbb{C}^\times$ be a character. Then $\tilde{\pi}_{\chi\otimes\tilde{\rho}} = \chi \otimes \tilde{\pi}_{\tilde{\rho}}$. If $a \in A$ is the image of $\tilde{s}$ and f is a function on $\tilde{G}_z(F)$ supported on the $G_z(F)$-coset of $b \in A$, then $\mathrm{tr}(\chi \otimes \tilde{\rho})(\tilde{s}) = \chi(a)\mathrm{tr}(\tilde{\rho})(\tilde{s})$, while $\Theta_{\chi\otimes\tilde{\pi}_{\tilde{\rho}}}(f) = \chi(b)\Theta_{\tilde{\pi}_{\tilde{\rho}}}(f)$. From this it follows that the right hand side above is zero if f is supported only on cosets for $b \in A$ such that $ab \neq 1$ in A^{ab}.

5 The conjecture for rigid inner forms

{sec:rigid}

In the preceding section, we introduced a refined local Langlands conjecture for pure inner forms of quasi-split disconnected groups. Those are inner forms of quasi-split disconnected groups $G \rtimes A$ that arise from $H^1(\Gamma, G)$. A general inner form of $G \rtimes A$ arises from $H^1(\Gamma, G/Z(G)^A)$ and in this section we are going to extend the conjecture from pure inner forms to general inner forms. Just like in the connected setting, the notion of an inner form needs to be rigidified. For this we can use the cohomology set $H^1(u \to W, Z(G)^A \to G)$ defined in [Kal16b], see also [Kal18]. However, in order to normalize the transfer factors in the disconnected case, we shall need a generalization of this cohomology set to complexes of tori of length 2, as well as a Tate-Nakayama duality theorem for this generalization. This will be the concern in the first two subsections below. Thankfully, what is needed is little more than a combination of the arguments of [Kal16b] and [KS99, App. A].

5.1 Definitions of hyper(co)homology groups

{sub:coho}

Consider a complex $Z \to T \to U$, where T and U are tori, Z is finite, and $Z \to T$ is injective. We write f for the map $T \to U$, and leave the map $Z \to T$ unnamed. Let $\bar{T}$ be the quotient T/Z. The map f induces a map $\bar{f} : \bar{T} \to U$.

We shall first define and study a cohomology group $H^1(u \to W, Z \to T \to U)$ that combines the group $H^1(u \to W, Z \to T)$ of [Kal16b] and the group $H^1(\Gamma, T \to U)$ of [KS99, App. A]. Define $Z^1(u \to W, Z \to T \to U)$ to consists of pairs $z \in Z^1(u \to W, Z \to T)$ and $c \in C^0(\Gamma, U)$ such that $\bar{f}(\bar{z}) = \partial c$, where $\bar{z} \in Z^1(\Gamma, \bar{T})$ is the image of z. Define $H^1(u \to W, Z \to T \to U)$ to be the quotient of $Z^1(u \to W, Z \to T \to U)$ by the subgroup $B^1(\Gamma, T \to U)$ consisting of $\{(t^{-1}\sigma(t), f(t)) | t \in T(\bar{F})\}$.

This definition involves a particular choice of extension $1 \to u \to W \to \Gamma \to 1$ in the distinguished isomorphism class. Just like in the case of $H^1(u \to W, Z \to G)$, the cohomology set $H^1(u \to W, Z \to T \to U)$ is independent of that choice, in that there is a unique isomorphism between the two versions of it coming from two choices of extensions. The argument is as follows. It is enough to show that an automorphism of the extension W acts trivially on $H^1(u \to W, Z \to T \to U)$. The vanishing of $H^1(\Gamma, u)$ asserted in [Kal16b, Theorem 3.1] implies that such an automorphism is of the form $\mathrm{Ad}(x)$ for some $x \in u$. An element $(z, u) \in Z^1(u \to W, Z \to T \to U)$ is sent by $\mathrm{Ad}(x)$ to (z', u) where $z'(w) = z(xwx^{-1}) = z(x \cdot \sigma(x^{-1}))z(w) = z(x) \cdot \sigma(z(x)))^{-1} \cdot z(w)$, where $\sigma \in \Gamma$ is the image of w. So the difference between (z, u) and (z', u) is measured by $(z(x) \cdot \sigma(z(x))^{-1}, 1) \in B^1(\Gamma, T \to U)$. We are using here that $f|_Z = 1$.

We have the following analog of [Kal16b, (3.6)]:

$$\begin{array}{ccccccccc}
 & & \bar{T}(F) & = & \bar{T}(F) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & H^1(\Gamma, Z) & \xrightarrow{\mathrm{Inf}} & H^1(u \to W, Z \to Z) & \xrightarrow{\mathrm{Res}} & \mathrm{Hom}(u, Z)^\Gamma & & \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & H^1(\Gamma, T \to U) & \xrightarrow{\mathrm{Inf}} & H^1(u \to W, Z \to T \to U) & \xrightarrow{\mathrm{Res}} & \mathrm{Hom}(u, Z)^\Gamma & \to & H^2(\Gamma, T \to U) \\
 & & \| & & \downarrow{\scriptstyle a} & & \downarrow & & \| \\
 & & H^1(\Gamma, T \to U) & \to & H^1(\Gamma, \bar{T} \to U) & \to & H^2(\Gamma, Z) & \to & H^2(\Gamma, T \to U) \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{5.1}$$

{eq:bfd2}

We also have the following analog of the long exact sequence [KS99, (A.1.1)]

$$\begin{aligned}
0 &\to H^0(\Gamma, Z) \to H^0(\Gamma, T) \to H^0(\Gamma, U) \to \\
&\to H^1(u \to W, Z \to T \to U) \to H^1(u \to W, Z \to T) \to H^1(\Gamma, U) \to \\
&\to H^2(\Gamma, \bar{T} \to U) \to H^2(\Gamma, \bar{T}) \to H^2(\Gamma, U) \to \\
&\to \dots
\end{aligned}$$

Note that the kernel of $H^1(u \to W, Z \to T \to U) \to H^1(u \to W, Z \to T)$ lies in the subgroup $H^1(\Gamma, T \to U)$ of $H^1(u \to W, Z \to T \to U)$ and the map $H^1(u \to W, Z \to T) \to H^1(\Gamma, U)$ factors through the surjection $H^1(u \to W, Z \to T) \to H^1(\Gamma, \bar{T})$. The difference between [KS99, (A.1.1)] and (5.1) is that we have replaced $H^1(\Gamma, T \to U)$ by $H^1(u \to W, Z \to T \to U)$, $H^i(\Gamma, T \to U)$ by $H^i(\Gamma, \bar{T} \to U)$, and $H^i(\Gamma, T)$ by $H^i(\Gamma, \bar{T})$, for $i > 1$.

Finally let K and C be the kernel and cokernel of f, respectively, so that we have an exact sequence $1 \to K \to T \to U \to C \to 1$ of diagonalizable groups. By assumption $Z \subset K$ and thus we also have $1 \to \bar{K} \to \bar{T} \to U \to C$, where

$\bar{K} = K/Z$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\Gamma, K) & \longrightarrow & H^1(\Gamma, T \to U) & \longrightarrow & H^0(\Gamma, C) & \longrightarrow & H^2(\Gamma, K) \\
 & & \downarrow & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & H^1(u \to W, Z \to K) & \longrightarrow & H^1(u \to W, Z \to T \to U) & \longrightarrow & H^0(\Gamma, C) & \longrightarrow & H^2(\Gamma, \bar{K}) \\
 & & \downarrow & & \downarrow & & \| & & \| \\
0 & \longrightarrow & H^1(\Gamma, \bar{K}) & \longrightarrow & H^1(\Gamma, \bar{T} \to U) & \longrightarrow & H^0(\Gamma, C) & \longrightarrow & H^2(\Gamma, \bar{K})
\end{array}$$

Next we shall define a functor $\bar{Y}_{+,\text{tor}}(Z \to T \to U)$ that combines the functor $\bar{Y}_{+,\text{tor}}(Z \to T)$ of [Kal16b] and the homology groups $H_0(W_{K/F}, X_*(T) \to X_*(U))_0$ of [KS99, App. A]. Consider the homomorphism $f_* : X_*(T) \to X_*(U)$. The assumption $Z \subset \ker(f)$ implies that this homomorphism extends (necessarily uniquely) to $f_* : X_*(\bar{T}) \to X_*(U)$. We consider this as a complex placed in degrees 0 and 1. For every finite Galois extension K/F splitting T and U we have the hyperhomology groups $H_0(W_{K/F}, X_*(T) \to X_*(U))$ and $H_0(W_{K/F}, X_*(\bar{T}) \to X_*(U))$, as well as their subgroups $H_0(-)_0$ defined in [KS99, App. A.3]. Let us recall some details. The group of inhomogenous n-chains $C_n(W_{K/F}, X_*(T))$ consists of all set-theoretic maps $W_{K/F}^n \to X_*(T)$ with finite support. If y is such a map, its differential $\partial y : W_{K/F}^{n-1} \to X_*(T)$ is given by

$$\begin{aligned}
\partial y(w_1, \ldots, w_{i-1}) &= \sum_x x^{-1} y(x, w_1, \ldots, w_{n-1}) \\
&+ \sum_{i=1}^{n-1} (-1)^i \sum_x y(w_1, \ldots, w_{i-1}, w_i x^{-1}, x, w_{i+1}, \ldots, w_{n-1}) \\
&+ (-1)^n \sum_x y(w_1, \ldots, w_{n-1}, x),
\end{aligned}$$

where x runs over $W_{K/F}$. The group $Z_0(W_{K/F}, X_*(T) \to X_*(U))$ has the explicit description as the set of pairs $\{(\lambda, \mu_1) | \lambda \in C_0(W_{K/F}, X_*(T)), \mu_1 \in C_1(W_{K/F}, X_*(U)), f_*(\lambda) = \partial\mu_1)$, while the group $B_0(W_{K/F}, X_*(T) \to X_*(U))$ is given by $\{(\partial\lambda_1, f_*(\lambda_1) - \partial\mu_2) | \lambda_1 \in C_1(W_{K/F}, X_*(T)), \mu_2 \in C_2(W_{K/F}, X_*(U))\}$. Then $H_0 = Z_0/B_0$. The subgroup $Z_0(-)_0$ consists of those (λ, μ_1) satisfying in addition $N_{K/F}\lambda = 0$, and $H_0(-)_0 = Z_0(-)_0/B_0$. Note that $H_0(W_{K/F}, X_*(T))_0 = H^{-1}_{\text{Tate}}(\Gamma_{K/F}, X_*(T)) = [X_*(T)/IX_*(T)]_{\text{tor}}$, where I is the augmentation ideal in $\Gamma_{K/F}$, or equivalently in Γ. In [Kal16b] we used the notation $Y_{\text{tor}}(T)$ for this finite abelian group.

{fct:tn++esy1}

Fact 5.1. *We have the exact sequence*

$$\begin{aligned}
&H_1(W_{K/F}, X_*(T)) \to H_1(W_{K/F}, X_*(U)) \to \\
\to &H_0(W_{K/F}, X_*(T) \to X_*(U))_0 \to Y_{tor}(T) \to Y_{tor}(U).
\end{aligned}$$

Proof. Left to the reader. □

We define $H_0(W_{K/F}, X_*(T) \to X_*(\bar{T}) \to X_*(U))_0 = Z_0(W_{K/F}, X_*(\bar{T}) \to X_*(U))_0 / B_0(W_{K/F}, X_*(T) \to X_*(U))$.

{fct:tn++esy2}

Fact 5.2. *We have the exact sequence*

$$\begin{aligned} &H_1(W_{K/F}, X_*(T)) \to H_1(W_{K/F}, X_*(U)) \to \\ \to &H_0(W_{K/F}, X_*(T) \to X_*(\bar{T}) \to X_*(U))_0 \to \bar{Y}_{+,tor}(Z \to T) \to Y_{tor}(U). \end{aligned}$$

Proof. Left to the reader. □

There is a coinflation map $C_n(W_{L/F}, X_*(T)) \to C_n(W_{K/F}, X_*(T))$ for a tower $L/K/F$ defined by

$$\text{coinf} y(w_1, \dots, w_n) = \sum_{\dot{w}_i \in p^{-1}(w_i)} y(\dot{w}_1, \dots, \dot{w}_n),$$

where $p : W_{L/F} \to W_{K/F}$ is the natural projection. This map respects differentials and induces a corresponding map $H_n(W_{L/F}, X_*(T) \to X_*(U)) \to H_n(W_{K/F}, X_*(T) \to X_*(U))$. It maps $H_0(W_{L/F}, X_*(T) \to X_*(U))_0$ to $H_0(W_{K/F}, X_*(T) \to X_*(U))_0$, this relies on the torsion-freeness of $X_*(T)$.

{fct:ks1}

Fact 5.3. *Consider a tower of finite Galois extensions $L/K/F$ and assume K splits T and U. Then the following diagram commutes*

$$\begin{array}{ccc} H_0(W_{L/F}, X_*(T) \to X_*(U))_0 & \longrightarrow & H^1(K/F, T(K) \to U(K)) \\ \downarrow & & \uparrow \\ H_0(W_{K/F}, X_*(T) \to X_*(U))_0 & \longrightarrow & H^1(L/F, T(L) \to U(L)) \end{array}$$

where the left map is coinflation, the right map is inflation, and the horizontal maps are the isomorphisms [KS99, (A.3.4)]. Both vertical maps are isomorphisms.

Proof. This is diagram [KS99, (A.3.11)], and its commutativity is proved there. The fact that inflation is an isomorphisms follows from the 5-lemma applied to the exact sequence

$$T(F) \to U(F) \to H^1(K/F, T(K) \to U(K)) \to H^1(K/F, T(K)) \to H^1(K/F, U(K))$$

and its L/F-analog. The fact that coinflation is an isomorphism follows from the commutativty of the above diagram. □

The coinflation map induces a map

$$H_0(W_{L/F}, X_*(T) \to X_*(\bar{T}) \to X_*(U))_0 \to H_0(W_{K/F}, X_*(T) \to X_*(\bar{T}) \to X_*(U))_0.$$

Fact 5.4. *This is an isomorphism.*

Proof. We apply the 5-lemma to the exact sequence

$$H_1(X_*(T)) \to H_1(X_*(U)) \to H_0(X_*(T) \to X_*(\bar{T}) \to X_*(U))_0 \to \bar{Y}_{+,\text{tor}}(T) \to Y_{\text{tor}}(U),$$

where we take homology of $W_{L/F}$, and then map it, via the coinflation map, to the same exact sequence but for $W_{K/F}$. For the last two terms coinflation induces the identity. For the first two terms, it is an isomorphism due to Fact 5.3 applied to the complexes $1 \to T$ and $1 \to U$. □

We define $\bar{Y}_{+,\text{tor}}(Z \to T \to U)$ as the inverse limit of $H_0(W_{K/F}, X_*(T) \to X_*(\bar{T}) \to X_*(U))_0$ with respect to coinflation.

{fct:tn++esy3}

Fact 5.5. *Let $H_1(X_*(T))$ denote the inverse limit of $H_1(W_{K/F}, X_*(T))$ with respect to coinflation. We have the exact sequence*

$$H_1(X_*(T)) \to H_1(X_*(U)) \to \bar{Y}_{tor}(Z \to T \to U) \to \bar{Y}_{+,tor}(T) \to Y_{tor}(U).$$

Finally, we consider the dual homomorphism $\widehat{f} : \widehat{U} \to \widehat{T}$. It lifts (uniquely) to a homomorphism $\widehat{\bar{f}} : \widehat{U} \to \widehat{\bar{T}}$. Let $\widehat{Z}$ be the kernel of the isogeny $\widehat{\bar{T}} \to \widehat{T}$, and let $\widehat{K}$ and $\widehat{C}$ be the kernel and cokernel of $\widehat{f}$. Then $[\widehat{\bar{f}}]^{-1}(\widehat{Z}) = \widehat{K}$.

We define the group $Z^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})$ to consist of the pairs $(z, \dot{c})$, where $z \in Z^1_{\text{cts}}(W_F, \widehat{U})$ and $\dot{c} \in \widehat{\bar{T}}$ satisfying $\partial c = \widehat{f}(z)$, where $c \in \widehat{T}$ is the image of $\dot{c}$. We define $B^1(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})$ to consist of $(\partial u, \widehat{\bar{f}}(u))$ for $u \in \widehat{U}$, and $H^1 = Z^1/B^1$. This group fits into the exact sequence

$$H^1_{\text{cts}}(W_F, \widehat{T}) \leftarrow H^1_{\text{cts}}(W_F, \widehat{U}) \leftarrow H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U}) \leftarrow [\widehat{\bar{T}}]^+ \leftarrow \widehat{U}^\Gamma.$$

Define $H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})_{\text{red}}$ to be the quotient of $H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})$ by the image of $[\widehat{\bar{T}}]^{+,\circ}$. Then we obtain the exact sequence

$$H^1_{\text{cts}}(W_F, \widehat{T}) \leftarrow H^1_{\text{cts}}(W_F, \widehat{U}) \leftarrow H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})_{\text{red}} \leftarrow \pi_0([\widehat{\bar{T}}]^+) \leftarrow \pi_0(\widehat{U}^\Gamma). \tag{5.2}$$

{eq:tn++esd1}

We introduce on $H^1(u \to W, Z \to T \to U)$ the unique topology that makes the homomorphism $U(F) \to H^1(u \to W, Z \to T \to U)$ continuous and open. Analogously, we introduce on $\bar{Y}_{\text{tor}}(Z \to T \to U)$ the unique topology that makes the homomorphisms $H_1(W_{K/F}, X_*(U)) \to \bar{Y}_{\text{tor}}(Z \to T \to U)$ continuous and open. Here $H_1(W_{K/F}, X_*(U))$ is topologized to make the Langlands isomorphism a homeomorphism.

5.2 Generalized Tate-Nakatyama duality

{sub:tnd++}

We shall now define a perfect pairing

$$H^1(u \to W, Z \to T \to U) \otimes H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})_{\text{red}} \to \mathbb{C}^\times \tag{5.3}$$

{eq:tnd++}

that generalizes the pairing [KS99, (A.3.12),(A.3.16)], which can be seen as the special case $Z = 1$. We do this in two steps – first introducing a pairing of elementary nature between $H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})$ and $\bar{Y}_{+,\text{tor}}(Z \to T \to U)$, and then an isomorphism of arithmetic nature $\bar{Y}_{+,\text{tor}}(Z \to T \to U) \to H^1(u \to W, Z \to T \to U)$.

Given $(z, \dot{c}) \in Z^1_{\text{cts}}(W_{K/F}, \widehat{Z} \to \widehat{\bar{T}} \to \widehat{U})$ and $(\bar{\lambda}, \mu_1) \in Z_0(W_{K/F}, X_*(\bar{T}) \to X_*(U))_0$ define $\langle (z, \dot{c}), (\bar{\lambda}, \mu_1)\rangle_K \in \mathbb{C}^\times$ as

$$\langle \dot{c}, \bar{\lambda}\rangle_{\bar{T}} \cdot \prod_{w \in W_{K/F}} \langle z(w), \mu_1(w)\rangle_U^{-1},$$

where $\langle -, -\rangle_{\bar{T}}$ is the pairing $\widehat{\bar{T}} \times X_*(\bar{T}) \to \mathbb{C}^\times$ and $\langle -, -\rangle_U$ is the analogous pairing for U. It is immediate that if $L/K/F$ is a tower of Galois extensions and z is inflated from $W_{K/F}$ we have $\langle (z, \dot{c}), \text{coinf}(\bar{\lambda}, \mu_1)\rangle_K = \langle (z, \dot{c}), (\bar{\lambda}, \mu_1)\rangle_L$. It is immediately checked that this pairing annihilates the (co)boundaries on both sides, as well as the image of $[\widehat{\bar{T}}]^{+,\circ}$, and therefore induces a pairing

$$H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \to \widehat{U})_{\text{red}} \otimes \bar{Y}_{\text{tor}}(Z \to T \to U) \to \mathbb{C}^\times \tag{5.4}$$ {eq:elempair}

functorial in $Z \to T \to U$.

Recall the pairing $H^1_{\text{cts}}(W_F, \widehat{U}) \otimes H_1(W_F, X_*(U)) \to \mathbb{C}^\times$ that underlies the Langlands isomorphism $H^1_{\text{cts}}(W_F, \widehat{U}) \to \text{Hom}_{\text{cts}}(U(F), \mathbb{C}^\times)$ and the pairing $\pi_0(\widehat{T}^\Gamma) \otimes Y_{\text{tor}}(T) \to \mathbb{C}^\times$. The latter was generalized to $\pi_0([\widehat{\bar{T}}]^+) \otimes \bar{Y}_{+,\text{tor}}(Z \to T) \to \mathbb{C}^\times$ in [Kal16b, Prop. 5.3].

{fct:tn++d1}

Fact 5.6. *The pairing* (5.4) *is compatible with the pairing* $\pi_0([\widehat{\bar{T}}]^+) \otimes \bar{Y}_{+,tor}(Z \to T) \to \mathbb{C}^\times$, *as well as the* negative *of the pairing* $H^1(W_F, \widehat{U}) \otimes H_1(W_F, X_*(U)) \to \mathbb{C}^\times$, *and induces an isomorphism*

$$H^1_{cts}(W_F, \widehat{Z} \to \widehat{\bar{T}} \leftarrow \widehat{U})_{red} \to Hom_{cts}(\bar{Y}_{tor}(Z \to T \to U), \mathbb{C}^\times).$$

Proof. The compatibility of the three pairings is immediate from the explicit formula defining (5.4). The compatibility with the negative Langlands pairing together with the definition of the topology on $\bar{Y}_{\text{tor}}(Z \to T \to U)$ implies that the image of the resulting homomorphism $H^1_{\text{cts}}(W_F, \widehat{Z} \to \widehat{\bar{T}} \to \widehat{U}) \to \text{Hom}(\bar{Y}_{\text{tor}}(Z \to T \to U), \mathbb{C}^\times)$ lies in $\text{Hom}_{\text{cts}}(\dots)$. Applying the functor $\text{Hom}_{\text{cts}}(-, \mathbb{C}^\times)$ to the exact sequence of Fact 5.5 produces an exact sequence: for $\text{Hom}(-, \mathbb{C}^\times)$ this is because $\mathbb{C}^\times$ is an injective abelian group, and passing from abstract to continuous homomorphisms doesn't ruin exactness due to the definition of the topology on $\bar{Y}_{\text{tor}}(Z \to T \to U)$. This exact sequence maps to the exact sequence (5.2), with the first two maps being the negative Langlands pairing, the middle map being (5.4), and the fourth and fifth map coming from [Kal16b, Proposition 5.3]. All maps except for the middle one are known to be isomorphisms, and the 5-lemma applies. □

We now turn to the isomorphism $\bar{Y}_{+,\text{tor}}(Z \to T \to U) \to H^1(u \to W, Z \to T \to U)$. We fix as in [Kal16b, §4.4ff] an exhaustive tower E_k/F of finite Galois extensions, compatible sections $s_k : \Gamma_{E_k/F} \to W_{E_k/F}$ and $\zeta_k : \Gamma_{E_k/F} \to$

$\Gamma_{E_{k+1}/F}$, a co-final sequence n_k of natural numbers, a compatible sequence $l_k : \bar{F}^\times \to \bar{F}^\times$ of n_k-roots. Define $c_k(\sigma,\tau) = \mathrm{rec}_k^{-1}(s_k(\sigma)s_k(\tau)s_k(\sigma\tau)^{-1})$. Then $\xi_k = dl_k c_k \sqcup_{E_k/F} \delta_e \in Z^2(\Gamma, u_k)$ gives rise to the extension $W_k = u_k \boxtimes_{\xi_k} \Gamma$ of Γ by u_k. We have the 1-cochain $\alpha_k \in C^1(\Gamma, u_k)$ of [Kal16b, (4.8)] leading to the surjective group homomorphism $f_k : W_{k+1} \to W_k$ defined by $f_k(x \boxtimes \sigma) = p(x)\alpha_k(\sigma) \boxtimes \sigma$, where $p : u_{k+1} \to u_k$ is the surjective group homomorphism of [Kal16b, (3.2)]. Then $W = \varprojlim_k W_k$ is an extension of Γ by u in the distinguished isomorphism class.

We are now going to construct the isomorphism by refining and merging together the constructions of [KS99, §A.3] and [Kal16b, §4.6]. More precisely, a central role in the constructions of [KS99, §A.3] is played by two maps $\phi = \phi_{T,k} : C_1(W_{E_k/F}, X_*(T)) \to T(E_k)$ and $\psi = \psi_{T,k} : C_0(W_{E_k/F}, X_*(T))_0 \to Z^1(\Gamma_{E_k/F}, T(E_k))$, where $C_0(W_{E_k/F}, X_*(T))_0$ is simply the kernel of the norm map for the action of $\Gamma_{E_k/F}$ on $X_*(T)$. They are functorial in T and satisfy $\phi \circ \partial = 0$ and $\partial \circ \phi = \psi \circ \partial$. We shall now recall these maps and give a refinement $\dot\psi$ of ψ using some material from [Kal16b, §4.6].

Fix k such that E_k splits both T and U and $\mathrm{ord}(Z)$ divides n_k. Consider $\bar\lambda \in X_*(\bar T)$ and $\mu_1 : W_{E_k/F} \to X_*(U)$ such that $(\bar\lambda, \mu_1) \in Z_0(W_{E_k/F}, X_*(\bar T) \to X_*(U))_0$. As in [KS99, §A.3] define $\phi_U(\mu_1) \in U(E_k)$ by

$$\phi_{U,k}(\mu_1) = \prod_{\sigma,\tau,a} \sigma(\mu_1(as(\tau)))(c_k(\sigma,\tau)^{-1}\sigma(a)^{-1}),$$

the product running over $\Gamma_{E_k/F} \times \Gamma_{E_k/F} \times E_k^\times$. As explained there, this is an explicit formula for the restriction map of 1-chains $C_1(W_{E_k/F}, X_*(U)) \to C_1(E_k^\times, X_*(U))$ composed with the isomorphism $C_1(E_k^\times, X_*(U)) \to X_*(U) \otimes_{\mathbb{Z}} E_k^\times = U(E_k)$. Furthermore, we define $\dot\psi_T(\bar\lambda) \in Z^1(u \to W, Z \to T)$ as the inflation along $W \to W_k = u_k \boxtimes_{\xi_k} \Gamma$ of the element $z_{\bar\lambda,k}$ of [Kal16b, Lemma 4.7], which we recall is defined as

$$x \boxtimes \rho \mapsto \phi_{\bar\lambda,k}(x) \cdot (l_k c_k \sqcup_{E_k/F} n_k\bar\lambda)(\rho) = \phi_{\bar\lambda,k}(x) \cdot \prod_{\sigma \in \Gamma_{K/F}} \rho\sigma(n_k\bar\lambda)(l_k c_{\rho,\sigma}).$$

The image $\bar z_{\bar\lambda,k} \in Z^1(\Gamma, \bar T)$ of $z_{\bar\lambda,k}$ is given by $c_k \cup \bar\lambda = \psi_{\bar T}(\bar\lambda)$ and hence satisfies the equation $f(\bar z_{\bar\lambda,k}) - \partial\phi_U(\mu_1) = f(\psi_{\bar T}(\bar\lambda)) - \partial\phi_U(\mu_1) = \psi_U(f_*(\bar\lambda)) - \psi_U(\partial\mu_1) = 0$, due to the functoriality of ψ. We conclude that $(z_{\bar\lambda,k}, \phi_U(\mu_1)) \in Z^1(u \to W, Z \to T \to U)$.

Now consider $(\partial\lambda_1, f_*(\lambda_1) - \partial\mu_2) \in B_0(W_{E_k/F}, X_*(T) \to X_*(U))$. Then we have $\dot\psi_T(\partial\lambda_1) = \psi_T(\partial\lambda_1) = \partial\phi_T(\lambda_1)$, and hence $(\dot\psi_T(\partial\lambda_1), \phi_U(f_*(\lambda_1) - \partial\mu_2)) = (\partial\phi_T(\lambda_1), f_*(\phi_T(\lambda_1)))$ is a coboundary.

We conclude that we have defined a group homomorphism

$$H_0(W_{E_k/F}, X_*(T) \to X_*(\bar T) \to X_*(U))_0 \to H^1(u \to W, Z \to T \to U).$$

Next, we consider the composition of this homomorphism with the coinflation map

$$H_0(W_{E_{k+1}/F}, X_*(T) \to X_*(\bar T) \to X_*(U))_0 \to H_0(W_{E_k/F}, X_*(T) \to X_*(\bar T) \to X_*(U))_0.$$

In [KS99, §A.3] a homomorphism $c : C_0(W_{E_{k+1}/F}, X_*(T))_0 \to C^0(E_{k+1}/F, T(E_{k+1}))$ is defined, and it is shown that

$$\inf \circ \phi_k \circ \mathrm{coinf} = \phi_{k+1} + c\partial,$$

$$\inf \circ \psi_k \circ \mathrm{coinf} = \psi_{k+1} + \partial c.$$

The homomorphism c is defined by the formula

$$c(\lambda) = \prod_{\sigma \in \Gamma_{E_k/F}} (\sigma\lambda) \left(\prod_{\nu \in \Gamma_{E_{k+1}/E_k}} c_{k+1}(v, \zeta_k(\sigma)) \right).$$

The compatibility of the chosen sections s_k and s_{k+1} implies, via [Kal16b, Lemma 4.4], that this homomorphism is trivial, because the inner product is equal to $c_k(1,\sigma) = 1$. It follows that for $\mu_1' : W_{E_{k+1}/F} \to X_*(U)$ the element $\phi_{U,k+1}(\mu_1') \in U(E_{k+1})$ is equal to the image of $\phi_{U,k}(\mathrm{coinf}(\mu_1')) \in U(E_k)$ under the natural inclusion $U(E_k) \to U(E_{k+1})$. On the other hand, the inflation of $z_{\bar\lambda,k}$ to W_{k+1} equals $z_{\bar\lambda,k+1}$ according to [Kal16b, Lemma 4.7], which in our notation here means $\dot\psi_{T,k}(\mathrm{coinf}(\bar\lambda)) = \dot\psi_{T,k+1}(\bar\lambda)$. This gives a commutative diagram

$$\begin{array}{ccc} Z_0(W_{E_{k+1}/F}, X_*(\bar T) \to X_*(U))_0 & & \\ \downarrow{\scriptstyle \mathrm{coinf}} & \searrow^{\dot\psi_{T,k+1},\phi_{U,k+1}} & \\ & & Z^1(u \to W, Z \to T \to U) \\ & \nearrow_{\dot\psi_{T,k},\phi_{U,k}} & \\ Z_0(W_{E_k/F}, X_*(\bar T) \to X_*(U))_0 & & \end{array}$$

already on the level of (co)cycles, and it in turn induces a commutative diagram on the level of (co)homology, leading to a homomorphism

$$\bar Y_{\mathrm{tor}}(Z \to T \to U) \to H^1(u \to W, Z \to T \to U). \tag{5.5}$$ {eq:arithiso}

{fct:tn++d2}

Proposition 5.7. *The homomorphism* (5.5) *is a functorial isomorphism. It is independent of the choices made in its construction.*

Proof. It is immediate from the construction that this homomorphism is functorial. The fact that it is an isomorphism follows from the 5-lemma, applied to the exact sequence just below diagram (5.1) and the corresponding exact sequence of Fact 5.2. The maps between the first two terms of these exact sequences are the Langlands isomorphism $H_1(X_*(T)) \to T(F)$ and its analog for U, the map between the third terms is (5.5), between the fourth terms it is the isomorphism $\bar Y_{+,\mathrm{tor}}(Z \to T) \to H^1(u \to W, Z \to T)$ of [Kal16b, §4], and between the fifth terms it is the Tate-Nakayama isomorphism $Y_{\mathrm{tor}}(U) \to H^1(\Gamma, U)$.

We next argue that this homomorphism is independent of the choices of sections s_k (and also ζ_k) and root maps l_k. For this, let ζ_k', s_k', and l_k' be other choices. We obtain $c'_k \in Z^2(\Gamma_{E_k/F}, E_k^\times)$, $\xi'_k \in Z^2(\Gamma, u_k)$, $W'_k = u_k \boxtimes_{\xi'_k} \Gamma$. Let $W' = \varprojlim W'_k$. The construction above gives a group homomorphism

$$\bar{Y}_{\text{tor}}(Z \to T \to U) \to H^1(u \to W', Z \to T \to U).$$

Every isomorphism $W' \to W$ of extensions induces the same isomorphism $H^1(u \to W, Z \to T \to U) \to H^1(u \to W', Z \to T \to U)$ and we need to show that the triangle

$$\begin{array}{ccc} & & H^1(u \to W, Z \to T \to U) \\ & \nearrow & \downarrow \\ \bar{Y}_{\text{tor}}(Z \to T \to U) & & \\ & \searrow & \downarrow \\ & & H^1(u \to W', Z \to T \to U) \end{array}$$

commutes. Define $\eta_k : \Gamma_{E_k/F} \to E_k^\times$ by $s'_k(\sigma) = \eta_k(\sigma)s_k(\sigma)$. Define $\alpha_{k',k} \in C^1(\Gamma, u_k)$ by

$$\alpha_{k',k}(\sigma) = (l'_k c'_k \cdot (l_k c_k)^{-1} \cdot (dl_k\eta_k)^{-1}) \sqcup_{E_k/F} \delta_e.$$

{lem:tn++i1}

Lemma 5.8. *The assignment $x \boxtimes \sigma \mapsto x\alpha_{k',k}(\sigma) \boxtimes \sigma$ defines an isomorphism of extensions $\bar{g}_k : W'_k \to W_k$ that satisfies $z_{\bar{\lambda},k} \circ \bar{g}_k = z'_{\bar{\lambda},k} \cdot d(l_k\eta_k \sqcup_{E_k/F} n_k\bar{\lambda})^{-1}$.*

Proof. This is a direct computation, using [Kal16b, Fact 4.3]. □

Consider the diagram

$$\begin{array}{ccc} W'_{k+1} & \xrightarrow{\bar{g}_{k+1}} & W_{k+1} \\ \downarrow{\scriptstyle f'_k} & & \downarrow{\scriptstyle f_k} \\ W'_k & \xrightarrow{\bar{g}_k} & W_k \end{array}$$

This diagram does not commute. Define $\beta_k : \Gamma_{E_k/F} \to \bar{F}^\times$ by

$$\beta_k(a) = l_k\eta_k(a)^{-1} \prod_{\substack{b \in \Gamma_{E_{k+1}/F} \\ b \mapsto a}} l_k\eta_{k+1}(b).$$

{lem:tn++i2}

Lemma 5.9. *1. $\beta_k(\sigma)^{n_k} = 1$ and hence $\beta_k \in u_k$;*

2. $f_k \circ \bar{g}_{k+1} = Ad(\beta_k^{-1}) \circ \bar{g}_k \circ f'_k$.

Proof. We begin with the second point. From the definitions of f_k and $\bar{g}_k$ we have

$$\begin{aligned} f_k(\bar{g}_{k+1}(x \boxtimes \sigma)) &= p(dl_{k+1}\eta_{k+1} \sqcup_{E_{k+1}/F} \delta_e)^{-1}(dl_k\eta_k \sqcup_{E_k/F} \delta_e) \cdot \bar{g}_k(f'_k(x \boxtimes \sigma)) \\ &= d[p(l_{k+1}\eta_{k+1} \sqcup_{E_{k+1}/F} \delta_e)^{-1}(l_k\eta_k \sqcup_{E_k/F} \delta_e)] \cdot \bar{g}_k(f'_k(x \boxtimes \sigma)). \end{aligned}$$

Recall the torus S_k defined as the quotient of $\mathrm{Res}_{E_k/F}\mathbb{G}_m$ by the diagonal copy of $\mathbb{G}_m$. Its subgroup $S_k[n_k]$ of n_k-torsion points is precisely u_k. We can compute $l_k\eta_k \sqcup_{E_k/F} \delta_e \in S_k$ explicitly and see that it is represented by the map $\Gamma_{E_k/F} \to \bar{F}^\times$ sending a to $l_k\eta_k(a)$. The analogous formula holds for $l_{k+1}\eta_{k+1} \sqcup_{E_{k+1}/F} \delta_e \in S_{k+1}$, whose image under p then sends a to $\prod_b l_k\eta_{k+1}(b)$, where b runs over the elements of $\Gamma_{E_{k+1}/F}$ mapping to a. Thus the argument of d is β_k^{-1} as claimed.

We come to the first point and need to prove that the function $\Gamma_{E_k/F} \to \bar{F}^\times$ defined by $a \mapsto \eta_k(a)^{-1}\prod_{b \mapsto a} \eta_{k+1}(b)$ represents the trivial element of S_k. For this we recall that the sections s_k and s_{k+1} were chosen to satisfy

$$s_{k+1}(y\zeta_k(x)) = s_{k+1}(y)s_{k+1}(\zeta_k(x)) \quad \text{and} \quad s_k(x) = \pi_k^W(s_{k+1}(\zeta_k(x))),$$

for $y \in \Gamma_{E_{k+1}/E_k}$ and $x \in \Gamma_{E_k/F}$, where π_k^W is the natural projection $W_{E_{k+1}/F} \to W_{E_k/F}$. From these we obtain via direct calculation the following identities

$$\eta_{k+1}(v\zeta'_k(a)) = \eta_{k+1}(v) \cdot {}^v\eta_{k+1}(\zeta'_k(a)) \quad \text{and} \quad \eta_k(a) = \prod_{v \in \Gamma_{E_{k+1}/E_k}} {}^v\eta_{k+1}(\zeta'_k(a)),$$

which imply $\eta_k(a)^{-1}\prod_{b \mapsto a} \eta_{k+1}(b) = \prod_v \eta_{k+1}(v)$. This is a constant function in a, hence represents the trivial element of S_k. □

Choose $\dot{\beta}_k \in u$ mapping to $\beta_k \in u_k$. Define $\dot{\beta}_{<k} = \prod_{i=1}^{k-1} \dot{\beta}_i$. Define $g_k : W'_k \to W_k$ as $\mathrm{Ad}(\dot{\beta}_{<k}) \circ \bar{g}_k$. Then $(g_k)_k$ commutes with the transition maps f_k and f'_k and induces an isomorphism $g : W' \to W$. We transport $z_{\bar{\lambda}}$ via g and obtain an element $z''_{\bar{\lambda}} \in Z^1(u \to W', Z \to S)$ that we want to compare with $z'_{\bar{\lambda}}$. Lemma 5.8 implies

$$\begin{aligned} z''_{\bar{\lambda},k}(x \boxtimes \sigma) &= z'_{\bar{\lambda},k}(x \boxtimes \sigma) \cdot \phi_{\bar{\lambda},k}(\dot{\beta}_{<k} \cdot {}^\sigma\dot{\beta}_{<k}{}^{-1}) \cdot d(l_k\eta_k \sqcup_{E_k/F} n_k\bar{\lambda})^{-1} \\ &= z'_{\bar{\lambda},k}(x \boxtimes \sigma) \cdot d(\phi_{\bar{\lambda},k}(\dot{\beta}_{<k}) \cdot l_k\eta_k \sqcup_{E_k/F} n_k\bar{\lambda})^{-1} \end{aligned}$$

On the other hand, the identity $\phi_{U,k}(\mu_1) = \phi'_{U,k}(\mu_1) - \eta_k \cup f_*(\bar{\lambda})$ was verified in [KS99, §A.3]. Since $f|_Z = 1$ we have $f(\phi_{\bar{\lambda},k}(\dot{\beta}_{<k}) \cdot l_k\eta_k \sqcup_{E_k/F} n_k\bar{\lambda}) = \bar{f}(\eta_k \cup \bar{\lambda}) = \eta_k \cup f_*(\bar{\lambda})$. It follows that $(\dot{\psi}_{T,k}(\bar{\lambda}), \phi_{U,k}(\mu_1))$ is cohomologous to $(\dot{\psi}'_{T,k}(\bar{\lambda}), \phi'_{U,k}(\mu_1))$, and so are their inflations.

Finally we argue that the homomorphism is independent of the choices of sequences n_k and E_k. If n'_k is another sequence, we may reduce to the special case $n_k|n'_k$ by comparing both n_k and n'_k to $n''_k = n_kn'_k$. In the special case $n_k|n'_k$ choose a compatible system l'_k with $l'^{n'_{k+1}/n'_k}_{k+1} = l'_k$ and define $l_k = l'^{n'_k/n_k}_k$. It is immediate to check that we have equality of cocycles $\xi_k = \xi'_k$ and $z_{\bar{\lambda},k} = z'_{\bar{\lambda},k}$

(we have of course choosen $\zeta_k = \zeta'_k$ and $s_k = s'_k$). This shows independence of the choice of n_k. For the choice of E_k, note first that passing to a co-final subsequence has no effect. If E'_k is another sequence, we may pass to co-final subsequences of both E_k and E'_k to arrange $E_k \subset E'_k \subset E_{k+1} \subset E'_{k+1}$. Define E''_k by $E''_{2k} = E_k$ and $E''_{2k+1} = E'_k$. Then E''_k is again an exhaustive sequence, of which both E_k and E'_k are co-final subsequences. This shows independence of the choice of E_k. □

{lem:tn++d3}

Lemma 5.10. *The isomorphism 5.5 satisfies the following compatibilities.*

1. *The maps $H_1(W_F, X_*(U)) \to \bar{Y}_{+,tor}(Z \to T \to U)$ and $H^1(u \to W, Z \to T \to U) \to H^0(\Gamma, U)$ translate the isomorphism 5.5 to the* negative *of the Langlands isomorphism $H_1(W_F, X_*(U)) \to H^0(\Gamma, U)$.*

2. *The maps $\bar{Y}_{+,tor}(Z \to T \to U) \to \bar{Y}_{+,tor}(Z \to T)$ and $H^1(u \to W, Z \to T \to U) \to H^1(u \to W, Z \to T)$ translate the isomorphism* (5.5) *to the isomorphism constructed in [Kal16b, §4].*

Proof. This follows by inspecting the construction of (5.5). Indeed, the definition of $\dot{\psi}_T(\bar{\lambda})$ as the inflation of $z_{\bar{\lambda},k}$ from W_k to W is the same as the construction in [Kal16b, §4.6]. On the other hand, the definition of ϕ_U used here is the same as the one in [KS99, §A.3]. The fact that it yields the negative of the Langlands isomorphism comes from the inverse in the formula $\prod_{a \in K^\times} x_a(a^{-1})$ appearing in the middle of page 131 in loc. cit. □

{cor:tn++d4}

Corollary 5.11. *The pairing* (5.3) *satisfies the following compatibilities.*

1. *The maps $H^0(\Gamma, U) \to H^1(u \to W, Z \to T \to U)$ and $H^1(W_F, \widehat{Z} \to \widehat{T} \leftarrow \widehat{U}) \to H^1(W_F, \widehat{U})$ translate the pairing* (5.3) *the the Langlands pairing.*

2. *The maps $H^1(u \to W, Z \to T \to U) \to H^1(u \to W, Z \to T)$ and $\pi_0([\widehat{\bar{T}}]^+) \to H^1(W_F, \widehat{Z} \to \widehat{T} \leftarrow \widehat{U})_{red}$ translate the pairing* (5.3) *to the pairing [Kal16b, Corollary 5.4].*

Proof. This follows directly from Fact 5.6 and Lemma 5.10. Note that in the case of the Langlands pairing both the Fact and the Lemma contain a negation, and the two cancel out. □

5.3 Rational classes and invariants for rigid inner forms

{sub:rigid_rat}

With the cohomological preliminaries out of the way, we can now extend the considerations of Section 4 to the case of general inner forms. In this subsection we extend the concepts of rational classes and their invariants.

We begin again with a quasi-split disconnected group $\tilde{G} = G \rtimes A$. More precisely, let G be a connected reductive group, defined and quasi-split over F. Let $(T, B, \{X_\alpha\})$ be an F-pinning of G and let A be a finite group that acts on G by pinned automorphisms. Assume given an action of Γ on A so that for $\sigma \in \Gamma$

we have $\sigma(a(g)) = \sigma(a)(\sigma(g))$. As we argued in Subsection 4.1 we may replace A by A^Γ and therefore assume that Γ acts trivially on A.

A given $\bar{z} \in Z^1(\Gamma, G/Z(G)^A)$ leads to the inner form $\tilde{G}_{\bar{z}}$ of $G \rtimes A$, where Γ acts on $\tilde{G}_{\bar{z}}(\bar{F})$ via the twisted action $\sigma \mapsto \mathrm{Ad}(\bar{z}(\sigma)) \rtimes \sigma$. The elements of $\tilde{G}_{\bar{z}}(F)$ are those $\tilde{\delta} \in (G \rtimes A)(\bar{F})$ that commute with $\bar{z}(\sigma) \rtimes \sigma$. Given a norm (S, γ) of $\tilde{\delta} = \delta \rtimes a$ we would like to define a cohomological invariant measing the relative position of (S, γ) and $\tilde{\delta}$. If we mimic the constructions of Subsection 4.3 we would arrive at an element $\mathrm{inv}(\gamma, (\bar{z}, \delta))$ of $H^1(\Gamma, S/Z(G)^A \xrightarrow{1-a} S)$, but that would be too crude for our purposes.

In order to define the right invariant, we need to work with $z \in Z^1(u \to W, Z(G)^A \to G)$ instead of $\bar{z} \in Z^1(\Gamma, G/Z(G)^A)$. Thus we consider the set of pairs $(z, \tilde{\delta})$, where $z \in Z^1(u \to W, Z(G)^A \to G)$, $\tilde{\delta} \in (G \rtimes A)(\bar{F})$, and $\tilde{\delta}$ commutes with $\bar{z}(\sigma) \rtimes \sigma$, where now $\bar{z} \in Z^1(\Gamma, G/Z(G)^A)$ is the image of z modulo $Z(G)^A$. This is the set of rational elements of rigid inner forms of $G \rtimes A$. The surjectivity of $Z^1(u \to W, Z(G)^A \to G) \to Z^1(\Gamma, G/Z(G)^A)$ asserted in [Kal16b, Proposition 3.6] implies that this set surjects onto the set of rational elements of inner forms considered above. Furthermore, the set of rational elements of pure inner forms of $G \rtimes A$ injects into the set of rational elements of rigid inner forms of $G \rtimes A$. The group G acts on the latter set by the same formula as in the case of pure inner forms, and the orbits of that action are the set of rational conjugacy classes of rational elements of rigid inner forms.

We can extend the cohomological notation of Subsection 4.2 as follows. Given two homomorphisms $(a, b) : G \rightrightarrows G$ and a central subgroup $Z \subset G$ that equalizes them, we consider the set $Z^1_{b,a}(u \to W, Z(G)^A \to G \rightrightarrows G)$ of pairs (z, δ), where $z \in Z^1(u \to W, Z \to G)$ and $\delta \in G$ satisfying $a(z(w)) = \delta^{-1} b(z(w)) \sigma_w(\delta)$, where $\sigma_w \in \Gamma$ is the image of $w \in W$. In our applications we will take $b = \mathrm{id}$ and abbreviate $Z^1_{b,a}$ to Z^1_a. As before, (z, δ) lies in Z^1_a if and only if $\tilde{\delta} = \delta \rtimes a$ commutes with $\tilde{z}(w) = z(w) \rtimes \sigma_w$, and we write $\tilde{Z}^1_a$ for the set of commuting pairs $(\tilde{z}, \tilde{\delta})$. The group G acts by conjugation on the set $\tilde{Z}^1_a$, or equivalently by $(g^{-1} z(w) \sigma_w(g), g^{-1} \delta a(g))$ on the set Z^1_a, and the sets of orbits under this action are denoted by $\tilde{H}^1_a$ respectively H^1_a. The set of rational elements of rigid inner forms of $G \rtimes A$ is $\bigcup_{a \in A} \tilde{Z}^1_a$, and the set of rational conjugacy classes of rational elements is the set $\bigcup_{a \in A} \tilde{H}^1_a$.

As in the case of pure inner forms, given a rational element $(\tilde{z}, \tilde{\delta})$ and a norm (S, γ) for the G-conjugacy class of $\tilde{\delta}$, we choose a representative $(\tilde{z}^*, \tilde{\delta}^*)$ of the G-orbit of $(\tilde{z}, \tilde{\delta})$ as in Lemma 4.1 and the same argument implies that $(\tilde{z}^*, \tilde{\delta}^*) \in \tilde{Z}^1_a(u \to W; Z(G)^A \to S \rightrightarrows S)$ and its cohomology class is independent of the choice of $(\tilde{z}^*, \tilde{\delta}^*)$. Moreover, $(z^{*,-1}, \delta^*)$ lies in the set $H^1(u \to W, Z(G)^A \to S \xrightarrow{1-a} S)$ defined in Subsection 5.1. We shall denote either of these classes by $\mathrm{inv}(\gamma, (z, \delta))$ or $\mathrm{inv}(\gamma, (\tilde{z}, \tilde{\delta}))$. The image of this invariant in $H^1(\Gamma, S/Z(G)^A \xrightarrow{1-a} S)$ is equal to the cruder invariant $\mathrm{inv}(\gamma, (\bar{z}, \delta))$ mentioned above.

5.4 Refined endoscopic data

{sub:ref_endo}

As in the case of connected groups, rigid inner forms require a refinement of the notion of endoscopic datum. The necessary refinement is directly analogous to that in the connected case. Namely, let $Z \subset Z(G)^A$ be finite, $\bar{G} = G/Z$, and $\widehat{\bar{G}} \to G$ the isogeny dual to $G \to \bar{G}$. Given an endoscopic datum $\mathfrak{e} = (G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, \tilde{s}^\mathfrak{e}, \xi^\mathfrak{e})$ in the sense of Subsection 4.7, a refinement consists of choosing a preimage $\dot{s}^\mathfrak{e} \in \widehat{\bar{G}} \rtimes A$ of $\tilde{s}^\mathfrak{e}$. The refined endoscopic datum is then $\dot{\mathfrak{e}} = (G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, \dot{s}^\mathfrak{e}, \xi^\mathfrak{e})$. An isomorphism $\dot{\mathfrak{e}} \to \dot{\mathfrak{e}}'$ of two such data is given by $g \in \widehat{G}$ satisfying $\xi^{\mathfrak{e}'} = \mathrm{Ad}(g) \circ \xi^\mathfrak{e}$ and $\dot{s}^{\mathfrak{e}'} = \mathrm{Ad}(g)\dot{s}^\mathfrak{e}$ modulo $Z(\widehat{\bar{G}})^\circ$.

5.5 Normalized transfer factors

{sub:rigid_tf}

Given a refined endoscopic datum $\dot{\mathfrak{e}}$ and a z-pair $\mathfrak{z}$ for $\mathfrak{e}$ we shall now define a normalized transfer factor: a function $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}, \mathfrak{z}]$ that assigns complex numbers to pairs $(\gamma^\mathfrak{z}, \tilde{\delta})$ of strongly regular semi-simple elements $\gamma^\mathfrak{z} \in G^\mathfrak{z}(F)$ and $\tilde{\delta} \in \tilde{G}_z(F)$. This factor is given by the same formula (4.2) as in the case of pure inner forms, but with a different construction of $\Delta_{KS}[\mathfrak{w}, \dot{\mathfrak{e}}, \mathfrak{z}]$, which depends on the refinement $\dot{\mathfrak{e}}$ of $\mathfrak{e}$. That in turn is given by the same formula (4.3), but we have to specify what $\Delta_{III}^{\text{new}}$ is. We shall now give this construction in the general case involving a z-pair.

The considerations are rather analogous to those of Subsection 4.11. We follow the notation there. Thus we have $\gamma^\mathfrak{z} \in S^\mathfrak{z}(F)$, $(\tilde{z}, \tilde{\delta}) \in \tilde{Z}^1_{b^{-1}}(u \to W, Z \to G \rightrightarrows G)$, a norm (S, γ) for $\tilde{\delta}$, and a representative $(\tilde{z}^*, \tilde{\delta}^*)$ of the G-conjugacy class of $(\tilde{z}, \tilde{\delta})$ with $\delta^* \in S(\bar{F})$ mapping to $\gamma \in S_b(F)$. The element $\delta^\mathfrak{z} = (\delta^*, \gamma^\mathfrak{z})$ lies in the fiber product $S_1^\mathfrak{z}$ of $S \to S_b \cong S^\mathfrak{e} \leftarrow S^\mathfrak{z}$. Under the homomorphism $(b_1^{-1} - 1) : S \to S_1^\mathfrak{z}$, the 1-cocycle $z^* \in Z^1(u \to W, Z \to S)$ maps to a 1-cocycle $(b_1^{-1} - 1)z^* \in Z^1(\Gamma, S_1^\mathfrak{z})$ that satisfies $(b_1^{-1} - 1)z^*(\sigma) = (\delta^\mathfrak{z})^{-1}\sigma(\delta^\mathfrak{z})$, and so $(z^{*,-1}, \delta^\mathfrak{z})$ belongs to $Z^1(u \to W, Z \to S \xrightarrow{1-b_1^{-1}} S_1^\mathfrak{z})$. The class $\mathrm{inv}(\gamma^\mathfrak{z}, (z, \delta))$ of this element is independent of the choice of (z^*, δ^*). Its image in $Z^1(u \to W, Z \to S \xrightarrow{1-b^{-1}} S)$ equals the class $\mathrm{inv}(\gamma, (z, \delta))$ defined in Subsection 5.3.

Next we define a class $\dot{A}_0 \in H^1(W_F, \widehat{Z} \to \widehat{S} \leftarrow \widehat{S}_1^\mathfrak{z})$ refining the class $A_0 \in H^1(W_F, \widehat{S} \to \widehat{S})$ of Subsection 4.11. Following the definition of A_0 we have the element $(a_S^{-1}, s_S) \in Z^1(W_F, (1 - b_1) : \widehat{S}_1^\mathfrak{z} \to \widehat{S})$. In addition to $\tilde{s}^\mathfrak{e} = \xi_S(s_S) \rtimes b$, we now also have $\dot{s}^\mathfrak{e} = \xi_S(\dot{s}_S) \rtimes b$, where we form $\bar{S} = S/Z$ and use the unique extension of ξ_S to ${}^L\bar{S} \to {}^L\bar{G}$ to define $\dot{s}_S \in \widehat{\bar{S}}$. Then $(a_S^{-1}, \dot{s}_S) \in Z^1(W_F, \widehat{Z} \to \widehat{S} \leftarrow \widehat{S}_1^\mathfrak{z})$ and its class is $\dot{A}_0$.

We now define $\Delta_{III}^{\text{new}}(\gamma^\mathfrak{z}, (z, \delta))$ to be the value of the pairing constructed in Subsection 5.2 at the classes $\mathrm{inv}(\gamma^\mathfrak{z}, (z, \delta))$ and $\dot{A}_0$.

5.6 The local correspondence and character identities

{sub:llc_rigid}

Let $\phi : L_F \to {}^LG$ be a tempered Langlands parameter. In subsection 4.6 we introduced the group of $\tilde G$-equivalences $\tilde S_\phi = \text{Cent}(\phi, \widehat G \rtimes A)$. It was part of an exact sequence

$$1 \to S_\phi \to \tilde S_\phi \to A^{[\phi]} \to 1,$$

where $A^{[\phi]}$ is the stabilizer in A of the G-equivalence class of ϕ. For a finite subgroup $Z \subset Z(G)^A$ we have the isogenies $G \to \bar G = G/Z$ and $\widehat{\bar G} \to \widehat G$ and we define $\tilde S_\phi^+$ to be the preimage in $\widehat{\bar G} \rtimes A$ of $\tilde S_\phi$. This is analogous to the definition of S_ϕ^+ as the preimage in $\widehat{\bar G}$ of S_ϕ given in [Kal16b, §5.4]. We have again the exact sequence

$$1 \to S_\phi^+ \to \tilde S_\phi^+ \to A^{[\phi]} \to 1.$$

We are now interested in the rigid inner form $\tilde G_z$ for some $z \in Z^1(u \to W, Z \to G)$. Let $A^{[z]}$ be the stabilizer of the class of z, and $A^{[\phi],[z]} = A^{[\phi]} \cap A^{[z]}$. Pulling back the above exact sequence along the inclusion $A^{[\phi],[z]} \to A^{[\phi]}$ we obtain the exact sequence

$$1 \to S_\phi^+ \to \tilde S_\phi^{+,[z]} \to A^{[\phi],[z]} \to 1.$$

In the case $F = \mathbb{R}$ set ${}^K\tilde G_z$ to be the associated K-group, i.e. the disjoint union of $\tilde G_{z'}$ for all z' in the image of $H^1(\mathbb{R}, G_{z,\text{sc}}) \to H^1(\mathbb{R}, G_z) \to H^1(u \to W, Z(G)^A \to G_z) \to H^1(u \to W, Z(G)^A \to G)$.

{cnj:llc_rigid}

Conjecture 5.12. 1. *The choice of an A-special Whittaker datum $\mathfrak{w}$ on G determines a bijection between the set of irreducible admissible G-tempered representations of $\tilde G_z(F)$ when $F/\mathbb{Q}_p$, or any member of ${}^K\tilde G_z(F)$ when $F = \mathbb{R}$, and the set of $\widehat G \rtimes A$-conjugacy classes of pairs $(\phi, \tilde\rho)$, where $\phi : L_F \to {}^LG$ is a tempered Langlands parameter, and $\tilde\rho \in Irr(\pi_0(\tilde S_\phi^{+,[z]}), [z])$. When $z = 1$ the representation corresponding to $(\phi, \tilde\rho)$ is $\mathfrak{w}$-generic if and only if $\tilde\rho = 1$.*

2. *This bijection satisfies the character identity* (4.4) *for a pair of functions f and $f^\mathfrak{s}$ as in Lemma 4.6, where now the transfer factor is the one constructed in Subsection 5.5.*

6 Change of Whittaker data

{sec:change_whit}

In [Kal13] we studied how the bijection $\text{Irr}(\pi_0(S_\varphi^+)) \to \Pi_\varphi$ of the refined local Langlands conjecture depends on the Whittaker datum $\mathfrak{w}$, in the case of a connected reductive group. Strictly speaking loc. cit. considered only pure and extended pure inner twists, but not rigid inner twists, which were unavailable at the time. In this section we shall extend these considerations to the case of rigid inner forms of quasi-split groups and may be connected or disconnected.

Consider first the connected case, which will serve primarily to recall notation from [Kal13]. Let G be a quasi-split connected reductive group defined over F and let $\mathfrak{w}_1, \mathfrak{w}_2$ be two Whittaker data. There is a unique element of $\mathrm{cok}(G(F) \to G_{\mathrm{ad}}(F))$ conjugating $\mathfrak{w}_1$ to $\mathfrak{w}_2$, which we denote by $(\mathfrak{w}_1, \mathfrak{w}_1)$. Recall from [Kal13, Lemma 4.1] that there is a natural injection

$$\mathrm{cok}\Big(G(F) \to G_{\mathrm{ad}}(F)\Big) \to \ker\Big(H^1(W_F, Z(\widehat{G}_{\mathrm{sc}})) \to H^1(W_F, Z(\widehat{G}))\Big)^D.$$

It essentially comes from Poitou-Tate duality

$$H^1(\Gamma, Z(G_{\mathrm{sc}})) \otimes H^1(\Gamma, X^*(Z(G_{\mathrm{sc}}))) \to H^2(\Gamma, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$$

and the identification $X^*(Z(G_{\mathrm{sc}})) = Z(\widehat{G}_{\mathrm{sc}})$ via the exponential map $\exp : X_*(\widehat{T}_{\mathrm{sc}}) \otimes_{\mathbb{Z}} \mathbb{C} \to \widehat{T}_{\mathrm{sc}}$ with kernel $X_*(\widehat{T}_{\mathrm{sc}})$. Given a tempered Langlands parameter $\phi : L_F \to {}^LG$ we endow the exact sequences

$$1 \to Z(\widehat{G}_{\mathrm{sc}}) \to \widehat{G}_{\mathrm{sc}} \to \widehat{G}_{\mathrm{ad}} \to 1, \qquad 1 \to Z(\widehat{G}) \to \widehat{G} \to \widehat{G}_{\mathrm{ad}} \to 1$$

with L_F-action via $\mathrm{Ad}(\phi(-))$. The actions on $Z(\widehat{G}_{\mathrm{sc}})$ and $Z(\widehat{G})$ are of course simply the Γ-action inflated to L_F and $H^1(L_F, -) = H^1(W_F, -)$ for these two groups. The connecting homomorphism $H^0(L_F, \widehat{G}_{\mathrm{ad}}) \to H^1(\Gamma, Z(\widehat{G}_{\mathrm{sc}}))$ is continuous and thus factors through the component group of the complex algebraic group $H^0(L_F, \widehat{G}_{\mathrm{ad}})$. We have $S_\phi = H^0(L_F, \widehat{G})$ and its image under that connecting homomorphism lands in $\ker(H^1(W_F, Z(\widehat{G}_{\mathrm{sc}})) \to H^1(W_F, Z(\widehat{G})))$. Therefore $(\mathfrak{w}_1, \mathfrak{w}_2)$ induces a character of $\pi_0(S_\phi/Z(\widehat{G})^\Gamma) = \pi_0(S_\phi^+/Z(\widehat{\bar{G}})^+)$. If

$$\iota_i : \mathrm{Irr}(S_\phi^+) \to \Pi_\phi(G)$$

are the two bijections of the refined local Langlands correspondence, where we are using compound L-packets encompassing all rigid inner forms, then according to [Kal13, (1.1)] we have

$$\iota_2(\rho) = \iota_1(\rho \otimes (\mathfrak{w}_1, \mathfrak{w}_2)).$$

We now turn to the disconnected case. Thus let $\tilde{G} = G \rtimes A$ be a quasi-split, (possibly) disconnected, reductive group, and let $\mathfrak{w}_1, \mathfrak{w}_2$ be A-special Whittaker data for G. Let $z \in Z^1(u \to W, Z(G)^A \to G)$. We denote by

$$\iota_i : \mathrm{Irr}(\pi_0(\tilde{S}_\phi^{+,[z]}), [z]) \to \Pi_\phi(\tilde{G}_z)$$

the bijections of Conjecture 5.12 with respect to the Whattaker data $\mathfrak{w}_i$. The approach to comparing these is the same as in the connected case. We use the exact sequences

$$1 \to Z(\widehat{G}_{\mathrm{sc}}) \to \widehat{G}_{\mathrm{sc}} \rtimes A \to \widehat{G}_{\mathrm{ad}} \rtimes A \to 1, \qquad 1 \to Z(\widehat{G}) \to \widehat{G} \rtimes A \to \widehat{G}_{\mathrm{ad}} \rtimes A \to 1$$

to obtain the connecting map $H^0(L_F, \widehat{G}_{\mathrm{ad}} \rtimes A) \to H^1(W_F, Z(\widehat{G}_{\mathrm{sc}}))$. This map is no longer a homomorphism, but rather a twisted homomorphism (i.e. a

1-cocycle) for the action for $\widehat{G}_{\text{ad}} \rtimes A$ on $Z(\widehat{G}_{\text{sc}})$ given by the projection to A and the natural action of A on $Z(\widehat{G}_{\text{sc}})$. Nonetheless, this map factors through $\pi_0(H^0(L_F, \widehat{G}_{\text{ad}} \rtimes A))$ and sends $\tilde{S}_\phi = H^0(L_F, \widehat{G} \rtimes A)$ to $\ker(H^1(W_F, Z(\widehat{G}_{\text{sc}})) \to H^1(W_F, Z(\widehat{G})))$. In this way we obtain a twisted homomorphism

$$\pi_0(\tilde{S}_\phi / Z(\widehat{G})^\Gamma) \to \ker(H^1(W_F, Z(\widehat{G}_{\text{sc}})) \to H^1(W_F, Z(\widehat{G}))).$$

Since the character $(\mathfrak{w}_1, \mathfrak{w}_2)$ of $\ker(H^1(W_F, Z(\widehat{G}_{\text{sc}})) \to H^1(W_F, Z(\widehat{G})))$ is A-invariant, its pull back under this twisted homomorphism is a character of $\pi_0(\tilde{S}_\phi / Z(\widehat{G})^\Gamma) = \pi_0(\tilde{S}_\phi^+ / Z(\widehat{\bar{G}})^+)$, which we may pull back further to $\pi_0(\tilde{S}_\phi^{+,[z]})$.

Proposition 6.1.

$$\iota_2(\rho) = \iota_1(\rho \otimes (\mathfrak{w}_1, \mathfrak{w}_2)).$$

Proof. Let $\tilde{s} = s \rtimes a \in \tilde{S}_\phi$. As in the proof of [Kal13, Theorem 4.3] it is enough to prove the identity

$$\Delta_{KS}[\mathfrak{w}_2] = \langle (\mathfrak{w}_1, \mathfrak{w}_2), s \rangle^{-1} \cdot \Delta_{KS}[\mathfrak{w}_1].$$

To prove this we choose an additive character ψ of F and A-invariant F-pinnings spl_i giving rise to $\mathfrak{w}_i$. We write Δ_{KS} according to (4.3) and note that only Δ_I depends on the pinnings. Let $g \in G_{\text{ad}}^A(F)$ conjugate spl_1 to spl_2 and let $x \in H^1(F, Z(G_{\text{sc}})^A)$ be the image of g under the connecting homomorphism for the exact sequence (cf. Fact 3.7)

$$1 \to Z(G_{\text{sc}})^A \to G_{\text{sc}}^A \to G_{\text{ad}}^A \to 1.$$

The argument of [LS87, (2.3.1)] shows that the twisted splitting invariant in $H^1(F, T_{\text{sc}}^a)$ with respect to spl_2 is the product of the twisted splitting invariant with respect to spl_1 with x (note that loc. cit. uses conjugation on the right, so their g is our g^{-1}). Therefore $\Delta_I[\mathfrak{w}_2] = \langle x, s \rangle^{-1} \Delta_I[\mathfrak{w}_1]$, as claimed. □

7 CHANGE OF COMPONENT GROUP

{sec:change_comp}

7.1 Restriction

{sub:comp_rest}

Assume now given a map of finite groups $B \to A$. We can consider the disconnected groups $G^A = G \rtimes A$ and $G^B = G \rtimes B$, where B acts on G via its map to A. We can consider restriction of representations along the map $G_z^B(F) \to G_z^A(F)$. Dually, the map $\widehat{G} \rtimes B \to \widehat{G} \rtimes A$ induces for each tempered Langlands parameter ϕ a map $\pi_0(S_\phi^{B,+,[z]}) \to \pi_0(S_\phi^{A,+,[z]})$ and we can consider restriction of representations along this map as well.

Let the G-tempered representation π^A of $G_z^A(F)$ correspond under Conjecture 5.12 to the pair (ϕ, ρ^A) with $\phi : L_F \to {}^LG$ and $\rho^A \in \text{Irr}(\pi_0(S_\phi^{A,+,[z]}), [z])$. Let the G-tempered representation π^B of $G_z^B(F)$ correspond under Conjecture 4.2 to the pair (ϕ', ρ^B) with $\phi' : L_F \to {}^LG$ and $\rho^B \in \text{Irr}(\pi_0(S_{\phi'}^{B,+,[z]}), [z])$. We thus have the multiplicity of π^B in $\operatorname{Res}\pi^A$. If the parameters ϕ' and ϕ are G^A-equivalent we may replace (ϕ, ρ^A) by an equivalent pair and assume $\phi = \phi'$. Then we also have the multiplicity of ρ^B in $\operatorname{Res}\rho^A$.

{cnj:llc_rest}

Conjecture 7.1. *The multiplicity of π^B in $\operatorname{Res}\pi^A$ is zero unless ϕ and ϕ' are G^A-equivalent. Assuming that and arranging $\phi = \phi'$, this multiplicity is equal to the multiplicty of ρ^B in $\operatorname{Res}\rho^A$.*

Remark 7.2. Let $\tilde{G} = G \rtimes A$ and $B = \{1\}$. Applying Conjecture 7.1 we obtain a complete description of the set $\tilde{\Pi}_{\phi,z}$ in terms of the L-packet $\Pi_{\phi,z}$ of the connected group $G_z(F)$: An irreducible G-tempered representation $\tilde{\pi}$ of $\tilde{G}_z(F)$ belongs to $\tilde{\Pi}_{\phi,z}$ if and only if its restriction to $G_z(F)$ intersects $\Pi_{\phi,z}$. Equivalently, the set $\tilde{\Pi}_{\phi,z}$ consists precisely of the irreducible constituents of the inductions to $\tilde{G}_z(F)$ of the elements of $\Pi_{\phi,z}$. Hence the content of Conjecture 5.12 is in the internal structure and character identities with normalized transfer factors. Note that, just like in the connected case, the packets $\tilde{\Pi}_{\phi,z}$ are disjoint and exhaust the set of isomorphism classes of irreducible admissible G-tempered representations, assuming this is the case for the packets $\Pi_{\phi,z}$.

7.2 Slicing by cosets

{sub:comp_coset}

In this section we assume the validity of the refined local Langlands correspondence for connected groups, as well as its functoriality as expressed in Conjecture A.1. Given a tempered parameter $\phi : L_F \to {}^LG$ we then have the L-packet $\Pi_\phi(G_z)$. If $\pi \in \Pi_\phi(G_z)$ and $\rho \in \operatorname{Irr}(\pi_0(S_\phi^+))$ corresponding to each other then Conjecture A.1 implies $A_\pi^{[z]} = A_\rho^{[\phi]}$. We have the elements $\alpha_\pi \in H^2(A_\pi^{[z]}, \mathbb{C}^\times)$ and $\alpha_\rho \in H^2(A_\rho^{[\phi]}, \mathbb{C}^\times)$ corresponding to the projective extension of π to $\tilde{G}_z(F)_\pi$ and of ρ to $\pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})$, respectively. The elements α_π and α_ρ are equal if and only if the representation $\pi \boxtimes \rho^\vee$ of $G_z(F) \times \pi_0(S_\phi^+)$ has an extension to $\tilde{G}_z(F)_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})$. Such an extension is then well-defined up to a character of $A_\pi^{[z]}$.

{cnj:coset}

Conjecture 7.3. *Let $\phi : L_F \to {}^LG$ be a tempered parameter. Let $\pi \in \Pi_\phi(G_z)$ and $\rho \in Irr(\pi_0(S_\phi^+))$ correspond to each other. The representation $\pi \boxtimes \rho^\vee$ of $G_z(F) \times \pi_0(S_\phi^+)$ has an extension $(\pi \boxtimes \rho^\vee)^{can}$ to $\tilde{G}_z(F)_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})$ such that for $a \in A$, $\tilde{f} \in \mathcal{C}_c^\infty([G \rtimes a]_z(F))$ and $\tilde{s} \in \tilde{S}_\phi$ mapping to a^{-1} we have*

$$S\Theta_{\phi^\mathfrak{z}}(f^\mathfrak{z}) = \sum_{\substack{\pi \in \Pi_\phi(G_z) \\ \pi \circ a \cong \pi}} tr\,(\pi \boxtimes \rho^\vee)^{can}(\tilde{f}, \tilde{s}^{-1}),$$

where $\mathfrak{e}$ is the endoscopic datum corresponding to the pair $(\tilde{s}, \phi)$ by the spectral construction of §4.8, $\mathfrak{z}$ is a z-pair for it, and $f^\mathfrak{z} \in \mathcal{C}_c^\infty(G^\mathfrak{z}(F))$ satisfies

$$SO_\gamma(f^\mathfrak{z}) = \sum_{\tilde{\delta} \in [G \rtimes a]_z(F)/G_z(F)-conj} \Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^\mathfrak{z}, \tilde{\delta}) \int_{x \in G_z(F)/G_z(F)_{\tilde{\delta}}} \tilde{f}(x\tilde{\delta}x^{-1}).$$

{rem:canext}

Remark 7.4. We note that the extension $(\pi \boxtimes \rho^\vee)^{\mathrm{can}}$ is unique if it exists, due to the character identities it is supposed to satisfy. Furthermore, these character identities imply that for any $b \in A$ the isomorphism

$$\tilde{G}_z(F)_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]}) \to \tilde{G}_z(F)_{b\pi} \times_{A_{b\pi}^{[z]}} \pi_0(\tilde{S}_{b\phi,b\rho}^{+,[z]})$$

(given up to an inner automorphism) of conjugation by b identifies the extension $(\pi \boxtimes \rho^\vee)^{\mathrm{can}}$ with $(b\pi \boxtimes b\rho^\vee)^{\mathrm{can}}$.

{pro:slice}

Proposition 7.5. *Conjecture 7.3 is equivalent to Conjectures 5.12 and 7.1.*

As a preparation for the proof we need the following elementary discussion. Consider an exact sequence of locally pro-finite groups

$$1 \to H \to \tilde{H} \to A \to 1$$

with A finite and an A-invariant subset X of the set of isomorphism classes of irreducible smooth representations of H. The group

$$\tilde{H} \times_A \tilde{H} = \{(\tilde{h}_1, \tilde{h}_2) \in \tilde{H} \times \tilde{H} | \tilde{h}_1 \in \tilde{h}_2 H\}$$

fits into the exact sequence

$$1 \to H \times H \to \tilde{H} \times_A \tilde{H} \to A \to 1.$$

{lem:ex2}

Lemma 7.6. *1. If $X = \{x\}$ there exists a projective representation $\tilde{x}$ of $\tilde{H}$ extending x and satisfying $\tilde{x}(h\tilde{h}) = x(h) \circ \tilde{x}(\tilde{h})$ for $h \in H$ and $\tilde{h} \in \tilde{H}$. The external tensor product $\tilde{x} \boxtimes \tilde{x}^\vee$ is a linear representation of $\tilde{H} \times_A \tilde{H}$ depending only on x, but not on $\tilde{x}$.*

2. *The isomorphism class of the representation*

$$\tilde{X} = \bigoplus_x \mathit{Ind}_{\tilde{H}_x \times_{A_x} \tilde{H}_x}^{\tilde{H} \times_A \tilde{H}} \tilde{x} \boxtimes \tilde{x}^\vee,$$

where x runs over a set of representatives for the A-orbits in X, is independent of that set and is an extension of $\bigoplus_{x \in X} x \boxtimes x^\vee$.

3. *We have*

$$\mathit{Ind}_{\tilde{H} \times_A \tilde{H}}^{\tilde{H} \times \tilde{H}} \tilde{X} = \bigoplus (\xi \boxtimes \xi^\vee),$$

where ξ runs over the set of irreducible representations of $\tilde{H}$ lying over X.

4. *Given a diagram of extensions*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & \tilde{H}_1 & \longrightarrow & A_1 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \tilde{H}_2 & \longrightarrow & A_2 & \longrightarrow & 1 \end{array}$$

and an A_2-invariant set X_2, let X_1 be the set X_2 with the action of A_1 restricted from that of A_2. The representation $\tilde{X}_1$ of $\tilde{H}_1 \times_{A_1} \tilde{H}_1$ is the pull-back of the representation $\tilde{X}_2$ of $\tilde{H}_2 \times_{A_2} \tilde{H}_2$.

{lem:ex1}

Lemma 7.7. *We are given two extensions $1 \to H_i \to \tilde{H}_i \to A \to 1$ of locally profinite groups with A finite. Let X be an A-set equipped with A-equivariant injections $X \to Irr(H_i)$. For $x \in X$ we write x_i for its image in $Irr(H_i)$ and $\alpha_{x_i} \in H^2(A_x, \mathbb{C}^\times)$ for the associated class. We assume $\alpha_{x_1} = \alpha_{x_2}$. Assume given an extension to $\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}$ of the representation $x_1 \boxtimes x_2^\vee$ of $H_1 \times H_2$, and call this extension $\tilde{x}$. Assume that for $a \in A$ and $y = ax$ we have $\tilde{y} = \tilde{x} \circ Ad(a^{-1})$. Then*

1. *If X is a single A-orbit the isomorphism class of the representation*

$$\tilde{X} := Ind_{\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}}^{\tilde{H}_1 \times_A \tilde{H}_2} \tilde{x}$$

is independent of the choice of $x \in X$. For general X set

$$\tilde{X} := \bigoplus_{X' \in X/A} \tilde{X}'.$$

2. *The representation $\tilde{X}$ is an extension to $\tilde{H}_1 \times_A \tilde{H}_2$ of the representation $\bigoplus_{x \in X} x_1 \boxtimes x_2^\vee$.*

3. *Let*

$$Ind_{\tilde{H}_1 \times_A \tilde{H}_2}^{\tilde{H}_1 \times \tilde{H}_2} \tilde{X} = \bigoplus (\xi_1 \boxtimes \xi_2^\vee)^{m(\xi_1, \xi_2)}$$

be the decomposition into irreducible pieces. Then $m(\xi_1, \xi_2) \leq 1$ and the correspondence $Irr(\tilde{H}_1) \leftrightarrow Irr(\tilde{H}_2)$ afforded by m is a bijection between the sets of irreducible representations of $\tilde{H}_i$ lying over the sets X_i.

4. *Let $A' \subset A$ be a subgroup and write $\tilde{H}_i' \subset \tilde{H}_i$ for the preimage of A'. All previous points can be applied to $\tilde{H}_i'$ in place of $\tilde{H}_i$. Let $\xi_i \in Irr(\tilde{H}_i)$ and $\xi_i' \in Irr(\tilde{H}_i')$ be such that $\xi_1 \leftrightarrow \xi_2$ under the bijection of point 3, and $\xi_1' \leftrightarrow \xi_2'$ under the analogous bijection. Then the multiplicity of ξ_1' in $\xi_1|_{\tilde{H}_1'}$ equals the multiplicity of ξ_2' in $\xi_2|_{\tilde{H}_2'}$.*

Proof of Lemma 7.7. Since all statements break up according to the orbits of A in X we assume for the rest of the proof that X is a single A-orbit. The independence of $\tilde{X}$ from $\tilde{x}$ follows from the assumption $\tilde{y} = \tilde{x} \circ \mathrm{Ad}(a^{-1})$ for $y = ax$ and the fact that X is a transitive A-set. The fact that $\tilde{X}$ is an extension of $\bigoplus x_1 \boxtimes x_2^\vee$ follows from the induction-restriction formula.

For the third claim we perform induction in stages

$$\mathrm{Ind}_{\tilde{H}_{1,x} \times \tilde{H}_{2,x}}^{\tilde{H}_1 \times \tilde{H}_2} \mathrm{Ind}_{\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}}^{\tilde{H}_{1,x} \times \tilde{H}_{2,x}} \tilde{x}$$

and consider first the inner induction. By assumption there exist projective representations $\tilde{x}_i$ of $\tilde{H}_{i,x}$ such that we have an equality $\alpha_{\tilde{x}_1} = \alpha_{\tilde{x}_2}$ of elements of $Z^2(A_x, \mathbb{C}^\times)$ and such that the restriction of $\tilde{x}_1 \boxtimes \tilde{x}_2^\vee$ from $\tilde{H}_{1,x} \times \tilde{H}_{2,x}$ to $\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}$ is equal to $\tilde{x}$. Write $\alpha_{\tilde{x}}$ for the common value of $\alpha_{\tilde{x}_i}$. Then

$$\mathrm{Ind}_{\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}}^{\tilde{H}_{1,x} \times \tilde{H}_{2,x}} \tilde{x} = \left(\mathrm{Ind}_{\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}}^{\tilde{H}_{1,x} \times \tilde{H}_{2,x}} \mathbf{1}\right) \otimes (\tilde{x}_1 \boxtimes \tilde{x}_2^\vee),$$

where on the right we are performing twisted induction with 2-cocycle $(\alpha_{\tilde{x}}^{-1}, \alpha_{\tilde{x}})$ in the left factor, and then tensoring with the $(\alpha_{\tilde{x}}, \alpha_{\tilde{x}}^{-1})$-projective representation $\tilde{x}_1 \boxtimes \tilde{x}_2^\vee$ to obtain a linear representation of $\tilde{H}_{1,x} \times \tilde{H}_{2,x}$. The representation $\mathrm{Ind}_{\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}}^{\tilde{H}_{1,x} \times \tilde{H}_{2,x}} \mathbf{1}$ of $\tilde{H}_{1,x} \times \tilde{H}_{2,x}$ is the inflation of the representation $\mathrm{Ind}_{A_x}^{A_x \times A_x} \mathbf{1}$, where A_x is embedded diagonally into $A_x \times A_x$. The latter is isomorphic to the twisted group algebra $\mathbb{C}[A_x]_{\alpha_{\tilde{x}}}$ seen as a left-right-bimodule over itself, and as such decomposes as the direct sum $\bigoplus_\tau \tau \boxtimes \tau^\vee$, where τ runs over the set of isomorphism classes of irreducible $\alpha_{\tilde{x}}$-projective representations of A_x. This shows that

$$\mathrm{Ind}_{\tilde{H}_{1,x} \times_{A_x} \tilde{H}_{2,x}}^{\tilde{H}_{1,x} \times \tilde{H}_{2,x}} \tilde{x} = \bigoplus_\tau (\tau \otimes \tilde{x}_1) \boxtimes (\tau \otimes \tilde{x}_2)^\vee.$$

As τ runs over the set of isomorphism classes of $\alpha_{\tilde{x}}$-projective representations of A_x, $\tau \otimes \tilde{x}_i$, runs over the set of irreducible linear representations of $\tilde{H}_{i,x}$ whose restriction to H_i contains x_i, and $\mathrm{Ind}_{\tilde{H}_{i,x}}^{\tilde{H}_i}$ runs over the set of irreducible linear representations of $\tilde{H}_i$ whose restriction to H_i contains x_i.

For the fourth claim we write $\xi_i = \mathrm{Ind}_{\tilde{H}_{i,x}}^{\tilde{H}_i} \tilde{x}_i \otimes \tau$, where $\tilde{x}_i$ is an extension of x_i to a projective representation of $\tilde{H}_{i,x}$ with 2-cocycle $\alpha_{\tilde{x}} \in Z^2(A_x, \mathbb{C}^\times)$ and τ is a projective representation of A_x with 2-cocycle $\alpha_{\tilde{x}}^{-1}$. Write correspondingly $\xi_i' = \mathrm{Ind}_{\tilde{H}_{i,x}'}^{\tilde{H}_i'} \tilde{x}_i \otimes \tau'$, where we take the restriction of $\tilde{x}_i$ to $\tilde{H}_{i,x}'$ and τ' is a projective representation of A_x' with 2-cocycle given by the restriction of $\alpha_{\tilde{x}}^{-1}$. The Mackey formula shows that

$$\mathrm{Res}_{\tilde{H}_i'}^{\tilde{H}_i} \mathrm{Ind}_{\tilde{H}_{i,x}}^{\tilde{H}_i} \tilde{x}_i \otimes \tau = \bigoplus_{c \in A' \backslash A / A_x} \mathrm{Ind}_{\tilde{H}_{i,cx}'}^{\tilde{H}_i'} \mathrm{Res}_{\tilde{H}_{i,cx}'}^{\tilde{H}_{i,cx}} c(\tilde{x}_i \otimes \tau).$$

The summation index parameterizes the A'-orbits in the A-orbit of x. Since ξ_i' lies over the A'-orbit of x, the multiplicity of ξ_i' in the above restriction is zero for all summands except possibly the one indexed by $c = 1$. This summand decomposes into the irreducible representations as

$$\bigoplus_{\tau''} \mathrm{Ind}_{\tilde{H}_{i,x}'}^{\tilde{H}_i'} (\tilde{x}_i \otimes \tau'')^{m(\tau, \tau'')},$$

where τ'' runs over the irreducible projective representations of A_x' and $m(\tau, \tau'')$ is the multiplicity of τ'' in the restriction of τ. We conclude that $m(\xi_i, \xi_i') = m(\tau, \tau')$. □

Proof of Lemma 7.6. For the first point we may fix a set $\dot{A} \subset \tilde{H}$ of representatives for $A = \tilde{H}/H$, an isomorphism $\tilde{x}(\dot{a}) : V_x \to V_x$ of complex vector spaces with the intertwining property $\tilde{x}(\dot{a})\circ x(\dot{a}^{-1}h\dot{a}) = x(h)\circ\tilde{x}(\dot{a})$ for all $h \in H$, and define $\tilde{x}(h\dot{a}) = x(h) \circ \tilde{x}(\dot{a})$ for all $h \in H$ and $a \in A$. This has the required properties. We define the automorphism $\tilde{x}^\vee(\tilde{h}) := \tilde{x}(\tilde{h})^{*,-1}$ of $V_x^\vee$. The linearity of $\tilde{x} \boxtimes \tilde{x}^\vee$ follows from the fact that the 2-cocycle of $\tilde{x}$ is inflated from A and the 2-cocycle of $\tilde{x}^\vee$ is its inverse. Keeping $\dot{A}$ fixed, the independence from $\tilde{x}$ follows because another choice is of the form $c \cdot \tilde{x}$ for $c \in C^1(A, \mathbb{C}^\times)$. The independence of the choice of $\dot{A}$ now follows readily.

If $y = x \circ \mathrm{Ad}(a)$ choose $\tilde{h}_a \in \tilde{H}$ mapping to a and define $\tilde{y} := \tilde{x} \circ \mathrm{Ad}(\tilde{h}_a)$. Then $\tilde{y}$ is a projective extension of y and satisfies the conditions of point 1. We have $\tilde{y}\boxtimes\tilde{y}^\vee = (\tilde{x}\boxtimes\tilde{x}^\vee)\circ\mathrm{Ad}(\tilde{h}_a)$ and the second point follows from Lemma 7.7.

The third point follows from the proof of the previous lemma, for we see from the argument given there that the right hand side is

$$\bigoplus_\tau \mathrm{Ind}_{\tilde{H}_x\times\tilde{H}_x}^{\tilde{H}\times\tilde{H}}(\tilde{x}\otimes\tau)\boxtimes(\tilde{x}\otimes\tau)^\vee,$$

where τ runs over the irreducible $\alpha_{\tilde{x}}$-projective representations of A_x.

For the fourth point we note that the right square in the diagram is automatically cartesian. Since pull-back is transitive it is enough to treat the extreme cases when $A_1 \to A_2$ is injective respectively surjective, a property that is then inherited by the maps $\tilde{H}_1 \to \tilde{H}_2$ and $\tilde{H}_1 \times_{A_1} \tilde{H}_1 \to \tilde{H}_2 \times_{A_2} \tilde{H}_2$. We may assume that X_2 is a single A_2-orbit. Choose $x \in X_2$. In the injective case we apply point 2 and the Mackey formula to see that

$$\mathrm{Res}_{\tilde{H}_1\times_{A_1}\tilde{H}_1}^{\tilde{H}_2\times_{A_2}\tilde{H}_2}\mathrm{Ind}_{\tilde{H}_{2,x}\times_{A_{2,x}}\tilde{H}_{2,x}}^{\tilde{H}_2\times_{A_2}\tilde{H}_2}\tilde{x}\boxtimes\tilde{x}^\vee$$

is given by

$$\bigoplus_{a\in A_1\setminus A_2/A_{2,x}}\mathrm{Ind}_{\tilde{H}_{1,ax}\times_{A_{1,ax}}\tilde{H}_{1,ax}}^{\tilde{H}_1\times_{A_1}\tilde{H}_1}\mathrm{Res}_{\tilde{H}_{1,ax}\times_{A_{1,ax}}\tilde{H}_{1,ax}}^{\tilde{H}_{2,ax}\times_{A_{2,ax}}\tilde{H}_{2,ax}}a(\tilde{x}\boxtimes\tilde{x}^\vee).$$

We have already argued that $a\tilde{x}$ is a projective extension of ax to $\tilde{H}_{2,ax}$ with the required property for point 1, and it is clear that its restriction to $\tilde{H}_{1,ax}$ is such as well. Since $A_1 \setminus A_2/A_{2,x}$ parameterizes the A_1-orbits in X, the claim follows again from point 2.

In the surjective case the kernel N of $A_1 \to A_2$ is also the kernel of the surjective maps $\tilde{H}_1 \to \tilde{H}_2$, $\tilde{H}_{1,x} \to \tilde{H}_{2,x}$, $\tilde{H}_1 \times_{A_1} \tilde{H}_1 \to \tilde{H}_2 \times_{A_2} \tilde{H}_2$, and $\tilde{H}_{1,x} \times_{A_{1,x}} \tilde{H}_{1,x} \to \tilde{H}_{2,x} \times_{A_{2,x}} \tilde{H}_{2,x}$. The set X_1 is a single A_1-orbit. We apply again point 2. If $\tilde{x}$ is a projective extension of x to $\tilde{H}_{2,x}$ satisfying the condition of point 1, then its pull-back to $\tilde{H}_{1,x}$ is a projective extension of x that satisfies the same condition. We have

$$\mathrm{Inf}_{\tilde{H}_1\times_{A_1}\tilde{H}_1}^{\tilde{H}_2\times_{A_2}\tilde{H}_2}\mathrm{Ind}_{\tilde{H}_{2,x}\times_{A_{2,x}}\tilde{H}_{2,x}}^{\tilde{H}_2\times_{A_2}\tilde{H}_2}\tilde{x}\boxtimes\tilde{x}^\vee = \mathrm{Ind}_{\tilde{H}_{1,x}\times_{A_{1,x}}\tilde{H}_{1,x}}^{\tilde{H}_1\times_{A_1}\tilde{H}_1}\mathrm{Inf}_{\tilde{H}_{1,x}\times_{A_{1,x}}\tilde{H}_{1,x}}^{\tilde{H}_{2,x}\times_{A_{2,x}}\tilde{H}_{2,x}}\tilde{x}\boxtimes\tilde{x}^\vee,$$

where Inf stands for inflation, i.e. pull-back. □

Proof of Proposition 7.5. Assume conjecture 7.3. We consider the extensions $1 \to G_z(F) \to \tilde{G}_z(F)^{[\phi]} \to A^{[\phi],[z]} \to 1$ and $1 \to \pi_0(S_\phi^+) \to \pi_0(\tilde{S}_\phi^{+,[z]}) \to A^{[\phi],[z]} \to 1$. Let $X_1 = \Pi_\phi(G_z)$ and $X_2 = \mathrm{Irr}(\pi_0(S_\phi^+),[z])$. By Conjecture A.1 these sets are in an $A^{[\phi],[z]}$-equivariant bijection. Define $\Pi_\phi(\tilde{G}_z^{[\phi]})$ to be the set of irreducible representations of $\tilde{G}_z(F)^{[\phi]}$ whose restriction to $G_z(F)$ meets X_1. Conjecture 7.3 fulfills the assumptions of Lemma 7.7, see also Remark 7.4. Point 3 of that Lemma provides a bijection between $\Pi_\phi(\tilde{G}_z^{[\phi]})$ and $\mathrm{Irr}(\pi_0(S_\phi^{+,[z]},[z]))$, and point 4 asserts that this bijection preserves multiplicities upon restriction along a map $B \to A$. By Conjecture A.1 the stabilizer in $\tilde{G}_z(F)$ of any element of $\Pi_\phi(\tilde{G}_z^{[\phi]})$ is contained in $\tilde{G}_z(F)^{[\phi]}$. Therefore induction gives a bijection from $\Pi_\phi(\tilde{G}_z^{[\phi]})$ to the set $\Pi_\phi(\tilde{G}_z)$ of irreducible representations of $\tilde{G}_z(F)$ whose restriction to $G_z(F)$ meets $\Pi_\phi(G_z)$, hence the first point of Conjecture 5.12 holds. Multiplicities are still preserved, hence Conjecture 7.1 holds.

Conversely, assume Conjecture 7.1 and the first point of Conjecture 5.12. Let $\pi \in \Pi_\phi(G_z)$ correspond to $\rho \in \mathrm{Irr}(\pi_0(S_\phi^+))$. Consider the group $\tilde{G}_z(F)_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})$. We claim that this group arises by taking F-points of a disconnected algebraic group that fits in the framework discussed in this paper. Indeed, let $\tilde{G}_\pi$ denote the preimage in $\tilde{G}$ of $A_\pi^{[z]}$. We have the isomorphism of algebraic groups

$$G \rtimes_A \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]}) \to \tilde{G}_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]}), \qquad (g \rtimes \tilde{s}) \mapsto (g \rtimes a_{\tilde{s}}) \times \tilde{s},$$

where on the left the subscript A indicates that the semi-direct product is taken for the action of $\pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})$ on G via the projection $\pi_0(\tilde{S}_\phi^{+,[z]}) \to A_\pi^{[z]}$, while on the right the group $\pi_0(\tilde{S}_\phi^{+,[z]})$ acts trivially on $\tilde{G}$. The above isomorphism is equivariant for the natural embedding of G into both sides and hence induces an isomorphism between the rational forms over F determined by the element $z \in Z^1(u \to W, Z \to G)$.

According to Conjectures 5.12 and 7.1 the extensions of the representation $\pi \boxtimes \rho^\vee$ of $G_z(F) \times \pi_0(S_\phi^+)$ to a representation of $\tilde{G}_z(F)_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})$ are in natural bijection with the extensions of the representation $\rho \boxtimes \rho^\vee$ of

$$\pi_0(\mathrm{Cent}(\phi, \widehat{G} \times \pi_0(S_\phi^+))^+) = \pi_0(S_\phi^+) \times \pi_0(S_\phi^+)$$

to the group

$$\pi_0(\mathrm{Cent}(\phi, \widehat{G} \rtimes_A \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]}))^{+,[z]}).$$

To compute this group we use the isomorphism

$$\widehat{\bar{G}} \rtimes_A \tilde{S}_{\phi,\rho}^{+,[z]} \to \widehat{\bar{G}} \rtimes_c \tilde{S}_{\phi,\rho}^{+,[z]}, \qquad (g, \tilde{s}) \mapsto (gs^{-1}, \tilde{s}),$$

where the subcript c on the right indicates that we are taking the semi-direct product with respect to the natural conjugation action of $\tilde{S}_{\phi,\rho}^{+,[z]} \subset \widehat{\bar{G}} \rtimes A$ on $\widehat{\bar{G}}$, and $\tilde{s} = s \rtimes a$. This isomorphism restricts to an isomorphism $\widehat{\bar{G}} \times S_\phi^+ \to \widehat{\bar{G}} \rtimes_c S_\phi^+$.

We now apply $\mathrm{Cent}(\phi,-)^{+,[z]}$ to both sides of the inclusion $\hat{\hat{G}} \rtimes_c S_\phi^+ \to \hat{\hat{G}} \rtimes_c \tilde{S}_{\phi,\rho}^{+,[z]}$ and obtain the natural inclusion

$$S_\phi^+ \rtimes_c S_\phi^+ \to S_\phi^+ \rtimes_c \tilde{S}_{\phi,\rho}^{+,[z]},$$

which under the isomorphism $(s,\tilde{s}) \mapsto (s\tilde{s},\tilde{s})$ becomes the natural inclusion

$$S_\phi^+ \times S_\phi^+ \to \tilde{S}_{\phi,\rho}^{+,[z]} \times_{A_\rho^{[\phi],[z]}} \tilde{S}_{\phi,\rho}^{+,[z]}.$$

Tracing through all identifications we see that we are looking for a natural extension of the representation $\rho \boxtimes \rho^\vee$ of the source of this inclusion to a representation of the target. But Lemma 7.6 provides just such an extension.

We come now to the character identities. The right hand side of the character identities in Conjecture 5.12 is

$$\sum_{\tilde{\pi}\in\Pi_\phi(\tilde{G}_z)} \mathrm{tr}\, \tilde{\pi} \boxtimes \tilde{\rho}_{\tilde{\pi}}^\vee(\tilde{f} \times \tilde{s}^{-1}).$$

This is the character of the representation $\bigoplus_{\tilde{\pi}} \tilde{\pi} \boxtimes \tilde{\rho}_{\tilde{\pi}}^\vee$ evaluated at the function $\tilde{f} \otimes \delta_{\tilde{s}^{-1}}$. By the preceding discussion that representation is equal to

$$\bigoplus_{\pi\in\Pi_\phi(G_z)/A^{[\phi],[z]}} \mathrm{Ind}_{\tilde{G}_z(F)_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})}^{\tilde{G}_z(F)\times\pi_0(\tilde{S}_\phi^{+,[z]})} \tilde{\pi}^{\mathrm{can}}.$$

Let $a \in A$ be the image of $\tilde{s}$. If $a \notin A_\pi$ the character of the corresponding induced representation is zero at $\tilde{f}\otimes\delta_{\tilde{s}^{-1}}$. Therefore we may restrict the sum by the condition $a \in A_\pi^{[z]}$, equivalently $a\pi \cong \pi$. Applying the Frobenius character formula we obtain

$$\sum_{\substack{\pi\in\Pi_\phi(G_z)/A^{[\phi],[z]}\\ a\pi\cong\pi}} \mathrm{tr}\, \tilde{\pi}^{\mathrm{can}} \left(\sum_x (\tilde{f} \otimes \delta_{\tilde{s}^{-1}})^x \right),$$

where x runs over the coset space

$$(\tilde{G}_z(F) \times \pi_0(\tilde{S}_\phi^{+,[z]}))/(\tilde{G}_z(F)_\pi \times_{A_\pi^{[z]}} \pi_0(\tilde{S}_{\phi,\rho}^{+,[z]})) \cong A^{[z]} \times A^{[z],[\phi]}/A_\rho^{[z]}.$$

The compatibility of $\tilde{\pi}^{\mathrm{can}}$ with conjugation under A implies that the above sum becomes

$$\sum_{\substack{\pi\in\Pi_\phi(G_z)\\ a\pi\cong\pi}} \mathrm{tr}\, \tilde{\pi}^{\mathrm{can}} \left(\sum_{c\in A^{[z]}} \tilde{f}^c \otimes \delta_{\tilde{s}^{-1}} \right).$$

We conclude that

$$\sum_{\tilde{\pi}\in\Pi_\phi(\tilde{G}_z)} \mathrm{tr}\, \tilde{\pi} \boxtimes \tilde{\rho}^\vee(\tilde{f} \times \tilde{s}^{-1}) = \sum_{\substack{\pi\in\Pi_\phi(G_z)\\ a\pi\cong\pi}} \mathrm{tr}\, \tilde{\pi}^{\mathrm{can}} \left(\tilde{f}_0 \times \tilde{s}^{-1}\right),$$

where $\tilde{f}_0 = \sum_{c\in A^{[z]}} \tilde{f}^c$. Recalling from Lemma 4.6 that $\tilde{f}^{\mathfrak{z}} = \tilde{f}_0^{\mathfrak{z},KS}$ we see that the character identites in Conjectures 5.12 and 7.3 are equivalent. □

Remark 7.8. Proposition 7.5 reduces the proof of the endoscopic character identities to the case of a cyclic A. It does not completely reduce the internal structure of L-packets to the case of cyclic A, because in the case when A_π is not cyclic one still needs to show the existence of the extension $\tilde{\pi}^{\text{can}}$.

7.3 The cyclic case

{sub:comp_cyc}

In this subsection we revisit the classical setting where we have a connected reductive group equipped with an automorphism. We begin with a quasi-split connected reductive group G equipped with an automorphism θ fixing an F-pinning. We further assume θ is of finite order. Set $A = \langle\theta\rangle$ and $\tilde{G} = G \rtimes \langle\theta\rangle$. Let $z \in Z^1(u \to W, Z(G)^A \to G)$. The map $\tilde{G}_z(F) \to A$ is surjective if and only if the class $[z]$ is fixed by θ, which we assume from now on, for otherwise we can pass to a power of θ without changing $\tilde{G}_z(F)$. Fix an arbitrary $\tilde{\delta}_z \in \tilde{G}_z(F)$ mapping to θ and set $\theta_z = \text{Ad}(\tilde{\delta}_z)$. The twisted group we are interested in is G_z with automorphism θ_z.

Let $\phi : L_F \to {}^LG$ be such that its $\widehat{G}$-conjugacy class is fixed by θ. Then we have

$$1 \to \pi_0(S_\phi) \to \pi_0(\tilde{S}_\phi) \to A \to 1 \tag{7.1}$$ {eq:sphicyc}

and $\tilde{S}_\phi^{[z]} = \tilde{S}_\phi$. This isomorphism class of a representation $\pi \in \Pi_\phi(G_z)$ is θ_z-fixed if and only if the isomorphism class of the corresponding $\rho \in \text{Irr}(S_\phi^+, [z])$ is θ-fixed. Assuming that this is the case, there is a natural extension of the representation $\pi \boxtimes \rho^\vee$ of $G_z(F) \times \pi_0(S_\phi^+)$ to a representation of $\tilde{G}_z(F) \times_A \pi_0(\tilde{S}_\phi^+)$ given as follows: Since A is cyclic ρ extends to a representation $\tilde{\rho}$ of $\pi_0(\tilde{S}_\phi^+)$. By Conjecture 5.12 there is a corresponding extension $\tilde{\pi}$ of π to $\tilde{G}_z(F)$. Another extension of ρ is of the form $\tilde{\rho} \otimes \chi$ for some character χ of A. The representation of $\tilde{G}_z(F)$ corresponding to $\tilde{\rho} \otimes \chi$ is then $\tilde{\pi} \otimes \chi$. Therefore the representation $\tilde{\pi} \boxtimes \tilde{\rho}^\vee$ of $\tilde{G}_z(F) \times \pi_0(\tilde{S}_\phi^+)$, when pulled back to $\tilde{G}_z(F) \times_A \pi_0(\tilde{S}_\phi^+)$, is independent of the choice of $\tilde{\rho}$. This is $(\pi \boxtimes \rho^\vee)^{\text{can}}$ of Conjecture 7.3.

7.4 Passing from A to $A^{[z],[\phi]}$

{sub:comp_rest1}

Proposition 7.5 shows that, once $[\phi]$ and $[z]$ have been fixed and Conjecture A.1 has been assumed, Conjecture 5.12 for the group $G \rtimes A$ reduces to the same conjecture for the group $G \rtimes A^{[z],[\phi]}$. More explicitly, let $B = A^{[z],[\phi]}$ and write $G^A = G \rtimes A$ and $G^B = G \rtimes B$. Let $\pi^B \in \Pi_\phi(G_z^B)$. All members of $\text{Res}^{G_z^B(F)}_{G_z(F)} \pi^B$ belong to the packet $\Pi_\phi(G_z)$. An element of $G_z^A(F)$ that normalizes $G_z^B(F)$ and intertwines π^B must therefore lie in $G_z^B(F)$. Thus $\pi^A := \text{Ind}^{G_z^A(F)}_{G_z^B(F)} \pi^B$ is irreducible, and in this way one obtains a bijection $\Pi_\phi(G_z^B) \to \Pi_\phi(G_z^A)$. In the same way one obtain a bijection $\text{Irr}(\pi_0(\tilde{S}_\phi(G^B)^{+,[z]}), [z]) \to \text{Irr}(\pi_0(\tilde{S}_\phi(G^A)^{+,[z]}), [z])$.

7.5 Induction

{sub:ind}

Let $\tilde{G} = G \rtimes A$ be a quasi-split disconnected reductive group and $A \to B$ an embedding. Set $H = \mathrm{Ind}_A^B G$ and $\tilde{H} = H \rtimes B$. The purpose of this subsection is to show that Conjecture 7.3 for $G \rtimes A$ implies this same conjecture for $H \rtimes B$.

Let $a \in A$. An element $z_G \in Z^1(u \to W, Z(G)^A \to G)$ has a-invariant cohomology class if and only if there exists $g_a \in G$ such that

$$az_G(w) = g_a^{-1} z_G(w) \sigma_w(g_a). \tag{7.2}$$ {eq:ind1a}

This is equivalent to $g_a \rtimes a \in \tilde{G}_{z_G}(F)$. Assuming that, a representation π_G of $G_{z_G}(F)$ has an a-invariant isomorphism class if and only if there exists a vector space isomorphism $\tilde{\pi}_A(g_a \rtimes a) : V_{\pi_G} \to V_{\pi_G}$ satisfying

$$\pi_G(g_a \cdot a(g) \cdot g_a^{-1}) \circ \tilde{\pi}_G(g_a \rtimes a) = \tilde{\pi}_G(g_a \rtimes a) \circ \pi_G(g). \tag{7.3}$$ {eq:ind1b}

Note that $g_a \cdot a(g) \cdot g_a^{-1} = (g_a \rtimes a) \cdot g \cdot (g_a \rtimes a)^{-1}$, and further that the existence of $\tilde{\pi}_G(g_a \rtimes a)$ is independent of the choice of g_a, for any other choice will be of the form $g'_a g_a$ with $g'_a \in G_{z_G}(F)$ and we can take $\tilde{\pi}_G(g'_a g_a \rtimes a) = \pi_G(g'_a) \circ \tilde{\pi}_G(g_a \rtimes a)$.

We have $H = \{h : B \to G | h(ab) = ah(b)\}$ with pointwise multiplication. Let $z_H \in Z^1(u \to W, Z(H)^B \to H)$. Thus $z_H(w_1 w_2, b) = z_H(w_1, b) \cdot w_1 z_H(w_2, b)$ and $z_H(w, ab) = az_H(w, b)$. Let $b \in B$. The cohomology class of z_H is b-invariant if and only if there exists $h_b \in H$ satisfying the analog of Equation (7.2). Again this is equivalent to $\tilde{h} = h \rtimes b \in \tilde{H}$ lying in $\tilde{H}_{z_H}(F)$ and in terms of the function $h : B \to G$ means

$$z_H(w, b'b) = h_b^{-1}(b') z_H(w, b') \sigma_w(h_b(b')), \qquad \forall b' \in B. \tag{7.4}$$ {eq:ind1c}

A representation (π_H, V_{π_H}) of $H_{z_H}(F)$ can be represented as a collection of vector spaces $\{V_c | c \in A \setminus B\}$ and on each V_c a family of representations $\pi_H^{\dot{c}} : G_{z_H(-,\dot{c})}(F) \to \mathrm{Aut}_{\mathbb{C}}(V_c)$ indexed by $\dot{c} \in c$ satisfying the compatibility relation

$$\pi_H^{a\dot{c}}(ag) = \pi_H^{\dot{c}}(g)$$

for all $a \in A$ and $g \in G_{z_H(-,\dot{c})}(F)$. Then $V_{\pi_H} = \otimes_c V_c$ and $\pi_H(h) = \otimes_{c \in A \setminus B} \pi_H^c(h(c))$ and each factor is well-defined. We shall write $\pi_H = \boxtimes_c \pi_H^c$.

Assuming the existence of $h_b \rtimes b \in H_{z_H}(F)$, a representation π_H of $H_{z_H}(F)$ has a b-invariant isomorphism class if and only if there exists a vector space isomorphism $\tilde{\pi}_H(h_b \rtimes b) : V_{\pi_H} \to V_{\pi_H}$ satisfying the analog of Equation (7.3), which in terms of the data $\{V_c\}$ and $\{\pi_H^{\dot{c}}\}$ can be expressed as

$$\tilde{\pi}_H(h_b \rtimes b)(\otimes_c v_c) = \otimes_c \tilde{\pi}_H(h_b \rtimes b)_c(v_{cb}),$$

where

$$\tilde{\pi}_H(h_b \rtimes b)_c : V_{cb} \to V_c$$

is an isomorphism of vector spaces satisfying

$$\pi_H^{\dot{c}}(g) \circ \tilde{\pi}_H(h_b \rtimes b)_c = \tilde{\pi}_H(h_b \rtimes b)_c \circ \pi_H^{\dot{c}b}(h_b(\dot{c})^{-1} g h_b(\dot{c})) \tag{7.5}$$ {eq:ind1d}

for one, hence any, lift $\dot{c} \in B$ of $c \in A \setminus B$.

For $g \in G$ and $b \in B$ we define $g^{\delta_b} \in H$ to be the element supported on Ab and sending $ab \in Ab$ to $a(g) \in G$. Given a section $s : A \setminus B \to B$ of the natural projection (which we may view as a map $B \to B$ invariant under left multiplication by A) we define a map $r : B \to A$ by $b = r(b)s(b)$. We have $r(ab) = ar(b)$.

{lem:ind1}

Lemma 7.9. *1. Given $z_H \in Z^1(u \to W, Z(H)^B \to H)$ define $z_G(w) = z_H(w, 1)$. Then $[z_H] \mapsto [z_G]$ establishes a bijection between $H^1(u \to W, Z(H)^B \to H)^B$ and $H^1(u \to W, Z(G)^A \to G)^A$.*

2. Assume $z_G(w) = z_H(w, 1)$. Given a representation π_H of $H_{z_H}(F)$ define $\pi_G(g) = \pi_H^1(g^{\delta_1})$. Then $[\pi_H] \mapsto [\pi_G]$ establishes a bijection between the set of B-fixed isomorphism classes of irreducible representations of $H_{z_H}(F)$ and the set of A-fixed isomorphism classes of irreducible representations of $G_{z_G}(F)$.

3. Fix a section $s : A \setminus B \to B$. Given $z_G \in Z^1(u \to W, Z(G)^A \to G)$ define $z_H \in Z^1(u \to W, Z(H)^B \to H)$ by $z_H(w, as(c)) = az_G(w)$. Given a representation π_G of $G_{z_G}(F)$ define a representation of $H_{z_H}(F)$ by $\boxtimes_c \pi_H^c$, where $\pi_H^{s(c)} = \pi_G$. These assignments are inverses of the above bijections.

Proof. Let $z_H \in Z^1(u \to W, Z(H)^B \to H)$ and $\pi_H \in \mathrm{Irr}(H_{z_H}(F))$ have B-fixed classes. Then Equation (7.4) for $b' = 1$ and $b = a \in A$ shows that the class of $z_G(w) = z_H(w, 1)$ is a-fixed, while Equation (7.5) with $\dot{c} = 1$ and $b = a \in A$ shows that the class of $\pi_G(g) = \pi_H^1(g^{\delta_1})$ is a-fixed. It is clear that the classes of z_G and π_G depend only on those of z_H and π_H.

Consider conversely $z_G \in Z^1(u \to W, Z(G)^A \to G)$ and $\pi_G \in \mathrm{Irr}(G_{z_G}(F))$ whose classes are A-fixed and let g_a and $\tilde{\pi}_G(g_a \rtimes a)$ be chosen to satisfy Equations (7.2) and (7.3). Fix a section $s : A \setminus B \to B$ and let $r : B \to A$ be defined by $b = r(b)s(b)$ for all $b \in B$. Set $h_b(as(c)) = a(g_{r(s(c)b)})$. Then $h_b \in H$ and an easy calculation shows that Equation (7.2) implies Equation (7.4) for $z_H(w, as(c)) := az_G(w)$, and further that if $h \in H_{z_H}(F)$ then $h(s(c)) \in G_{z_G}(F)$ for all $c \in A \setminus B$. This allows us to define a representation π_H of $H_{z_H}(F)$ acting on the vector space $V_{\pi_G}^{\otimes_c}$ by $\pi_H(h) = \otimes_c \pi_G(h(s(c)))$. In other words, the representation π_H is given by the constant collection of vector spaces $\{V_c = V_{\pi_G} | c \in A \setminus B\}$ and for each $c \in A \setminus B$ we have $\pi_H^{as(c)}(g) = \pi_G(a^{-1}(g))$. Define $\tilde{\pi}_H(h_b \rtimes b)_c : V_{cb} \to V_c$ to be given by $\tilde{\pi}_G(g_a \rtimes a) : V_{\pi_G} \to V_{\pi_G}$ for $a = r(s(c)b)$. Then Equation (7.5) for $\dot{c} = s(c)$ follows from Equation (7.3). It is clear that the classes of z_H and π_H depend only on those of z_G and π_G.

We have thus established the desired maps in both directions and must now check that they are mutually inverse. Starting with z_G and π_G and constructing z_H and π_H it is immediate that $z_H(w, 1) = z_G(w)$ and $\pi_H^1(g^{\delta_1}) = \pi_G(g)$. Conversely start with z_H and π_H and define $z_G(w) = z_H(w, 1)$ and $\pi_G(g) = \pi_H^1(g^{\delta_1})$. Let now $z_H^0(w, as(c)) = az_G(w)$ and $\pi_H^0(h) = \otimes_c \pi_G(h(s(c)))$. We need to show that the classes of z_H and z_H^0 are equal, and the classes of π_H and π_H^0 are equal.

For z_H and z_H^0 we need to show the existence of $h \in H$ such that for all $a \in A$ and $c \in A \setminus B$ we have

$$z_H^0(w, as(c)) = h(as(c))^{-1} z_H(w, as(c)) \sigma(h(as(c))),$$

which due to the A-equivariance of all terms and the definition of z_H^0 reduces to

$$z_H(w, 1) = h(s(c))^{-1} z_H(w, s(c)) \sigma_w(h(s(c))),$$

which follows from Equation (7.4) with $h = h_b$, $b = s(c)^{-1}$ and $b' = s(c)$. Thus the element h we are looking for is given by $h(as(c)) := ah_{s(c)^{-1}}(s(c))$.

Before we can compare the classes of π_H and π_H^0 we note that the former is a representation of $H_{z_H}(F)$, while the latter is a representation of $H_{z_H^0}(F)$. We must therefore precompose π_H with the isomorphism $\mathrm{Ad}(h) : H_{z_H^0}(F) \to H_{z_H}(F)$. Thus we need to show the existence of a $(\pi_H \circ \mathrm{Ad}(h), \pi_H^0)$-equivariant vector space isomorphism $V_{\pi_H} \to V_{\pi_H^0}$. This reduces to finding for each $c \in A \setminus B$ a vector space isomorphism $V_c \to V_1$ translating the action of $G_{z_G}(F)$ on V_c given by $\pi_H^{s(c)} \circ \mathrm{Ad}(h_{s(c)^{-1}}(s(c)))$ to the action of on V_1 given by π_H^1. According to Equation (7.5) such an isomorphism is given by $\tilde{\pi}_H(h_b \rtimes b)_c^{-1}$ for $b = s(c)^{-1}$. □

Lemma 7.10. *Under the bijection $\pi_G \leftrightarrow \pi_H$ of Lemma 7.9 the element of $H^2(B, \mathbb{C}^\times)$ corresponding to π_H is the corestriction of the element of $H^2(A, \mathbb{C}^\times)$ corresponding to π_G.* {lem:ind2}

More precisely, let $z_G \in Z^1(u \to W, Z(G)^A \to G)$ and $\pi_G \in Irr(G_{z_G}(F))$ have A-fixed classes. For each $a \in A$ fix $g_a \in G$ and $\tilde{\pi}_G(g_a \rtimes a)$ satisfying Equations (7.2) *and* (7.3)*, so that we have the element*

$$\alpha(a_1, a_2) = \tilde{\pi}_G(g_{a_1} \rtimes a_1) \circ \tilde{\pi}_G(g_{a_2} \rtimes a_2) \circ \tilde{\pi}_G(g_{a_1} \rtimes a_1 \cdot g_{a_2} \rtimes a_2)^{-1}$$

of $Z^2(A, \mathbb{C}^\times)$ representing the class associated to π_G, where the third term is defined via the rule $\tilde{\pi}_G(gg_a \rtimes a) = \pi_G(g)\tilde{\pi}_G(g_a \rtimes a)$ for $g \in G_{z_G}(F)$. Define $z_H(w, as(c)) = az_G(w)$. Define the representation π_H of $H_{z_H}(F)$ as $\pi_H = \boxtimes_c \pi_H^c$, $\pi_H^{s(c)} = \pi_G$. For each $b \in B$ define the element $h_b \in H$ by $h_b(as(c)) = a(g_{r(s(c)b)})$ and the isomorphism $\tilde{\pi}_H(h_b \rtimes b) : \otimes_c V_{\pi_G} \to \otimes_c V_{\pi_G}$ by $\tilde{\pi}_H(h_b \rtimes b)(\otimes_c v_c) = \otimes_c \tilde{\pi}_G(g_{r(s(c)b)} \rtimes r(s(c)b))(v_{cb})$. Then h_b and $\tilde{\pi}_H(h_b \rtimes b)$ satisfy Equations (7.4) *and* (7.5) *and the associated element*

$$\beta(b_1, b_2) = \tilde{\pi}_H(h_{b_1} \rtimes b_1) \circ \tilde{\pi}_H(h_{b_2} \rtimes b_2) \circ \tilde{\pi}_H(h_{b_1} \rtimes b_1 \cdot h_{b_2} \rtimes b_2)^{-1}$$

of $Z^2(B, \mathbb{C}^\times)$ is obtained from α by applying the cochain formula for corestriction with respect to the section s.

Proof. That Equations (7.4) and (7.5) are satisfied was already discussed in the proof of Lemma 7.9. It remains to prove the corestriction claim, which is the following identity:

$$\beta(b_1, b_2) = \prod_c \alpha(r(s(c)b_1), r(s(cb_1)b_2)).$$

The (scalar) endomorphism

$$\tilde{\pi}_H(h_{b_1} \rtimes b_1) \circ \tilde{\pi}_H(g_{b_2} \rtimes b_2) \circ \tilde{\pi}_H(h_{b_1} \rtimes b_1 \cdot h_{b_2} \rtimes b_2)^{-1}$$

of the vector space $\otimes_c V_{\pi_G}$ is by definition a tensor product of endomorphisms of V_{π_G} and it is enough to show that the endomorphism of the tensor factor indexed by c is given by multiplication by the scalar $\alpha(r(s(c)b_1), r(s(cb_1)b_2))$. By definition this endomorphism is given by

$$\tilde{\pi}_H(h_{b_1} \rtimes b_1)_c \circ \tilde{\pi}_H(h_{b_2} \rtimes b_2)_{cb_1} \circ \tilde{\pi}_H(h_{b_1} \rtimes b_1 \cdot h_{b_2} \rtimes b_2)_c^{-1},$$

where the subscript notation is as in the proof of Lemma 7.9. We compute

$$\begin{aligned}
& \tilde{\pi}_H(h_{b_1} \rtimes b_1 \cdot h_{b_2} \rtimes b_2)_c \\
= \quad & \pi_H(h_{b_1} \cdot b_1 h_{b_2} \cdot h_{b_1 b_2}^{-1}) \circ \tilde{\pi}_H(h_{b_1 b_2} \rtimes b_1 b_2)_c \\
= \quad & \pi_G(h_{b_1}(s(c)) \cdot h_{b_2}(s(c)b_1) h_{b_1 b_2}(s(c))^{-1}) \circ \tilde{\pi}_G(g_{r(s(c)b_1 b_2)} \rtimes r(s(c)b_1 b_2)) \\
= \quad & \tilde{\pi}_G(g_{r(s(c)b_1)} \cdot r(s(c)b_1) g_{r(s(cb_1)b_2)} \cdot g^{-1}_{r(s(c)b_1 b_2)} \cdot g_{r(s(c)b_1 b_2)} \rtimes r(s(c)b_1 b_2)) \\
= \quad & \tilde{\pi}_G(g_{r(s(c)b_1)} \rtimes r(s(c)b_1) \cdot g_{r(s(cb_1)b_2)} \rtimes r(s(cb_1)b_2)).
\end{aligned}$$

With this we see that the endomorphism of the tensor factor indexed by c is given by

$$\begin{aligned}
\tilde{\pi}_G(g_{r(s(c)b_1)} \rtimes r(s(c)b_1)) \circ \tilde{\pi}_G(g_{r(s(cb_1)b_2)} \rtimes r(s(cb_1)b_2)) \\
\circ\, \tilde{\pi}_G(g_{r(s(c)b_1)} \rtimes r(s(c)b_1) \cdot g_{r(s(cb_1)b_2)} \rtimes r(s(cb_1)b_2)),
\end{aligned}$$

which is precisely $\alpha(r(s(c)b_1), r(s(cb_1)b_2))$. □

We consider the group homomorphism

$$\mathrm{ev}_1 : H \to G, \qquad h \mapsto h(1).$$

It is A-equivariant, hence extends to a group homomorphism

$$\mathrm{ev}_1 : H \rtimes A \to G \rtimes A, \qquad h \rtimes a \mapsto \mathrm{ev}_1(h) \rtimes a.$$

It also respects rational structures under the convention $z_G(w) = z_H(w, 1)$ that has been used so far.

More generally we consider $b \in B$ and a section $l : A\backslash B/\langle b\rangle \to B$. For every $d \in A \setminus B/\langle b\rangle$ let n_d be the size of the orbit of the element $Al(d)$ of $A \setminus B$ for the action of b on $A\backslash B$ by right multiplication. Equivalently, n_d is the smallest non-negative number n satisfying $b^n \in l(d)^{-1}Al(d)$. Write $A_d := l(d)^{-1}Al(d) \subset B$ so that $b^{n_d} \in A_d$, and write $a_d = l(d)b^{n_d}l(d)^{-1} \in A$. We obtain a section $s : A \setminus B \to B$ by

$$s(Al(d)b^i) = l(d)b^i, \qquad i = 0, \ldots, n_d - 1. \tag{7.6}$$ {eq:sec1}

The group homomorphism

$$\mathrm{ev}_{l(d)} : H \to G, \qquad h \mapsto h(l(d))$$

satisfies $\mathrm{ev}_{l(d)}(l(d)^{-1}al(d)h) = a\mathrm{ev}_{l(d)}(h)$ for $a \in A$ and $h \in H$ and therefore extends to a group homomorphism

$$\mathrm{ev}_{l(d)} : H \rtimes A_d \to G \rtimes A, \qquad h \rtimes l(d)^{-1}al(d) \mapsto h(l(d)) \rtimes a,$$

which is defined over F under the assumption $z_H(w, l(d)) = z_G(w)$, which is implied by the assumption $z_H(w, s(c)) = z_G(w)$ for all $c \in A \setminus B$.

{lem:indprod0}

Lemma 7.11. *Let $z_G^d \in Z^1(u \to W, Z(G)^A \to G)$. Define $z_H \in Z^1(u \to W, Z(H)^B \to B)$ by $z_H(w, al(d)b^i) = az_G^d(w)$. The map*

$$H \to \prod_d \prod_{i=0}^{n_d-1} G, \qquad h \mapsto \prod_d \prod_{i=0}^{n_d-1} h(l(d)b^i)$$

is an isomorphism of algebraic groups. It respects the quasi-split rational structures on both sides, as well as their twists by z_H and $(z_G^d)_d$ respectively. It translates the action by b to the action by $(\Theta_d)_d$, where $\Theta_d(g_{d,0}, \dots, g_{d,n_d-1}) = (g_{d,1}, \dots, g_{n_d-1}, a_d(g_0))$.

Proof. This is an immediate computation. □

Note that $h(l(d)b^i) = \mathrm{ev}_{l(d)}(b^ih) = \mathrm{ev}_{l(d)}(\mathrm{Ad}(1 \rtimes b)^ih)$. Since the action of b on H, as well as the action of a_d on G, need not respect the rational structures given by z_H and z_G^d, respectively, the following slight variation of the above isomorphism will also be useful.

{lem:indprod}

Lemma 7.12. *Let $\tilde{h} \in [H \rtimes b]_{z_H}(F)$. The map*

$$H \to \prod_d \prod_{i=0}^{n_d-1} G, \qquad h \mapsto \prod_d \prod_{i=0}^{n_d-1} ev_{l(d)}(Ad(\tilde{h})^ih)$$

is an isomorphism of algebraic groups that respects the twists of the quasi-split rational structures by z_H and $(z_G^d)_d$, respectively. It translates the action of conjugation by $\tilde{h}$ to the action sending $(g_{d,0}, \dots, g_{d,n_d-1})_d$ to $(g_{d,1}, \dots, g_{d,n_d-1}, Ad(\tilde{g}_d)g_0)_d$, where $\tilde{g}_d = ev_{l(d)}(\tilde{h}^{n_d}) \in [G \rtimes a_d]_{z_G}(F)$.

Proof. This is an immediate computation. □

For a moment we consider the following situation that encapsulates each factor in the first product of above lemma.

{lem:basictwist}

Lemma 7.13. *Let J be a locally profinite group with an automorphism θ and consider the locally profinite group $I = J \times J \times \cdots \times J$ with the automorphism $\Theta(j_0, \dots, j_{n-1}) = (j_1, \dots, j_{n-1}, \theta(j_0))$. Consider the maps*

$$m, p_0 : I \to J, \quad m(j_0, \dots, j_{n-1}) = j_0 \dots j_{n-1}, \quad p_0(j_0, \dots, j_{n-1}) = j_0$$

as well as

$$\Delta, i_0 : J \to I, \quad \Delta(j) = (j, \dots, j), \quad i_0(j) = (j, 1, \dots, 1).$$

1. *We have* $m(g_I^{-1} \cdot \delta_I \cdot \Theta(g_I)) = p_0(g_I)^{-1} \cdot m(\delta_I) \cdot \theta(p_0(g_I))$ *for* $g_I, \delta_I \in I$.

2. *The map* m *induces a bijection from the set of* Θ*-twisted conjugacy classes in* I *to the set of* θ*-twisted conjugacy classes in* J *with inverse given by* i_0.

3. *The map* p_0 *induces an isomorphism of groups* $Cent_\Theta(\delta_I, I) \to Cent_\theta(m(\delta_I), J)$ *whose inverse sends* s *to* $Ad(g_I)\Delta(s)$, *where* $g_I = (g_0, \dots, g_{n-1})$ *and* $g_i = (\delta_0 \dots \delta_{i-1})^{-1}$.

4. *If* $f_0, \dots, f_{n-1} \in \mathcal{C}_c^\infty(J)$, $f_I = f_0 \otimes \dots \otimes f_{n-1}$, *and* $\delta_I \in I$, *then*

$$TO_{\delta_I}^{I,\Theta}(f_I) = TO_{m(\delta_I)}^{J,\theta}(f_0 * f_1 * \dots * f_{n-1}),$$

where the convolution $f_0 * \dots * f_{n-1} \in \mathcal{C}_c^\infty(J)$ *is defined by* $f_0 * \dots * f_{n-1}(x) = \int f_0(h_1) f_1(h_1^{-1} h_2) \dots f_{n-2}(h_{n-2}^{-1} h_{n-1}) f_{n-1}(h_{n-1}^{-1} x) dh_1 \dots dh_{n-1}$.

5. *Let* π *be an admissible representation of* J *and let* $\tilde{\pi} : \pi \circ \theta^{-1} \to \pi$ *be an isomorphism. Then* $\pi_I = \pi \boxtimes \dots \boxtimes \pi$ *is an admissible representation of* I *and* $\tilde{\pi}_I(v_0 \otimes \dots \otimes v_{n-1}) = v_1 \otimes \dots \otimes v_{n-1} \otimes \tilde{\pi}(v_0)$ *is an isomorphism* $\pi_I \circ \Theta^{-1} \to \pi_I$. *We have*

$$tr(\pi_I(f_I) \circ \tilde{\pi}_I) = tr(\pi(f_0 * \dots * f_{n-1}) \circ \tilde{\pi}).$$

Proof. The first point is an immediate computation. It follows that m induces a map between the sets of twisted conjugacy classes, and p_0 induces a map between the twisted centralizers. The fact that i_0 respects twisted conjugacy follows from $i_0(g^{-1}\delta\theta(g)) = g_I^{-1} i_0(\delta) \Theta(g_I)$ for $g_I = (g, \theta(g), \dots, \theta(g))$. The fact that i_0 is inverse to m as maps between twisted conjugacy classes follows from the trivial relation $m(i_0(\delta)) = \delta$ and the relation $i_0(m(\delta_I)) = g_I^{-1} \delta_I \Theta(g_I)$ for $\delta_I = (\delta_0, \dots, \delta_{n-1})$ and $g_I = (g_0, \dots, g_{n-1})$ with $g_0 = 1$ and $g_i = \delta_i \dots \delta_{n-1}$ for $i > 0$. The fact that p_0 has the given inverse as maps between twisted centralizers is immediate.

For the equality of twisted orbital integrals we take $\delta_I = (\delta_0, \dots, \delta_{n-1})$ and write out the left-hand side as

$$\int_{\mathrm{Cent}_\Theta(\delta_I, I) \backslash I} f_0(g_0^{-1} \delta_0 g_1) f_1(g_1^{-1} \delta_1 g_2) \dots f_{n-1}(g_{n-1}^{-1} \delta_{n-1} \theta(g_0)) dg_0 \dots dg_{n-1},$$

where the integration variable is $g_I = (g_0, \dots, g_{n-1})$, and use the substitution $h_0 = g_0$, $h_i = g_0^{-1} \delta_0 \dots \delta_{i-1} g_i$ for $i > 0$.

The fact that $\tilde{\pi}_I$ is an isomorphism $\pi_I \circ \Theta^{-1} \to \pi_I$ is immediate. To verify the equality of traces write $\phi_i = \pi(f_i) \in \mathrm{End}_{\mathbb{C}}(V_\pi)$. Then $\pi_I(f_I)$ is the operator $\phi_0 \otimes \dots \otimes \phi_{n-1}$ of $V_{\pi_I} = V_\pi \otimes \dots \otimes V_\pi$, while $\pi(f_0 * \dots * f_{n-1})$ is the operator $\phi_0 \circ \dots \circ \phi_{n-1}$ of V_π. Therefore the claimed equality is

$$\mathrm{tr}(\phi_0 \otimes \dots \otimes \phi_{n-1} \circ \tilde{\pi}_I | V_{\pi_I}) = \mathrm{tr}(\phi_0 \circ \dots \circ \phi_{n-1} \circ \tilde{\pi} | V_\pi).$$

Both sides are continuous and n-linear in the ϕ_i, which allows us to reduce the proof first to the case that ϕ_i has finite rank, and then to the case that it has rank 1, i.e. $\phi_i = \lambda_i \otimes w_i$ for $\lambda_i \in V_\pi^*$ and $w_i \in V_\pi$. The operator on the left has rank 1 and is given by $((\lambda_0 \otimes \cdots \otimes \lambda_{n-1}) \circ \tilde{\pi}_I) \otimes (w_0 \otimes \cdots \otimes w_{n-1})$. Its trace is therefore $\lambda_0(w_1)\lambda_1(w_2)\ldots\lambda_{n-1}(\tilde{\pi}(w_0))$. The operator on the right also has rank 1 and is given by $(\lambda_0(w_1)\lambda_1(w_2)\ldots\lambda_{n-2}(w_{n-1}))\cdot w_0 \otimes \lambda_{n-1}\circ\tilde{\pi}$. Its trace is therefore given by $\lambda_0(w_1)\ldots\lambda_{n-2}(w_{n-1})\lambda_{n-1}(\tilde{\pi}(w_0))$. Thus the two traces are equal. □

We now return the group $H \rtimes B$. We fix $\tilde{h} \in [H \rtimes b]_{z_H}(F)$ and let $\tilde{g}_d = \mathrm{ev}_{l(d)}(\tilde{h}^{n_d}) \in [G \rtimes a_d]_{z_G}(F)$. Consider given functions $f_{d,i} \in \mathcal{C}_c^\infty(G_{z_G}(F))$ for $d \in A \setminus B/\langle b\rangle$ and $i = 0, \ldots, n_d - 1$. The tensor product $\otimes_{d,i} f_{d,i}$ becomes, via the isomorphism of Lemma 7.12, a function $f_H \in \mathcal{C}_c^\infty(H_{z_H}(F))$. Write $\tilde{f}_H = R_{\tilde{h}}^{-1} f_H \in \mathcal{C}_c^\infty([H \rtimes b]_{z_H}(F))$ for the function $\tilde{f}_H(h \cdot \tilde{h}) = f_H(h)$. Analogously we obtain for each d the function $R_{\tilde{g}_d}^{-1}(f_{d,0} * \cdots * f_{d,n_d-1}) \in [G \rtimes a_d]_{z_G}(F)$.

Fix a collection $(\pi_d)_{d\in A\setminus B/\langle b\rangle}$ of representations of $G_{z_G}(F)$ and isomorphisms $\tilde{\pi}_d : \pi_d \circ \mathrm{Ad}(\tilde{g}_d)^{-1} \to \pi_d$. Via the isomorphism of Lemma 7.12 we can transport $\boxtimes_d \pi_d^{\boxtimes n_d}$ to a representation π_H of $H_{z_H}(F)$ and $\tilde{\pi}_H(\otimes_d(v_{d,0} \otimes \cdots \otimes v_{d,n_d-1})) = \otimes_d(v_{d,1} \otimes \cdots \otimes v_{d,n_d-1} \otimes \tilde{\pi}_d(v_{d,0}))$ to an isomorphism $\pi_H \circ \mathrm{Ad}(\tilde{h})^{-1} \to \pi_H$.

{cor:indorb}

Corollary 7.14. *1. The map $[H \rtimes b]_{z_H}(F) \to \prod_d [G \rtimes a_d]_{z_G}(F)$ sending $\tilde{h}'$ to $(\tilde{g}'_d)_d$ defined as $\tilde{g}'_d = ev_{l(d)}(\tilde{h}'^{n_d})$, induces a bijection between the set of $H_{z_H}(F)$-conjugacy classes in $[H \rtimes b]_{z_H}(F)$ and the set of $G_{z_G}(F)^d$-conjugacy classes in $\prod_d [G \rtimes a_d]_{z_G}(F)$.*

2. If $\tilde{h}' \in [H \rtimes b]_{z_H}(F)$ maps to $(\tilde{g}'_d)_d \in \prod_d [G \rtimes a_d]_{z_G}(F)$ then

$$O^H_{\tilde{h}'}(R_{\tilde{h}}^{-1} f_H) = \prod_d O^{G^d}_{\tilde{g}'_d}(R^{-1}_{\tilde{g}_d}(f_{d,0} * \cdots * f_{d,n_d-1})).$$

3. We have

$$tr(\pi_H(f_H) \circ \tilde{\pi}_H) = \prod_d tr(\pi_d(f_{d,0} * \cdots * f_{d,n_d-1}) \circ \tilde{\pi}_d).$$

Proof. We have the bijection $H_{z_H}(F) \to [H \rtimes b]_{z_H}(F)$ sending h to $h \cdot \tilde{h}$. It translates the conjugation action of $H_{z_H}(F)$ on $[H \rtimes b]_{z_H}(F)$ to the twisted conjugation action of $H_{z_H}(F)$ on itself, with respect to the automorphism $\mathrm{Ad}(\tilde{h})$. The isomorphism of Lemma 7.12 identifies the group $H_{z_H}(F)$ with the group $\prod_d \prod_i G_{z_G}(F)$ and the automorphism $\mathrm{Ad}(\tilde{h})$ with the automorphism sending $(g_{d,0}, \ldots, g_{d,n_d-1})$ to $(g_{d,1}, \ldots, g_{d,n_d-1}, \mathrm{Ad}(\tilde{g}_d)g_{d,0})$. According to Lemma 7.13, the map sending $\tilde{h}' = h' \cdot \tilde{h} \in [H \rtimes b]_{z_H}(F)$ to

$$(\mathrm{ev}_{l(d)}(h' \cdot (\tilde{h}h'\tilde{h}^{-1}) \ldots (\tilde{h}^{n_d-1}h'\tilde{h}^{1-n_d})))_d = (\mathrm{ev}_{l(d)}(\tilde{h}'^{n_d}) \cdot \mathrm{ev}_{l(d)}(\tilde{h}^{n_d})^{-1}$$

is a bijection from the set of $H_{z_H}(F)$-conjugacy classes in $[H \rtimes b]_{z_H}(F)$ to the set of $G_{z_G}(F)^d$-twisted conjugacy classes in $G_{z_G}(F)^d$ with respect to the automorphism $(\mathrm{Ad}(\tilde{g}_d))_d$. Composing this bijection with the bijection $G_{z_G}(F) \to [G \rtimes a_d]_{z_G}(F)$ sending g' to $g'\tilde{g}_d$ we obtain the first claim. With this translation set up, the other two claims follow readily from Lemma 7.13. □

We now turn to the dual side. We can take $\widehat{H} = \mathrm{Ind}_A^B \widehat{G}$. More precisely, if T is a torus with A-action, we have the identification $\mathrm{Ind}_A^B X_*(T) = X_*(\mathrm{Ind}_A^B T)$ sending an element $\lambda^B : B \times \mathbb{G}_m \to T$ of $\mathrm{Ind}_A^B X_*(T)$ to the element $x \mapsto \lambda^B(-, x)$ of $X_*(\mathrm{Ind}_A^B T)$. The pairing

$$\langle \chi^B, \lambda^B \rangle = \prod_{c \in A \backslash B} \langle \chi^B(c), \lambda^b(c) \rangle \tag{7.7}$$

{eq:ind_duality}

between $\mathrm{Ind}_A^B X^*(T)$ and $\mathrm{Ind}_A^B X_*(T)$ is perfect and equivariant for Γ and B and identifies $\mathrm{Ind}_A^B X^*(T)$ with $X^*(\mathrm{Ind}_A^B T)$ as Γ-modules with B-action. If (T, C) and $(\widehat{T}, \widehat{C})$ are Γ-invariant Borel pairs for G and $\widehat{G}$ respectively, then $(\mathrm{Ind}_A^B T, \mathrm{Ind}_A^B C)$ and $(\mathrm{Ind}_A^B T, \mathrm{Ind}_A^B C)$ are such pairs for $\mathrm{Ind}_A^B G$ and $\mathrm{Ind}_A^B \widehat{G}$, respectively. The duality between $X_*(T)$ and $X_*(\widehat{T})$ that realizes the duality between G and $\widehat{G}$ induces, via the above pairing, a duality between $\mathrm{Ind}_A^B X_*(T) = X_*(\mathrm{Ind}_A^B T)$ and $\mathrm{Ind}_A^B X_*(\widehat{T}) = X_*(\mathrm{Ind}_A^B X_*(\widehat{T}))$, and therefore realizes the duality between $\mathrm{Ind}_A^B G$ and $\mathrm{Ind}_A^B \widehat{G}$.

Let $a \in A$. A Langlands parameter $\phi_G : L_F \to {}^L G$, which we represent as $\phi_G(x) = \phi_{G,0}(x) \rtimes x$ with $\phi_{G,0} : L_G \to \widehat{G}$, has a-invariant $\widehat{G}$-conjugacy class if and only if there exists an element $\breve{g}_a \in \widehat{G}$ satisfying

$$a\phi_{G,0}(x) = \breve{g}_a^{-1} \phi_{G,0}(x) \sigma_x(\breve{g}_a). \tag{7.8}$$

{eq:ind1ad}

This is equivalent to $\breve{g}_a \rtimes a \in \tilde{S}_{\phi_G}$. Assuming this, a representation (ρ_G, V_G) of S_{ϕ_G} has an a-invariant isomorphism class if and only if there is a vector space isomorphism $\tilde{\rho}_G(\breve{g}_a \rtimes a) : V_{\rho_G} \to V_{\rho_G}$ satisfying

$$\rho_G(\breve{g}_a \cdot a(\breve{g}) \cdot \breve{g}_a^{-1}) \circ \tilde{\rho}_G(\breve{g}_a \rtimes a) = \tilde{\rho}_G(\breve{g}_a \rtimes a) \circ \rho_G(\breve{g}). \tag{7.9}$$

{eq:ind1bd}

Let $b \in B$. A Langlands parameter $\phi_H : L_F \to {}^L H$, which we again represent as $\phi_H(x) = \phi_{H,0}(x) \rtimes x$ with $\phi_{H,0} : L_F \to \widehat{H}$, has a b-invariant $\widehat{H}$-conjugacy class if and only if there exists an element $\check{h}_b \in \widehat{H}$ satisfying

$$\phi_{H,0}(x, b'b) = \check{h}_b^{-1}(b') \phi_{H,0}(x, b') \sigma_x(\check{h}_b(b')), \qquad \forall b' \in B. \tag{7.10}$$

{eq:ind1cd}

Again this means $\check{h}_b \rtimes b \in \tilde{S}_{\phi_H}$. Assuming this, a representation (ρ_H, V_{ρ_H}) of S_{ϕ_H}, represented again by the collection $\{V_c | c \in A \backslash B\}$ of vector spaces and the collection $\rho_H^{\dot{c}}$ of representations $\rho_H^{\dot{c}} : S_{\phi_H(-,\dot{c})} \to \mathrm{Aut}(V_c)$ for all $\dot{c} \in c$, subject to $\rho_H^{a\dot{c}}(as) = \rho_H^{\dot{c}}(s)$, has a b-invariant isomorphism class if and only if there exists a vector space isomorphism $\tilde{\rho}_H(\check{h}_b \rtimes b) : V_{\rho_H} \to V_{\rho_H}$ satisfying

$$\tilde{\rho}_H(\check{h}_b \rtimes b)(\otimes_c v_c) = \otimes_c \tilde{\rho}_H(\check{h}_b \rtimes b)_c(v_{cb}),$$

where

$$\tilde{\rho}_H(\check{h}_b \rtimes b)_c : V_{cb} \to V_c$$

is an isomorphism of vector spaces satisfying

$$\rho_H^{\dot{c}}(\breve{g}) \circ \tilde{\rho}_H(\check{h}_b \rtimes b)_c = \tilde{\rho}_H(\check{h}_b \rtimes b)_c \circ \rho_H^{\dot{c}b}(\check{h}_b(\dot{c})^{-1} \breve{g} \check{h}_b(\dot{c})) \tag{7.11}$$

{eq:ind1dd}

for one, hence any, lift $\dot{c} \in B$ of $c \in A \setminus B$.

{lem:ind1d}

Lemma 7.15. *1. The assignment $\phi_{G,0}(x) = \phi_{H,0}(x,1)$ establishes a bijection between the A-invariant $\widehat{G}$-conjugacy classes of parameters $\phi_G(x) = \phi_{G,0}(x) \rtimes x$ and the B-invariant $\widehat{H}$-conjugacy classes of parameter $\phi_H(x) = \phi_{H,0}(x) \rtimes x$.*

2. *Assume $\phi_{G,0}(x) = \phi_{H,0}(x,1)$. The assignment $\rho_G(\check{g}) = \rho_H^1(\check{g}^{\delta_1})$ establishes a bijection between the set of B-fixed isomorphism classes of irreducible representations of S_{ϕ_H} and the set of A-fixed isomorphism classes of irreducible representations of S_{ϕ_G}.*

3. *Fix a section $s : A \setminus B \to B$. The assignments $\phi_{H,0}(x, as(c)) = a\phi_{G,0}(x)$ and $\rho_H(\check{h}) = \otimes_c \rho_G(\check{h}(s(c))) \in End_{\mathbb{C}}(V_{\rho_G}^{\otimes_c})$ are inverses of the above bijections.*

Proof. The proof is the same as for Lemma 7.9 so we will not repeat it. □

{lem:ind2d}

Lemma 7.16. *Under the bijection $\rho_G \leftrightarrow \rho_H$ of Lemma 7.15 the element of $H^2(B, \mathbb{C}^\times)$ corresponding to ρ_H is the corestriction of the element of $H^2(A, \mathbb{C}^\times)$ corresponding to ρ_G.*

More precisely, let ϕ_G and $\rho_G \in Irr(S_{\phi_G})$ have A-fixed classes. For each $a \in A$ fix $\check{g}_a \in \widehat{G}$ and $\tilde{\rho}_G(\check{g}_a \rtimes a)$ satisfying Equations (7.8) *and* (7.9)*, so that we have the element*

$$\alpha(a_1, a_2) = \tilde{\rho}_G(\check{g}_{a_1} \rtimes a_1) \circ \tilde{\rho}_G(\check{g}_{a_2} \rtimes a_2) \circ \tilde{\rho}_G(\check{g}_{a_1} \rtimes a_1 \cdot \check{g}_{a_2} \rtimes a_2)^{-1}$$

of $Z^2(A, \mathbb{C}^\times)$ representing the class associated to π_G. Define $\phi_{H,0}(x, as(c)) = a\phi_{G,0}(x)$ and the representation ρ_H of S_{ϕ_H} on $\otimes_c V_{\rho_G}$ by $\rho_H(\check{h}) = \otimes_c \rho_G(\check{h}(s(c)))$. For each $b \in B$ define the element $\check{h}_b \in \widehat{H}$ by $\check{h}_b(as(c)) = a(\check{g}_{r(s(c)b)})$ and the isomorphism $\tilde{\rho}_H(\check{h}_b \rtimes b) : \otimes_c V_{\rho_G} \to \otimes_c V_{\rho_G}$ by $\tilde{\rho}_H(\check{h}_b \rtimes b)(\otimes_c v_c) = \otimes_c \tilde{\rho}_G(\check{g}_{r(s(c)b)} \rtimes r(s(c)b))(v_{cb})$. Then $\check{h}_b$ and $\tilde{\rho}_H(\check{h}_b \rtimes b)$ satisfy Equations (7.10) *and* (7.11) *and the associated element*

$$\beta(b_1, b_2) = \tilde{\rho}_H(\check{h}_{b_1} \rtimes b_1) \circ \tilde{\rho}_H(\check{h}_{b_2} \rtimes b_2) \circ \tilde{\rho}_H(\check{h}_{b_1} \rtimes b_1 \cdot \check{h}_{b_2} \rtimes b_2)^{-1}$$

of $Z^2(B, \mathbb{C}^\times)$ is obtained from α by applying the cochain formula for the corestriction with respect to the section s.

Proof. The proof is the same as for Lemma 7.10, so we will not repeat it. □

Consider again $b \in B$ and a section $l : A \backslash B / \langle b \rangle \to B$. Let n_d and $s : A \backslash B \to B$ be as in (7.6). Recall $a_d = l(d) b^{n_d} l(d)^{-1} \in A$.

{lem:indprodd0}

Lemma 7.17. *Let $\phi_{G,0}^d : L_F \to \widehat{G}$. Define $\phi_{H,0} : L_F \to \widehat{H}$ by $\phi_{H,0}(x, al(d)b^i) = a\phi_G^d(x)$. The map*

$$\widehat{H} \to \prod_d \prod_{i=0}^{n_d-1} \widehat{G}, \qquad \check{h} \mapsto \prod_d \prod_{i=0}^{n_d-1} \check{h}(l(d)b^i)$$

is a Γ-equivariant isomorphism of algebraic groups. It translates conjugation by $\phi_H(x)$ to conjugation by $(\phi_G^d(x))_d$. It translates the action by b to the action by $(\Theta_d)_d$, where $\Theta_d(\check{g}_{d,0}, \dots, \check{g}_{d,n_d-1}) = (\check{g}_{d,1}, \dots, \check{g}_{n_d-1}, a_d(\check{g}_0))$.

Proof. This is an immediate computation. ☐

{lem:indprodd}

Lemma 7.18. *Let $\check{h}_b \rtimes b \in \tilde{S}_{\phi_H}$. The map*

$$\widehat{H} \to \prod_d \prod_{i=0}^{n_d-1} \widehat{G}, \qquad \check{h} \mapsto \prod_d \prod_i ev_{l(d)}(Ad(\check{h}_b \rtimes b)^i h)$$

is an isomorphism of algebraic groups. It translates the action of $Ad(\phi_H(w))$ to the diagonal action of $(Ad(\phi_G^d(w)))_d$. It translates the action of conjugation by $\check{h}_b \rtimes b$ to the action sending $(\check{g}_{d,0}, \dots, \check{g}_{d,n_d-1})_d$ to $(g_{d,1}, \dots, g_{d,n_d-1}, Ad(\check{g}_d \rtimes a_d)\check{g}_{d,0})_d$, where $\check{g}_d \rtimes a_d = ev_{l(d)}((\check{h}_b \rtimes b)^{n_d})$.

Proof. Direct computation. ☐

{lem:basictwistd}

Lemma 7.19. *Let J be a quasi-split connected reductive group with a pinned automorphism θ. Consider $I = J \times \cdots \times J$ with the pinned automorphism Θ defined by $\Theta(g_0, \dots, g_{n-1}) = (g_1, \dots, g_{n_d-1}, \theta(g_0))$. Let $\widehat{\theta}$ and $\widehat{\Theta}$ denote the duals of θ and Θ. We have $\widehat{I} = \widehat{J} \times \cdots \times \widehat{J}$ and $\widehat{\Theta}(\check{g}_0, \dots, \check{g}_{n-1}) = (\widehat{\theta}(\check{g}_{n-1}), \check{g}_0, \dots, \check{g}_{n-2})$.*

1. *If $(J^\mathfrak{e}, \mathcal{J}^\mathfrak{e}, \tilde{s}_J^\mathfrak{e}, \xi_J^\mathfrak{e})$ is an endoscopic datum for $J \rtimes \theta$ and we write $\tilde{s}_J^\mathfrak{e} = \check{s}_J^\mathfrak{e} \rtimes \widehat{\theta}$ and $\xi_J^\mathfrak{e}(\jmath) = \xi_{J,0}^\mathfrak{e}(\jmath) \rtimes w_\jmath$, and define $I^\mathfrak{e} = J^\mathfrak{e}$, $\mathcal{I}^\mathfrak{e} = \mathcal{J}^\mathfrak{e}$, $\tilde{s}_I^\mathfrak{e} = \check{s}_I^\mathfrak{e} \rtimes \Theta$, $\check{s}_I^\mathfrak{e} = (\check{s}_J^\mathfrak{e}, 1, \dots, 1)$, $\xi_I^\mathfrak{e}(\jmath) = (\xi_{J,0}^\mathfrak{e}(\jmath), \dots, \xi_{J,0}^\mathfrak{e}(\jmath)) \rtimes w_\jmath$. Then $(I^\mathfrak{e}, \mathcal{I}^\mathfrak{e}, \tilde{s}_I^\mathfrak{e}, \xi_I^\mathfrak{e})$ is an endoscopic datum for $I \rtimes \Theta$.*

2. *If $(I^\mathfrak{e}, \mathcal{I}^\mathfrak{e}, \tilde{s}_I^\mathfrak{e}, \xi_I^\mathfrak{e})$ is an endoscopic datum for $I \rtimes \Theta$ and we write $\tilde{s}_I^\mathfrak{e} = \check{s}_I^\mathfrak{e} \rtimes \widehat{\Theta}$, $\check{s}_I^\mathfrak{e} = (\check{s}_0, \dots, \check{s}_{n-1})$, $\xi_I^\mathfrak{e}(\imath) = (\xi_0^\mathfrak{e}(\imath), \dots, \xi_{n-1}^\mathfrak{e}(\imath)) \rtimes w_\imath$, and define $J^\mathfrak{e} = I^\mathfrak{e}$, $\mathcal{J}^\mathfrak{e} = \mathcal{I}^\mathfrak{e}$, $\tilde{s}_J^\mathfrak{e} = \check{s}_J^\mathfrak{e} \rtimes \widehat{\theta}$, $\check{s}_J^\mathfrak{e} = \check{s}_{n-1} \dots \check{s}_0$, $\xi_J^\mathfrak{e}(\imath) = \xi_{n-1}^\mathfrak{e}(\imath) \rtimes w_\imath$, then $(J^\mathfrak{e}, \mathcal{J}^\mathfrak{e}, \tilde{s}_J^\mathfrak{e}, \xi_J^\mathfrak{e})$ is an endoscopic datum for $J \rtimes \theta$.*

3. *The above constructions given mutually inverse bijections between the sets of isomorphism classes of endoscopic data for $J \rtimes \theta$ and $I \rtimes \Theta$.*

4. *Let $\phi_{J,0} : L_F \to \widehat{J}$ and define $\phi_{I,0} = (\phi_{J,0}, \dots, \phi_{J,0})$. In the above constructions we have $\tilde{s}_I \in \tilde{S}_{\phi_I}$ if and only if $\tilde{s}_J \in \tilde{S}_{\phi_J}$ and the isomorphism classes of the resulting endoscopic data correspond under the above bijections.*

5. *Let $z_J \in Z^1(u \to W, Z(J)^\theta \to J)$ and define $z_I = (z_J, \dots, z_J) \in Z^1(u \to W, Z(I)^\Theta \to I)$. If $\mathfrak{e}$ is an endoscopic datum, both for $I \rtimes \Theta$ and for $J \rtimes \theta$ via the above bijections, $\mathfrak{z}$ is a z-pair for $\mathfrak{e}$, $\mathfrak{w}$ is a θ-special Whittaker datum for J and hence a Θ-special Whittaker datum for I, $\delta_I = (\delta_0, \dots, \delta_{n-1}) \in I_{z_I}(F)$, and $\delta_J = \delta_0 \dots \delta_{n-1}$ we have*

$$\Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^\mathfrak{z}, \delta_I \rtimes \Theta) = \Delta_{KS}[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}](\gamma^\mathfrak{z}, \delta_J \rtimes \theta).$$

Proof. For the first two points we only need to check parts (4.7.6) and (4.7.7) of the definition of endscopic datum in §4.7. These verifications are immediate and left to the reader. For the third point, the assignment $(J^\mathfrak{e}, \mathcal{J}^\mathfrak{e}, \tilde{s}_J^\mathfrak{e}, \xi_J^\mathfrak{e}) \mapsto$

$(I^\mathfrak{e},\mathcal{I}^\mathfrak{e},\tilde s_I^\mathfrak{e},\xi_I^\mathfrak{e})\mapsto(J^\mathfrak{e},\mathcal{J}^\mathfrak{e},\tilde s_J^\mathfrak{e},\xi_J^\mathfrak{e})$ is the identity on data, even before taking isomorphism classes. On the other hand, the element $(\check g_0,\dots,\check g_{n-1})\in\widehat{I}$ with $\check g_i=\check s_{n-1}\dots\check s_{i+1}$ gives an isomorphism between the soruce and target of the assignment $(I^\mathfrak{e},\mathcal{I}^\mathfrak{e},\tilde s_I^\mathfrak{e},\xi_I^\mathfrak{e})\mapsto(J^\mathfrak{e},\mathcal{J}^\mathfrak{e},\tilde s_J^\mathfrak{e},\xi_J^\mathfrak{e})\mapsto(I^\mathfrak{e},\mathcal{I}^\mathfrak{e},\tilde s_I^\mathfrak{e},\xi_I^\mathfrak{e})$. For the fourth point it is enough to start with $(\tilde s_J,\phi_J)$, let $\tilde s_I$ be as in the first point, produce from $(\tilde s_J,\phi_J)$ respectively $(\tilde s_I,\phi_I)$ endoscopic data $(J^\mathfrak{e},\mathcal{J}^\mathfrak{e},\tilde s_J^\mathfrak{e},\xi_J^\mathfrak{e})$ respectively $(I^\mathfrak{e},\mathcal{I}^\mathfrak{e},\tilde s_I^\mathfrak{e},\xi_I^\mathfrak{e})$ via the spectral construction of §4.8, and then verify that these two data are related by the construction of the first point. This is immediate and left to the reader.

The remainder of the proof will be concerned with the equality of transfer factors. We consider each individual term in the product (4.3)

$$\Delta_{KS}=e([I\rtimes\Theta]_{\bar z_I})\epsilon_L(V,\psi)(\Delta_I^{\text{new}})^{-1}\Delta_{II}(\Delta_{III}^{\text{new}})^{-1}\Delta_{IV}.$$

These terms were recalled in §4.9, except $\Delta_{III}^{\text{new}}$, for which we follow the construction given in §5.5. These terms depend on various auxiliary data recalled in §4.9 and their comparison requires that we compare this auxiliary data for the group I and the group J.

We fix a θ-invariant F-pinning $(T_J,B_J,\{X_\alpha\})$ of J and a non-trivial character $\psi:F\to\mathbb{C}^\times$. Thaking the product of this pinning gives a Θ-invariant F-pinning of I and all Θ-invariant pinnings of I arise this way. In this way θ-special Whittaker data for J correspond to Θ-special Whittaker data for I. We fix a norm (S_I,γ) for $\delta_I\rtimes\Theta$. Here $S_I\subset I_{z_I}$ is a maximal torus defined over F, invariant under Θ, and contained in a Borel subgroup $C_I\subset I$ defined over $\bar F$ and invariant under Θ. Moreover $\gamma\in[S_I]_\Theta(F)$ and there exists $g_I\in I$ such that $g_I^{-1}(\delta_I\rtimes\Theta)g_I=\delta_I^*\rtimes\Theta$ with $\delta_I^*\in S_I$ whose image in $[S_I]_\Theta$ is γ. It is immediate that $S_I=S_J^n$ and $C_I=C_J^n$ for a θ-invariant Borel pair $S_J\subset C_J\subset J$, with S_J defined over F. Moreover, the product map $S_I\to S_J$ induces an isomorphism $[S_I]_\Theta\to[S_J]_\theta$. If we write $g_I=(g_0,\dots,g_{n-1})$ then $\delta_I^*=(g_0^{-1}\delta_0g_1,\dots,g_{n-2}^{-1}\delta_{n-2}g_{n-1},g_{n-1}^{-1}\delta_{n-1}\theta(g_0))$ and its image in S_J is given by $\delta_J^*=g_0^{-1}\delta_0\dots\delta_{n-1}\theta(g_0)$. Therefore (S_J,γ) is a norm for $\delta_J\rtimes\theta$.

The set of Θ-orbits in $R(S_I,I)$ is in natural bijection with the set of θ-orbits in $R(S_J,J)$. In this way $R_{\text{res}}(S_I,I)=R_{\text{res}}(S_J,J)$. We fix a-data and χ-data for this set.

We can now compare the individual terms of Δ_{KS} for I and J. The term $\epsilon_L(V,\psi)$ for I is the root number of the virtual Galois representation $X^*(T_I)^\Theta_\mathbb{C}-X^*(T^\mathfrak{e})_\mathbb{C}$. But $X^*(T_I)^\Theta=X^*(T_J)^\theta$ so this equals the term $\epsilon_L(V,\psi)$ for J. The equality $e([I\rtimes\Theta]_{\bar z_I})=e([J\rtimes\theta]_{\bar z_J})$ is Lemma 3.11.

The term Δ_{II} is a fraction. The denominator for I equals the denominator for J by virtue of the identification $I^\mathfrak{e}=J^\mathfrak{e}$. The numerator for I is a product over the Γ-orbits in $R_{\text{res}}(S_I,I)$ and the factor corresponding to α_{res} involves the quantity $N_\Theta\alpha_I(\delta_I^*)$, where $\alpha_I\in R(S_I,I)$ represents α_{res}. Now $R(S_I,I)=R(S_J,J)\cup\dots\cup R(S_J,J)$ and if $\alpha_J\in R(S_J,J)$ represents α_{res} then $N_\Theta\alpha_I(\delta_I^*)=N_\theta\alpha_J(\delta_J^*)$. Therefore the numerators of Δ_{II} for I matches the numerator of Δ_{II} for J.

The term Δ_{IV} is discussed in the same way as the term Δ_{II}.

The term Δ_I^{new} for J is defined as the Tate-Nakayama pairing applied to an element $t_J \in H^1(\Gamma, S^{\theta,\circ})$ with an element $\check{s}_{J,\theta} \in \pi_0([\widehat{S}]_\theta^\Gamma)$. The diagonal inclusions $J \to I$ and $S_J \to S_I$ become isomorphisms $J^{\theta,\circ} \to I^{\Theta,\circ}$ and $S_J^{\theta,\circ} \to S_I^{\Theta,\circ}$. Tracing through the construction we see that under the isomorphism $H^1(\Gamma, S_J^{\theta,\circ}) \to H^1(\Gamma, S_I^{\Theta,\circ})$ the elements t_J and t_I are identified.

The element $\check{s}_{I,\theta} \in \pi_0([\widehat{S}_I]_\Theta^\Gamma)$ is obtained by recognizing that $\check{s}_I$ lies in the image of a certain embedding $\widehat{S}_I \to \widehat{I}$, so that it can be transported to $\widehat{S}_I$ under that embedding and then mapped to $[\widehat{S}_I]_\Theta$. Dual to the isomorphism $S_J^{\theta,\circ} \to S_I^{\Theta,\circ}$ is the isomorphism $[\widehat{S}_I]_\Theta \to [\widehat{S}_J]_\theta$ induced by the product map $\widehat{S}_I = \widehat{S}_J \times \cdots \times \widehat{S}_J \to \widehat{S}_J$. Since the image of $\check{s}_I \in \widehat{S}_I$ under the product map produces the element $\check{s}_J \in \widehat{S}_J$, we see that the term Δ_I^{new} for I equals the term Δ_I^{new} for J.

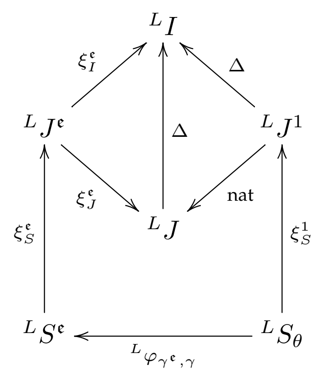
We come to the term $\Delta_{III}^{\text{new}}$. We shall give the proof in the special case of pure inner forms and no z-pair. The proof in the general case is the same, but with more cumbersome notation that obscures the main point. This term for J is given by the Tate-Nakayama pairing of the element $\text{inv}(\gamma, (z_J, \delta_J)) \in H^1(\Gamma, S_J \xrightarrow{1-\theta} S_J)$ with the element $A_{0,J} \in H^1(W_F, \widehat{S}_J \xrightarrow{1-\widehat{\theta}} \widehat{S}_J)$.

We consider the two dual commutative diagrams

$$\begin{array}{ccc} S_I & \xrightarrow{p_0} & S_J \\ {\scriptstyle 1-\Theta}\downarrow & & \downarrow{\scriptstyle 1-\theta} \\ S_I & \xrightarrow[m]{} & S_J \end{array} \qquad\qquad \begin{array}{ccc} \widehat{S}_I & \xleftarrow{i_0} & \widehat{S}_J \\ {\scriptstyle 1-\widehat{\Theta}}\uparrow & & \uparrow{\scriptstyle 1-\widehat{\theta}} \\ \widehat{S}_I & \xleftarrow[\Delta]{} & \widehat{S}_J \end{array}$$

Where m is the multiplication map, Δ is the diagonal inclusion, i_0 is the inclusion into the first coordinate, and p_0 is the projection onto the first coordinate. These diagrams can be seen as morphisms of complexes of tori of length 2, the complexes being the vertical arrows and the morphisms being the horizontal arrows. It is immediate to check that these morphisms are quasi-isomorphisms. Therefore they induce isomorphisms $H^1(\Gamma, S_I \xrightarrow{1-\Theta} S_I) \to H^1(\Gamma, S_J \xrightarrow{1-\theta} S_J)$ and $H^1(W_F, \widehat{S}_J \xrightarrow{1-\widehat{\theta}} \widehat{S}_J) \to H^1(W_F, \widehat{S}_I \xrightarrow{1-\widehat{\Theta}} \widehat{S}_I)$. The element $\text{inv}(\gamma, (z_I, \delta_I \rtimes \Theta))$ is the pair $((g_I^{-1} z_I(\sigma)\sigma(g_I))^{-1}, \delta_I^*) \in Z^1(\Gamma, S_I \xrightarrow{1-\Theta} S_I)$. We had already noted that the image of δ_I^* under the multiplication map is δ_J^*. At the same time, the image of $g_I^{-1} z_I(\sigma)\sigma(g_I)$ under p_0 is $g_0^{-1} z_J(\sigma)\sigma(g_0)$. Thus the image of $\text{inv}(\gamma, (z_I, \delta_I \rtimes \Theta))$ under the first of these isomorphisms is indeed $\text{inv}(\gamma, (z_J, \delta_J \rtimes \Theta))$.

To compare $A_{0,J}$ and $A_{0,I}$ we consider the commutative diagram



The element $A_{0,J}$ is the class of $(a_{S_J}^{-1}, \check{s}_J)$, where $a_{S_J} : W_F \to \widehat{S}_J$ is the 1-cocycle measuring the difference between $\xi_J^{\mathfrak{e}} \circ \xi_S^{\mathfrak{e}} \circ {}^L\varphi_{\gamma^{\mathfrak{e}},\gamma}$ and $\text{nat} \circ \xi_S^1$. The element $A_{0,I}$ is the class of $(a_{S_I}^{-1}, \check{s}_I)$, where $a_{S_I} : W_F \to \widehat{S}_I$ is the 1-cocycle measuring the difference between $\xi_I^{\mathfrak{e}} \circ \xi_S^{\mathfrak{e}} \circ {}^L\varphi_{\gamma^{\mathfrak{e}},\gamma}$ and $\Delta \circ \circ\xi_S^1$. The commutativity of the diagram shows that $a_{S_I} = \Delta(a_{S_J})$. On the other hand $\check{s}_I = i_0(\check{s}_J)$ and we conclude that the image of $A_{0,J}$ under the isomorphism $H^1(W_F, \widehat{S}_J \xrightarrow{1-\widehat{\theta}} \widehat{S}_J) \to H^1(W_F, \widehat{S}_I \xrightarrow{1-\widehat{\Theta}} \widehat{S}_I)$ equals $A_{0,I}$. □

{cor:indprodd}

Corollary 7.20. *Let $\phi_H : L_F \to {}^L H$ and $\tilde{s}_H \in \tilde{S}_{\phi_H}$. Write $\tilde{s}_H = \check{s}_H \rtimes b$. Fix a section $l : A \setminus B/\langle b \rangle \to B$. For each $d \in A \setminus B/\langle b \rangle$ define $\tilde{s}_d = ev_{l(d)}(\tilde{s}_H^{n_d})$ and $\phi_{d,0} = ev_{l(d)} \circ \phi_{H,0}$. Then $\phi_d : L_F \to {}^L G$ and $\tilde{s}_d \in \tilde{S}_{\phi_d}$.*

1. *Let $\mathfrak{e}_H$ and $\mathfrak{e}_d$ be the endoscopic data associated to $(\tilde{s}_H, \phi_H)$ and $(\tilde{s}_d, \phi_d)$. There is a natural identification*
$$\mathfrak{e}_H = \prod_d \mathfrak{e}_d.$$

2. *Fix a z-pair $\mathfrak{z}_H$ for $\mathfrak{e}_H$ and express it as $\prod_d \mathfrak{z}_d$. If $\gamma^{\mathfrak{z}_H}$ corresponds to $(\gamma^{\mathfrak{z}_d})_d$ under this identification, and the stable class of $\tilde{h}' \in [H \rtimes b^{-1}]_{z_H}(F)$ corresponds to the stable class of $(\tilde{g}'_d)_d$ under the bijection of Corollary 7.14, then*
$$\Delta_{KS}(\gamma^{\mathfrak{z}_H}, \tilde{h}') = \prod_d \Delta_{KS}(\gamma^{\mathfrak{z}_d}, \tilde{g}'_d).$$

3. *The transfer of the function $R_{\tilde{h}}^{-1} f_H \in \mathcal{C}_c^\infty(\tilde{H}_{z_H}(F))$ to $\mathfrak{z}_H$ equals the tensor product over d of the transfers of the functions $R_{\tilde{g}_d}^{-1}(f_{d,0} * \cdots * f_{d,n_d-1}) \in \mathcal{C}_c^\infty(\tilde{G}_{z_G}(F))$ to $\mathfrak{z}_d$.*

Proof. We consider the isomorphism
$$H \to \prod_d \prod_{i=0}^{n_d-1} G, \qquad h \mapsto \prod_d \prod_{i=0}^{n_d-1} h(l(d)b^{-i})$$

of Lemma 7.11 and the isomorphism

$$\widehat{H} \to \prod_d \prod_{i=0}^{n_d-1} \widehat{G}, \qquad \check{h} \mapsto \prod_d \prod_{i=0}^{n_d-1} \check{h}(l(d)b^{-i})$$

of Lemma 7.17, both applied to the element b^{-1}. They are dual to each other (cf. (7.7)). The first of them translates b^{-1} to $(\Theta_d)_d$ and hence a_d^{-1} to θ_d, while the second translates b to $(\widehat{\Theta}_d)_d$ and a_d to $\widehat{\theta}_d$.

Since the $\widehat{H}$-conjugacy class of ϕ_H is b-invariant, we may conjugate $(\phi_H, \tilde{s}_H)$ by $\widehat{H}$ to arrange that the image of ϕ_H^0 under the second isomorphism is of the form $(\phi_d^0, \dots, \phi_d^0)_d$ for parameters $\phi_d : L_F \to {}^LG$. Let $(\check{s}_{d,i})$ be the image of $\check{s}_H$. Then Lemma 7.19 gives the identification $\mathfrak{e}_H = \prod_d \mathfrak{e}'_d$, where $\mathfrak{e}'_d$ is the endoscopic datum associated to $(\tilde{s}_d, \phi_d)$ with $\tilde{s}'_d = \check{s}'_d \rtimes a_d$ and

$$\check{s}'_d = \check{s}_{d,n_d-1} \dots \check{s}_{d,0} = \check{s}_H(l(d)b^{1-n_d}) \cdot \check{s}_H(l(d)b^{2-n_d}) \dots \check{s}_H(l(d)).$$

The element $(\check{s}_{d,n_d-1} \dots \check{s}_{d,1})^{-1}$ conjugates $\tilde{s}'_d$ to $\tilde{s}_d$ and hence provides an isomorphism $\mathfrak{e}'_d \to \mathfrak{e}_d$.

Let $\tilde{h}' = h' \rtimes b^{-1}$ and let $(g_{d,i})$ be the image of h'. It is immediate that $g_{d,0} \dots g_{d,n_d-1} \rtimes a_d^{-1} = \mathrm{ev}_{l(d)}(\tilde{h}') = \tilde{g}'_d$, therefore Lemma 7.19 (and Fact 3.10) implies

$$\Delta_{KS}(\gamma^{\mathfrak{z}}, \tilde{h}') = \prod_d \Delta_{KS}(\gamma^{\mathfrak{z}_d}, \tilde{g}'_d).$$

The identification of transfers of functions follows from the equation

$$\begin{aligned} SO_{\gamma^{\mathfrak{z}_H}}((R_{\tilde{h}}^{-1} f_H)^{\mathfrak{e}_H}) &= \sum_{\tilde{h}'} \Delta_{\mathrm{KS}}(\gamma^{\mathfrak{z}_H}, \tilde{h}') O_{\tilde{h}'}^H(R_{\tilde{h}}^{-1} f_H) \\ &= \prod_d \sum_{\tilde{g}'_d} \Delta_{\mathrm{KS}}(\gamma^{\mathfrak{z}_d}, \tilde{g}'_d) O_{\tilde{g}'_d}^G(R_{\tilde{g}_d}^{-1}(f_{d,0} * \dots * f_{d,n_d-1})) \\ &= \prod_d SO_{\gamma^{\mathfrak{z}_d}}((R_{\tilde{g}_d}^{-1}(f_{d,0} * \dots * f_{d,n_d-1}))^{\mathfrak{e}_d}). \end{aligned}$$

Here $\tilde{h}'$ runs over the set of $H_{z_H}(F))$-classes in $[H \rtimes b^{-1}]_{z_H}(F)$. We have used Corollary 7.14 for b^{-1} to identify this set with the set $(\tilde{g}'_d)_d$ of $G_{z_G}(F)^d$-conjugacy classes in $\prod_d [G \rtimes a_d^{-1}]_{z_G}(F)$, and to related the corresponding orbital integrals. □

We continue with z_G and ϕ_G whose equivalence classes are A-fixed and consider $\pi_G \in \Pi_{\phi_G}$ corresponding to $\rho_G \in \mathrm{Irr}(\pi_0(S_{\phi_G}^+))$ whose equivalence classes are also A-fixed (provided these exist). Recall that z_G and ϕ_G determine B-fixed equivalence classes of z_H and ϕ_H, and that π_G and ρ_G determine B-fixed isomorphism classes of representations π_H of $H_{z_H}(F)$ and ρ_H of $\pi_0(\tilde{S}_{\phi_H}^+)$. Assume given an extension of $\pi_G \boxtimes \rho_G^\vee$ from $G_{z_G}(F) \times \pi_0(S_{\phi_G}^+)$ to $\tilde{G}_{z_G}(F) \times_A \pi_0(\tilde{S}_{\phi_G}^+)$. We claim that it determines an extension of $\pi_H \boxtimes \rho_H^\vee$ from $H_{z_H}(F) \times$

$\pi_0(S^+_{\phi_H})$ to $\tilde{H}_{z_H}(F)\times_B \pi_0(\tilde{S}^+_{\phi_H})$. To see this, fix $g_a\in G$ and $\tilde\pi_G(g_a\rtimes a): V_{\pi_G}\to V_{\pi_G}$ satisfying Equations (7.2) and (7.3), and fix analogously $\check{g}_a\in\widehat{G}$ and $\tilde\rho_G(\check{g}_a\rtimes a): V_{\rho_G}\to V_{\rho_G}$ satisfying Equations (7.8) and (7.9). We demand that these choices are made in such a way that the restriction to $\tilde{G}_{z_G}(F)\times_A\pi_0(\tilde{S}^+_{\phi_G})$ of the exterious tensor product $\tilde\pi_G\boxtimes\tilde\rho_G^\vee$ is the given extension of $\pi_G\boxtimes\rho_G^\vee$. We fix a section $s: A\setminus B\to B$ and according to Lemma 7.9 we can take $z_H(w,as(c)) = az_G(w)$ and $\pi_H(h)=\otimes_c\pi_G(h(s(c)))$ acting on $V_{\pi_G}^{\otimes_c}$, and according to Lemma 7.16 we can take $\phi_{H,0}(x,as(c)) = a\phi_{G,0}(x)$ and $\rho_H(\check{h}) = \otimes_c\rho_G(\check{h}(s(c)))$. We then define for each $b\in B$ the element $h_b\in H$ and the isomorphism $\tilde\pi_H(h_b\rtimes b)$ as in Lemma 7.10 and the element $\check{h}_b\in\widehat{H}$ and the isomorphism $\tilde\rho_H(\check{h}_b\rtimes b)$ as in Lemma 7.16 and consider $\tilde\pi_H\boxtimes\tilde\rho_H^\vee$ restricted to $\tilde{H}_{z_H}(F)\times_B\pi_0(\tilde{S}^+_{\phi_H})$. According to Lemmas 7.10 and 7.16 this is a linear representation and extends $\pi_H\boxtimes\rho_H^\vee$.

{lem:ind4}

Lemma 7.21. *The restriction of $\tilde\pi_H\boxtimes\tilde\rho_H^\vee$ to $\tilde{H}_{z_H}(F)\times_B\pi_0(\tilde{S}^+_{\phi_H})$ is independent of the choices of g_a, $\tilde\pi_G(g_a\rtimes a)$, $\check{g}_a$, and $\tilde\rho_G(\check{g}_a\rtimes a)$.*

Proof. Keeping $\{g_a\}$ and $\{\check{g}_a\}$ fixed, for any a the isomorphism $\tilde\pi_G(g_a\rtimes a)$ can only be changed to $z_a\tilde\pi_G(g_a\rtimes a)$ for some $z_a\in\mathbb{C}^\times$. Since the restriction of $\tilde\pi_G\boxtimes\tilde\rho_G$ to $\tilde{G}_{z_G}(F)\times_A\pi_0(\tilde{S}^+_{\phi_G})$ is fixed, this means that $\tilde\rho_G(\check{g}_a\rtimes a)$ must be changed to $z_a^{-1}\tilde\rho_G(\check{g}_a\rtimes a)$. Now $\tilde\pi_H(h_b\rtimes b)$ is multiplied by z_a raised to the power of the cardinality of $\{c|a=r(s(c)b)\}$, while $\tilde\rho_H(\check{h}_b\rtimes b)$ is multiplied by z_a^{-1} raised to the same power, so we see that the restriction of $\tilde\pi_H\boxtimes\rho_H^\vee$ to $\tilde{H}_{z_H}(F)\times_B\pi_0(\tilde{S}^+_{\phi_H})$ remains unchanged.

Replace now g_a by $g_a' = g_a^0 g_a$ and $\check{g}_a$ by $\check{g}_a' = \check{g}_a^0\check{g}_a$ for $g_a^0\in G_{z_G}(F)$ and $\check{g}_a^0\in S^+_{\phi_G}$. By the previous argument the choices of $\tilde\pi_G(g_a'\rtimes a)$ and $\tilde\rho_G(\check{g}_a'\rtimes a)$ will not influence the construction. We choose them to be $\pi_G(g_a^0)\circ\tilde\pi_G(g_a\rtimes a)$ and $\rho_G(\check{g}_a^0)\circ\tilde\rho_G(\check{g}_a\rtimes a)$ respectively. We claim that $\tilde\pi_H$ and $\tilde\rho_H$ are both unchanged. Indeed, h_b is now replaced by h_b' defined by $h_b'(as(c)) = a(g'_{r(s(c)b)}) = a(g^0_{r(s(c)b)})\cdot a(g_{r(s(c)b)})$. Define $h^0\in H$ by $h^0(as(c)) = a(g^0_{r(s(c)b)})$. Then $h^0\in H_{z_H}(F)$ and $h_b' = h^0h_b$. The new choices now stipulate

$$\tilde\pi_H(h_b'\rtimes b)_c = \tilde\pi_G(g'_{r(s(c)b)}\rtimes r(s(c)b)) = \pi_G(g^0_{r(s(c)b)})\circ\tilde\pi_G(g_{r(s(c)b)}\rtimes r(s(c)b)),$$

while according to the old choices we have

$$\tilde\pi_H(h_b'\rtimes b) = \pi_H(h^0)\circ\tilde\pi_H(h_b\rtimes b)$$

and we see that both of these values for $\tilde\pi_H(h_b'\rtimes b)_c$ agree. The argument for $\tilde\rho_H$ is analogous. ☐

Thus far we have focused on the setting in which the objects z_H, π_H, ϕ_H, ρ_H have equivalence classes fixed by B. In general this will not be the case, but one can reduce to that case by a simple application of Mackey theory, at the expense of introducing rather cumbersome notation. This is what we turn to next.

Let $B'\subset B$ be a subgroup. Fix a section $l: A\setminus B/B'\to B$ of the natural projection. For each $d\in A\setminus B/B'$ let $A_d = l(d)^{-1}Al(d)$, $A_d' = A_d\cap B'$, and write G^d for the group G with action of A_d defined by $(l(d)^{-1}al(d)) \cdot_d g = ag$. Fix a section $s_d : A'_d \setminus B' \to B'$ of the natural projection. Then each element of B has a unique expression of the form $b = al(d)s_d(c'_d)$ for $d \in A \setminus B/B'$ and $c'_d \in A'_d \setminus B'$. This gives a section $A \setminus B \to B$.

{cor:ind6}

Corollary 7.22. *1. Let $z_H \in Z^1(u \to W, Z(H)^{B'} \to H)$ have a B'-fixed class. For each $d \in A \setminus B/B'$ we obtain an element of $Z^1(u \to W, Z(G^d)^{A'_d} \to G^d)$ by $z_{G^d}(w) = z_H(w, l(d))$, and the map $z_H \mapsto (z_{G^d})_d$ is a bijection between $H^1(u \to W, Z(H)^{B'} \to H)^{B'}$ and $\prod_d H^1(u \to W, Z(G^d)^{A'_d} \to G^d)^{A'_d}$.*

2. *Let π_H be an irreducible representation of $H_{z_H}(F)$ whose class is B'-fixed. For each $d \in A \setminus B/B'$ we obtain an irreducible representation π_{G^d} of $G^d_{z_{G^d}}(F)$ on $V_{l(d)}$ by $\pi_{G^d}(g) = \pi_H^{l(d)}(g)$. The map $\pi_H \mapsto (\pi_{G^d})_d$ is a bijection between $Irr(H_{z_H}(F))^{B'}$ and $\prod_d Irr(G^d_{z_{G^d}}(F))^{A'_d}$.*

3. *Let $\phi_H(x) = \phi_{H,0}(x) \rtimes x$ be a Langlands parameter for H whose class is B'-fixed. For each $d \in A \setminus B/B'$ we obtain a Langlands parameter $\phi_{G^d,0}$ for G^d by $\phi_{G^d,0}(x) = \phi_{H,0}(x, l(d))$. The map $\phi_H \mapsto (\phi_{G^d})_d$ is a bijection between $\Phi(H)^{B'}$ and $\prod_d \Phi(G^d)^{A'_d}$.*

4. *Let ρ_H be an irreducible representation of $\pi_0(S^+_{\phi_H})$. For each $d \in A \setminus B/B'$ we obtain an irreducible representation of $\pi_0(S^+_{\phi_{G^d}})$ by $\rho_{G^d}(\check{g}) = \rho_H^{l(d)}(\check{g})$. The map $\rho_H \mapsto (\rho_{G^d})_d$ is a bijection between the sets $Irr(\pi_0(S^+_{\phi_H}))^{B'}$ and $\prod_d Irr(\pi_0(S^+_{\phi_{G^d}}))^{A'_d}$.*

5. *The inverse of the above bijections are given by $z_H(w, al(d)s_d(c'_d)) = az_{G,d}(w)$, $\phi_H(x, al(d)s_d(c'_d)) = a\phi_{G^d}(x)$, $\pi_H(h) = \otimes_d \otimes_{c'_d} \pi_{G^d}(h(l(d)s_d(c'_d)))$, and $\rho_H(\check{h}) = \otimes_d \otimes_{c'_d} \rho_{G^d}(\check{h}(l(d)s_d(c'_d)))$.*

6. *For each $d \in A \setminus B/B'$ and $a'_d \in A'_d$ choose $g_{a'_d} \in G^d$ and $\tilde{\pi}_{G^d}(g_{a'_d} \rtimes a'_d) : V_{\pi_{G^d}} \to V_{\pi_{G^d}}$ satisfying Equations* (7.2) *and* (7.3) *for $G^d \rtimes A'_d$. For each $b' \in B'$ define $h_{b'} \in H$ by $h_{b'}(al(d)s_d(c'_d)) = ag_{r_d(s_d(c'_d)b')}$ and*

$$\tilde{\pi}_H(h_{b'} \rtimes b') : \otimes_d \otimes_{c'_d} V_{d,c'_d} \to \otimes_d \otimes_{c'_d} V_{d,c'_d}$$

by

$$\tilde{\pi}_H(h_{b'} \rtimes b')(\otimes_d \otimes_{c'_d} v_{d,c'_d}) = \otimes_d \otimes_{c'_d} \tilde{\pi}_{G^d}(g_{r_d(s_d(c'_d)b')} \rtimes r_d(s_d(c'_d)b'))(v_{d,c'_d \cdot b'}).$$

Then these satisfy Equations (7.4) *and* (7.5). *Moreover, the 2-cocycle $\beta \in Z^2(B', \mathbb{C}^\times)$ for $\tilde{\pi}_H$ is the product over $d \in A \setminus B/B'$ of the co-restrictions (computed with respect to the sections s_d) of the 2-cocycles $\alpha_d \in Z^2(A'_d, \mathbb{C}^\times)$ for $\tilde{\pi}_{G^d}$.*

7. *For each $d \in A \setminus B/B'$ and $a'_d \in A'_d$ choose $\check{g}_{a'_d} \in \widehat{G}^d$ and $\tilde{\rho}_{G^d}(\check{g}_{a'_d} \rtimes a'_d) : V_{\rho_{G^d}} \to V_{\rho_{G^d}}$ satisfying Equations* (7.8) *and* (7.9) *for $\widehat{G}^d \rtimes A'_d$. For each $b' \in B'$*

define $\check{h}_{b'} \in \widehat{H}$ *by* $\check{h}_{b'}(al(d)s_d(c'_d)) = a\check{g}_{r_d(s_d(c'_d)b')}$ *and*

$$\tilde{\rho}_H(\check{h}_{b'} \rtimes b') : \otimes_d \otimes_{c'_d} V_{d,c'_d} \to \otimes_d \otimes_{c'_d} V_{d,c'_d}$$

by

$$\tilde{\rho}_H(h_{b'} \rtimes b')(\otimes_d \otimes_{c'_d} v_{d,c'_d}) = \otimes_d \otimes_{c'_d} \tilde{\rho}_{G^d}(g_{r_d(s_d(c'_d)b')} \rtimes r_d(s_d(c'_d)b'))(v_{d,c'_d \cdot b'}).$$

Then these satisfy Equations (7.10) *and* (7.11). *Moreover, the 2-cocycle* $\beta \in Z^2(B', \mathbb{C}^\times)$ *for* $\tilde{\rho}_H$ *is the product over* $d \in A \setminus B/B'$ *of the co-restrictions (computed with respect to the sections* s_d*) of the 2-cocycles* $\alpha_d \in Z^2(A'_d, \mathbb{C}^\times)$ *for* $\tilde{\rho}_{G^d}$.

8. *Let* $b_1 \in B$ *and set* $B'_1 = b_1 B' b_1^{-1}$. *Define* $l_1 : A \setminus B/B'_1 \to B$ *by* $l_1(d_1) = l(d)b_1^{-1}$ *for* $d_1 \in A \setminus B/B'_1$ *and* $d = d_1 b_1 \in A \setminus B/B'$. *Define* $s_{d_1} : A'_{d_1} \setminus B'_1 \to B'_1$ *by* $s_{d_1}(c'_{d_1}) = b_1 s_d(c'_d) b_1^{-1}$ *for* $c'_d \in A'_d \setminus B'$ *and* $c'_{d_1} = b_1 c'_d b_1^{-1} \in A'_{d_1} \setminus B'_1$. *If* (z_H, π_H) *corresponds to* $(z_{G^d}, \pi_{G^d})_d$ *via the choices of* l *and* (s_d), *then* $b(z_H, \pi_H)$ *corresponds to* $(z_{G^{d_1}}, \pi_{G^{d_1}})$ *via the choices of* l_1 *and* (s_{d_1}), *where* $z_{G^{d_1}}(w) = z_{G^d}(w)$ *and* $\pi_{G^{d_1}} = \pi_{G^d}$.

Proof. We have the Mackey isomorphism

$$\mathrm{Res}^B_{B'} H \to \prod_{d \in A \setminus B/B'} \mathrm{Ind}^{B'}_{A'_d} \mathrm{Res}^{A_d}_{A'_d} G^d.$$

It sends $h \in H$ to the collection $(h_d)_d$ given by $h_d(b') = h(l(d)b')$. Write $H_d = \mathrm{Ind}^{B'}_{A'_d} \mathrm{Res}^{A_d}_{A'_d} G^d$, so that $\mathrm{Res}^B_{B'} H = \prod H_d$.

The element z_H is mapped to the collection $(z_{H_d})_d$, where $z_{H_d} \in Z^1(u \to W, Z(H_d)^{B'} \to H_d)$. The class of each z_{H_d} is B'-invariant, as one checks by sending (7.4) through the Mackey isomorphism. In turn, z_{H_d} corresponds by Lemma 7.9 to $z_{G,d} \in Z^1(u \to W, Z(G^d)^{A'_d} \to G^d)$. Explicitly, we have $z_H(w, al(d)s_d(c'_d)) = az_{H_d}(w, s_d(c'_d)) = az_{G^d}(w)$.

According to the product $H = \prod_d H_d$, the representation π_H is given by $\otimes \pi_{H_d}$, where π_{H_d} is a representation of $H_{d, z_{H_d}}(F)$ on a vector space V_d. Thus π_H acts on $\otimes_d V_d$ as $\pi_H(h) = \otimes_d \pi_{H_d}(h_d)$. Therefore each π_{H_d} acts on $\otimes_{c'_d} V_{d,c'_d}$ as $\pi_{H_d}(h_d) = \otimes_{c'_d} \pi^{c'_d}_{H_d}(h_d(c'_d))$. Thus we have $\pi_H^{l(d)s_d(c'_d)} = \pi_{H_d}^{s_d(c'_d)}$ acting on V_{d,c'_d}. The class of each π_{H_d} is B'-invariant, so π_{H_d} corresponds to the representation π_{G^d} of G^d on the vector space $V_{d,1}$ given by $\pi_{G^d}(g) = \pi^1_{H_d}(g) = \pi^{l(d)}_H(g)$.

The statements concerning z_H and π_H now follow immediately from Lemma 7.9 by taking products over d. The statement about $\tilde{\pi}_H$ follows from Lemma 7.10. The argument for the dual side is analogous, using Lemmas 7.15 and 7.16 instead. □

{pro:ind}

Proposition 7.23. *Assume that Conjectures A.1 and 7.3 hold for* $G \rtimes A$. *Then they also hold for* $H \rtimes B$.

Proof. Let $\phi_H : L_F \to {}^L H$ be a tempered Langlands parameter, $z_H \in Z^1(u \to W, Z(H)^B \to B)$ and $\rho_H \in \mathrm{Irr}(S^+_{\phi_H}, [z_H])$. We are assuming the validity of the refined local Langlands correspondence, so there is a corresponding $\pi_H \in \Pi_{\phi_H}(H_{z_H})$.

Consider any subgroup $B' \subset B$ fixing the equivalence classes of z_H, π_H, ϕ_H, ρ_H. Choose a sections $l : A \setminus B/B' \to B$, as well as a section $s_d : A'_d \setminus B' \to B'$ for each $d \in A \setminus B/B'$, as in the discussion before the statement of Corollary 7.22. That Corollary provides collections (z_{G^d}),(π_{G^d}),(ϕ_{G^d}),(ρ_{G^d}) indexed by $d \in A \setminus B/B'$, where $\phi_{G^d} \in \Phi(G^d)$, $\rho_{G^d} \in \mathrm{Irr}(\pi_0(S^+_{\phi_{G^d}}))$, $z_{G^d} \in Z^1(u \to W, Z(G^d)^{A'_d} \to G^d)$, and $\pi_{G^d} \in \mathrm{Irr}(G^d_{z_{G^d}}(F))$. Since the refined local Langlands correspondence is compatible with products of reductive groups we see that for each d, (ϕ_{G^d}, ρ_{G^d}) corresponds to (z_{G^d}, π_{G^d}). The part of Corollary 7.22 that describes the compatibility of forming these collections with the action of B, applied to the case $B' = \{1\}$, shows that for any $b_1 \in B$ the pair $b_1(\phi_H, \rho_H)$ corresponds to the pair $b_1(z_H, \pi_H)$. That is, Conjecture A.1 holds for $H \rtimes B$.

In particular we see $B^{[\phi_H]}_{\rho_H} = B^{[z_H]}_{\pi_H}$. Take B' to be this group and apply the above discussion to obtain the collections (z_{G^d}),(π_{G^d}),(ϕ_{G^d}),(ρ_{G^d}). Let $\tilde\pi_{G_d} \boxtimes \tilde\rho^\vee_{G^d}$ be the extension of $\pi_{G^d} \boxtimes \rho^\vee_{G^d}$ to $\tilde G^d_{z_{G^d}}(F)_{\pi_{G^d}} \times_{A'_d} \pi_0(\tilde S^{+,[z_{G^d}]}_{\phi_{G^d},\rho_{G^d}})$ that Conjecture 7.3 for $G^d \rtimes A'_d$ provides. Taking the tensor product over d of the extensions provided by the construction prior to Lemma 7.21 gives an extension $\tilde\pi_H \boxtimes \tilde\rho^\vee_H$ of $\pi_H \boxtimes \rho^\vee_H$ to $\tilde H_{z_H}(F)_{\pi_H} \times_{B'} \pi_0(\tilde S^{+,[z_H]}_{\phi_H,\rho_H})$.

We now come to the character identity. Thus we fix $\check h_b \rtimes b \in \tilde S_{\phi_H}$, $h_b \rtimes b^{-1} \in \tilde H_{z_H}(F)$, a function $f_H \in \mathcal{C}^\infty_c(H_{z_H}(F))$, and $\check t \in S_{\phi_H}$, and consider

$$\sum_{\substack{\pi_H \in \Pi_{\phi_H}\\ \pi_H \circ b \cong \pi_H}} \mathrm{tr}(\tilde\pi_H \boxtimes \tilde\rho^\vee_H)(R^{-1}_{h_b \rtimes b^{-1}} f_H \times (\check t \check h_b \rtimes b)). \tag{7.12}$$ {eq:charidind1}

We apply Lemmas 7.12 and 7.18 to represent π_H as $\boxtimes_d \pi_d^{\boxtimes n_d}$ and ρ_H as $\boxtimes_d \rho_d^{\boxtimes n_d}$ where, for each $d \in A \setminus B/\langle b \rangle$, π_d is a representations of $G_{z_G}(F)$ invariant under $g_d \rtimes a_d^{-1} = \tilde g_d = \mathrm{ev}_{l(d)}((h_b \rtimes b^{-1})^{n_d})$ and ρ_d is a representation of S_{ϕ_G} invariant under $\check g_d \rtimes a_d = \mathrm{ev}_{l(d)}((\check h_b \rtimes b)^{n_d})$. We fix isomorphisms $\tilde\pi_d : \pi_d \circ \mathrm{Ad}(\tilde g_d)^{-1} \to \pi_d$ and $\tilde\rho_d : \rho_d \circ \mathrm{Ad}(\check g_d \rtimes a_d)^{-1} \to \rho_d$. We are interested in the canonical extension $\tilde\pi_H \boxtimes \tilde\rho^\vee_H$ of $\pi_H \boxtimes \rho^\vee_H$ to $[H \rtimes \langle b \rangle]_{z_H}(F) \times_{\langle b \rangle} [\widehat H \rtimes \langle b \rangle]_{\phi_H}$. Under Lemmas 7.12 and 7.18 this is the representation $\boxtimes_d (\tilde\pi_d \boxtimes \tilde\rho^\vee_d)^{\boxtimes n_d}$ of $\prod_d \prod_i [G \rtimes \langle a_d \rangle]_{z_G} \times_{\langle a_d \rangle} [\widehat G \rtimes \langle a_d \rangle]_{\phi_G}$, where $\tilde\pi_d \boxtimes \tilde\rho^\vee_d$ is the canonical extension of $\pi_d \boxtimes \rho^\vee_d$. We write $f_H = \otimes_d \otimes_{i=0}^{n_d-1} f_{d,i}$ and $\check t = \prod_d \prod_i \check s_{d,i}$ and then Lemma 7.13 implies that $\mathrm{tr}(\tilde\pi_H \boxtimes \tilde\rho^\vee_H)(R^{-1}_{h_b \rtimes b^{-1}} f_H \times (\check t \check h_b \rtimes b))$ equals

$$\prod_d \mathrm{tr}(\tilde\pi_d \boxtimes \tilde\rho^\vee_d)(R^{-1}_{g_d \rtimes a_d^{-1}} f_{d,0} * \cdots * f_{d,n_d-1}, \check s_{d,0} \ldots \check s_{d,n_d-1} \check g_d \rtimes a_d).$$

The set $\{\pi_H \in \Pi_{\phi_H} | \pi_H \circ b \cong \pi_H\}$ is translated to the set $\{\otimes_d \pi_d \in \otimes_d \Pi_{\phi_d} | \pi_d \circ$

$a_d \cong \pi_d\}$. Therefore (7.12) becomes

$$\prod_d \sum_{\substack{\pi_d \in \Pi_{\phi_d} \\ \pi_d \circ a_d \cong \pi_d}} \mathrm{tr}(\tilde{\pi}_d \boxtimes \tilde{\rho}_d^\vee)(R^{-1}_{g_d \rtimes a_d} f_{d,0} * \cdots * f_{d,n_d-1}, \check{s}_{d,0} \ldots \check{s}_{d,n_d-1} \breve{g}_d \rtimes a_d).$$

The parameter ϕ_d and the element $\check{s}_{d,0} \ldots \check{s}_{d,n_d-1} \breve{g}_d \rtimes a_d \in \widehat{G} \rtimes A$ lead to an endoscopic datum $\mathfrak{e}_d$ and parameter $\phi_{\mathfrak{e}_d}$. The character identities for $\tilde{G}$ imply that the above equals

$$\prod_d S\Theta_{\phi_{\mathfrak{e}_d}}(f^{\mathfrak{e}_d}_{\mathrm{KS}}),$$

where $f^{\mathfrak{e}_d}_{KS}$ is the transfer of $R^{-1}_{g_d \rtimes a_d} f_{d,0} * \cdots * f_{d,n_d-1}$ with respect to Δ_{KS}. By Corollary 7.20 the endoscopic datum for H and $\check{t}\check{h}_b \rtimes b$ and ϕ_H is $\prod_d \mathfrak{e}_d$, and the function $\otimes f^{\mathfrak{e}_d}_{KS}$ has KS-matching orbital integrals with f_H.

□

8 THE CASE OF TORI

{sec:tori}

In this section we are going to sketch the proof of Conjecture 5.12 in the case where the reductive group G is a torus. We will write T instead of G to emphasize this. Note that, while a torus T is tautologically quasi-split, an inner form of $T \rtimes A$ need not be quasi-split. This is the main source of complications we will have to deal with.

8.1 Initial considerations

{sub:init}

Let $\phi : W_F \to \widehat{T}$ and let $[\phi]$ denote both the equivalence class of ϕ and the corresponding character $T(F) \to \mathbb{C}^\times$. Let $Z \subset T$ be finite and defined over F, and $z \in Z^1(u \to W, Z \to T)$. Then

$$\tilde{T}_z(F) = (T(\bar{F}) \rtimes A)^{\bar{\tilde{z}}(\Gamma)} = \{\tilde{\delta} \in T(\bar{F}) \rtimes A | \mathrm{Ad}(\bar{\tilde{z}}(\sigma))\tilde{\delta} = \tilde{\delta}\ \forall \sigma \in \Gamma\}.$$

The group $\tilde{T}_z(F)$ is an extension

$$1 \to T(F) \to \tilde{T}_z(F) \to A^{[z]} \to 1.$$

The set $\tilde{\Pi}_{\phi,z}$ consists of those irreducible admissible representations of $\tilde{T}_z(F)$ whose restriction to $T(F)$ contains the character $[\phi]$. All these representations are finite-dimensional. They can be described as follows. We have the pull-back and push-out diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T(F) & \longrightarrow & (T(\bar{F}) \rtimes A)^{\bar{\bar{z}}(\Gamma)} & \longrightarrow & A^{[z]} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & T(F) & \longrightarrow & (T(\bar{F}) \rtimes A^{[\phi]})^{\bar{\bar{z}}(\Gamma)} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle [\phi]} & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{E}^z_{[\phi]} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1
\end{array}$$

The bottom extension is central. If we let $\mathrm{Irr}(\mathcal{E}^z_{[\phi]}, \mathrm{id})$ denote the set of irreducible representations of $\mathcal{E}^z_{[\phi]}$ whose central character restricts to the identity on $\mathbb{C}^\times$, then inflating an element of $\mathrm{Irr}(\mathcal{E}^z_{[\phi]}, \mathrm{id})$ to $(T(\bar{F}) \rtimes A^{[\phi]})^{\bar{\bar{z}}(\Gamma)}$ and then inducing it to $(T(\bar{F}) \rtimes A)^{\bar{\bar{z}}(\Gamma)}$ provides a canonical bijection

$$\mathrm{Irr}(\mathcal{E}^z_{[\phi]}, \mathrm{id}) \to \tilde{\Pi}_{\phi,z}.$$

Dually, we have $\tilde{S}^{[z]}_\phi = (\widehat{T} \rtimes A^{[z]})^{\phi(W_F)}$. Its preimage $\tilde{S}^{+,[z]}_\phi$ in $\widehat{\bar{T}} \rtimes A$ fits in the following push-out diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_0([\widehat{\bar{T}}]^+) & \longrightarrow & \pi_0(\tilde{S}^{+,[z]}_\phi) & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle [z]} & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{E}^\phi_{[z]} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1
\end{array}$$

and again there is a canonical bijection $\mathrm{Irr}(\mathcal{E}^\phi_{[z]}, \mathrm{id}) \to \mathrm{Irr}(\tilde{S}_{\phi,[z]}, [z])$ given simply by inflation.

While the extensions $\mathcal{E}^\phi_{[z]}$ and $\mathcal{E}^z_{[\phi]}$ are constructed from essentially the same data, their constructions are in some sense dual to each other. In Subsection 8.2 we will construct a natural isomorphism of extensions $\mathcal{E}^z_{[\phi]} \cong \mathcal{E}^\phi_{[z]}$. The resulting bijections

$$\mathrm{Irr}(\tilde{S}_{\phi,[z]}, [z]) \to \mathrm{Irr}(\mathcal{E}^\phi_{[z]}, \mathrm{id}) \cong \mathrm{Irr}(\mathcal{E}^z_{[\phi]}, \mathrm{id}) \to \tilde{\Pi}_{\phi,z} \qquad (8.1)$$ {eq:cnj1tori}

will imply Conjecture 4.2. We will then go on to verify the identity claimed in Conjecture 4.7.

8.2 The isomorphism $\mathcal{E}^z_{[\phi]} \cong \mathcal{E}^\phi_{[z]}$

{sub:eiso}

We will first realize the extensions $\mathcal{E}^z_{[\phi]}$ and $\mathcal{E}^\phi_{[z]}$ explicitly as twisted products of $\mathbb{C}^\times$ with $A^{[\phi],[z]}$.

For each element $a \in A^{[\phi],[z]}$ we choose $t_a \in T(\bar{F})$ and $s_a \in \widehat{T}$ such that

$$(z, t_a) \in Z^1_a(u \to W, Z \to T \rightrightarrows T) \qquad \text{and} \qquad (\phi, s_a \rtimes a) \in Z^1_a(W_F, \widehat{T} \rightrightarrows \widehat{T}).$$

More explicitly, if we write $\phi(w) = \phi_0(w) \rtimes w$, then the above can be reformulated as

$$z(w)^{-1} \cdot a(z(w)) = t_a^{-1} \cdot \sigma_w(t_a) \qquad \text{and} \qquad \phi_0(w)^{-1} \cdot a(\phi_0(w)) = s_a^{-1} \cdot \sigma_w(s_a),$$

where $\sigma_w \in \Gamma$ is the image of $w \in W$ in the first case, and of $w \in W_F$ in the second. The choices of $t_\bullet$ and $s_\bullet$ give us sections $a \mapsto t_a \rtimes a$ and $a \mapsto s_a \rtimes a$ of the two extensions

$$1 \to T(F) \to (T(\bar{F}) \rtimes A^{[\phi]})^z \to A^{[\phi],[z]} \to 1$$

and

$$1 \to \widehat{T}^\Gamma \to (\widehat{T} \rtimes A^{[z]})^\phi \to A^{[\phi],[z]} \to 1.$$

Choose a lift $\dot{s}_a \in \widehat{\bar{T}}$ for each s_a. Then $a \mapsto \dot{s}_a \rtimes a$ is a section of the extension

$$1 \to \pi_0([\widehat{\bar{T}}]^+) \to \tilde{S}_\phi^{+,[z]} \to A^{[\phi],[z]} \to 1.$$

The 2-cocycles corresponding to the sections $a \mapsto t_a \rtimes a$ and $a \mapsto \dot{s}_a \rtimes a$ are

$$\alpha(a,b) = t_a \cdot {}^a t_b \cdot t_{ab}^{-1} \qquad \text{and} \qquad \beta(a,b) = \dot{s}_a \cdot {}^a \dot{s}_b \cdot \dot{s}_{ab}^{-1}.$$

Let $\bar{\alpha} = [\phi] \circ \alpha$ and $\bar{\beta} = [z] \circ \beta$. Then we have

$$\mathcal{E}^z_{[\phi]} = \mathbb{C}^\times \boxtimes_{\bar{\alpha}} A^{[\phi],[z]} \qquad \text{and} \qquad \mathcal{E}^\phi_{[z]} = \mathbb{C}^\times \boxtimes_{\bar{\beta}} A^{[\phi],[z]}.$$

By construction, for each $a \in A^{[\phi],[z]}$, we have $(z_0^{-1}, t_{a^{-1}}) \in Z^1(u \to W, Z \to T \xrightarrow{1-a^{-1}} T)$ and $(\phi_0^{-1}, \dot{s}_a) \in Z^1(W_F, Z \to \widehat{T} \xleftarrow{1-a} \widehat{T})$. We put

$$h(a) := \bar{\alpha}(a^{-1}, a) \cdot \left\langle (z^{-1}, t_{a^{-1}}), (\phi_0^{-1}, \dot{s}_a) \right\rangle,$$

where the pairing $\langle -, - \rangle$ is (5.3).

Proposition 8.1. *The map $x \boxtimes a \mapsto xh(a) \boxtimes a$ is an isomorphism $\mathcal{E}^z_{[\phi]} \to \mathcal{E}^\phi_{[z]}$. It is independent of the choices of t_a and $\dot{s}_a$.* {pro:isoh}

Proof. It is obvious that the map is bijective, but we need to show that it is a homomorphism. This amounts to the equation

$$h(a)h(b)h(ab)^{-1} = \bar{\alpha}(a,b)\bar{\beta}(a,b)^{-1}. \tag{8.2}$$ {eq:h}

We choose for each $a \in A$ an element $(\bar{\lambda}_a, \mu_a) \in Z_0(W_{K/F}, X_*(\bar{T}) \xrightarrow{1-a} X_*(T))_0$ whose image in $H^1(u \to W, Z \to T \xrightarrow{1-a} T)$ under the isomorphism (5.5) equals (z^{-1}, t_a). Here K/F is a suitably large Galois extension. Note that all $\bar{\lambda}_a \in Z_0(W_{K/F}, X_*(\bar{T}))_0 = X_*(\bar{T})^{N_{K/F}}$ have the same image in $X_*(\bar{T})^N / IX_*(T)$, since their images under the isomorphism $X_*(\bar{T})^N / IX_*(T) \to H^1(u \to W, Z \to$

$T)$ all equal z^{-1}. Thus we may choose a single $\bar{\lambda} \in Z_0(W_{K/F}, X_*(T))_0$ and arrange, by modifying $(\bar{\lambda}_a, \mu_a)$ by a coboundary, that $\bar{\lambda}_a = \bar{\lambda}$ for all a. Then the pairing $\langle (z^{-1}, t_{a^{-1}}), (\phi_0^{-1}, \dot{s}_a)\rangle$ is equal to

$$\langle \bar{\lambda}, \dot{s}_a \rangle \cdot \prod_{w \in W_{K/F}} \langle \mu_{a^{-1}}(w), \phi_0(w) \rangle,$$

according to the definition of (5.3) as the composition of (5.4) and (5.5). Here the angle brackets denote the canonical pairing $X_*(T) \otimes \widehat{T} \to \mathbb{C}^\times$ and its analog for $\bar{T}$.

With this, we can now compute $h(a)h(b)h(ab)^{-1}$. For $h(b)$ and $h(ab)$ we simply plug in this formula. For $h(a)$, we shall replace $(\phi_0^{-1}, \dot{s}_a)$ by the element $({}^b\phi_0^{-1}, \dot{s}_a \cdot {}^a\dot{s}_b \cdot \dot{s}_b^{-1})$, which is easily seen to be cohomologous using the fact that $(\phi_0^{-1}, \dot{s}_b) \in Z^1(W_F, \widehat{Z} \to \widehat{T} \xleftarrow{1-b} \widehat{T})$. All together we obtain

$$\begin{aligned} h(a)h(b)h(ab)^{-1} &= \bar{\alpha}(a^{-1}, a)\bar{\alpha}(b^{-1}, b)\bar{\alpha}((ab)^{-1}, ab)^{-1} \cdot \\ &\quad \langle \bar{\lambda}, \dot{s}_a \cdot {}^a\dot{s}_b \cdot \dot{s}_{ab}^{-1} \rangle \cdot \prod_w \langle {}^{b^{-1}}\mu_{a^{-1}} + \mu_{b^{-1}} - \mu_{(ab)^{-1}}, \phi_0(w) \rangle. \end{aligned}$$

Using $(\phi_0^{-1}, \dot{s}_a) \in Z^1(W_F, \widehat{Z} \to \widehat{T} \xleftarrow{1-a} \widehat{U})$ and $(\bar{\lambda}, \mu_a) \in Z_0(W_{K/F}, X_*(\bar{T}) \xrightarrow{1-a^{-1}} X_*(T))$ one checks that $\dot{s}_a \cdot {}^a\dot{s}_b \cdot \dot{s}_{ab}^{-1} \in [\widehat{\bar{T}}]^+$ and ${}^{b^{-1}}\mu_{a^{-1}} + \mu_{b^{-1}} - \mu_{(ab)^{-1}} \in Z_1(W_{K/F}, X_*(T))$. The functoriality of the maps $\dot{\psi}$ and ϕ that make up the isomorphism (5.5) implies that we have

$$\begin{aligned} h(a)h(b)h(ab)^{-1} &= \bar{\alpha}(a^{-1}, a)\bar{\alpha}(b^{-1}, b)\bar{\alpha}((ab)^{-1}, ab)^{-1} \cdot \\ &\quad \langle (z^{-1}, \dot{s}_a \cdot {}^a\dot{s}_b \cdot \dot{s}_{ab}^{-1}), (\phi_0^{-1}, t_{b^{-1}} \cdot {}^{b^{-1}}t_{a^{-1}} \cdot t_{(ab)^{-1}}^{-1}) \rangle. \end{aligned}$$

where the pairing is now between $H^1(u \to W, Z \to T \xrightarrow{0} T)$ and $H^1(W_F, \widehat{Z} \to \widehat{T} \xleftarrow{0} \widehat{T})$. Using Corollary 5.11 we see

$$h(a)h(b)h(ab)^{-1} = \bar{\alpha}(a^{-1}, a)\bar{\alpha}(b^{-1}, b)\bar{\alpha}((ab)^{-1}, ab)^{-1}\bar{\alpha}(b^{-1}, a^{-1})^{-1} \cdot \bar{\beta}(a, b)^{-1}.$$

Finally, an elementary computation using the fact that $\bar{\alpha}$ is a cocycle shows that all terms involving $\bar{\alpha}$ combine to $\bar{\alpha}(a, b)$.

It remains to show that the isomorphism $\mathcal{E}^z_{[\phi]} \to \mathcal{E}^\phi_{[z]}$ we have just constructed is independent of the choices involved in its construction, that is of the choices of elements $t_a \in T(\bar{F})$ and $\dot{s}_a \in \widehat{\bar{T}}$. For this we need to check that if we replace t_a by $x_a t_a$ with $x_a \in T(F)$, then $h(a)$ is replaced by $\langle [\phi], x_a \rangle h(a)$, and if we replace $\dot{s}_a$ by $y_a \dot{s}_a$ with $y_a \in [\widehat{\bar{T}}]^+$, then $h(a)$ is replaced by $\langle [z], y_a \rangle^{-1} h(a)$. Both of these verifications are immediate. □

8.3 Remarks and generalizations

Before we continue with the proof of Conjecture 5.12 for tori, we would like to point out a beautiful symmetry between $\tilde{T}_z(F)$ and $\tilde{S}_\phi^{[z]}$ that may have become covered under the debris of generality. To see it more clearly, let us consider the special case where the Langlands parameter ϕ extends to the Galois group (thus it corresponds to a character of $T(F)$ whose composition with the norm map $N_{K/F} : T(K) \to T(F)$ is trivial for some finite extension K/F) and the inner form of $T \rtimes A$ we are considering is pure. As above we shall write $\phi : \Gamma \to \widehat{T} \rtimes \Gamma$ for the Langlands parameter, and $\phi_0 : \Gamma \to \widehat{T}$ for the corresponding cocycle, so that $\phi(\sigma) = \phi_0(\sigma) \rtimes \sigma$. We shall use the analogous notation $z : \Gamma \to T \rtimes \Gamma$ and $z_0 : \Gamma \to T$ for the pure inner form, slightly deviating from the notation of the rest of the paper, where we used z and $\tilde{z}$ instead. We are writing T for $T(\bar{F})$, in the same way we are writing $\widehat{T}$ for $\widehat{T}(\mathbb{C})$.

Now $\tilde{T}_z(F) = (T \rtimes A)^{z(\Gamma)}$ and $S_\phi = (\widehat{T} \rtimes A)^{\phi(\Gamma)}$. These fit into the extensions

$$1 \to T^{z(\Gamma)} \to (T \rtimes A)^{z(\Gamma)} \to A^{[z]} \to 1$$

and

$$1 \to \widehat{T}^{\phi(\Gamma)} \to (T \rtimes A)^{\phi(\Gamma)} \to A^{[\phi]} \to 1.$$

We have written $T^{z(\Gamma)}$ for $T^\Gamma = T(F)$ and $\widehat{T}^{\phi(\Gamma)} = \widehat{T}^\Gamma$ to emphasize the symmetry. Now $[\phi]$ is a character of $T^{z(\Gamma)}$ and $[z]$ is a character of $T^{\phi(\Gamma)}$. We pull back the above extensions along the inclusions of $A^{[z],[\phi]}$ into $A^{[z]}$ and $A^{[\phi]}$ and obtain the push-out diagrams

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^{z(\Gamma)} & \longrightarrow & (T \rtimes A)^{z(\Gamma),[\phi]} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle [\phi]} & & & & & & \\ & & \mathbb{C}^\times & & & & & & \end{array}$$

and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{T}^{\phi(\Gamma)} & \longrightarrow & (\widehat{T} \rtimes A)^{\phi(\Gamma),[z]} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle [z]} & & & & & & \\ & & \mathbb{C}^\times & & & & & & \end{array}$$

Which lead to the extensions $\mathcal{E}^z_{[\phi]}$ and $\mathcal{E}^\phi_{[z]}$ of $A^{[z],[\phi]}$ by $\mathbb{C}^\times$. The symmetry of the situation now makes it rather natural to expect that these two extensions are closely related.

Moving towards the opposite end on the spectrum of clarity, we are now going to formulate a situation a bit more general then the one considered in Subsection 8.2. We will not need this generalization in the present paper, but will need it in a forthcoming paper in a rather different set-up.

Let T be an algebraic torus T defined over F, and A a finite group acting on T by F-automorphisms. Let $Z \subset T$ be a finite subgroup defined over F and fixed pointwise by A. Let $\phi : W_F \to \widehat{T} \rtimes W_F$ and $z \in Z^1(u \to W, Z \to T)$. Write $\widehat{\bar{T}}$ for the complex dual group of $\bar{T} = T/Z$.

Instead of considering the split extensions $T \rtimes A$ and $\widehat{\bar{T}} \rtimes A$, we now assume given extensions $1 \to T \to \tilde{T} \to A \to 1$ and $1 \to \widehat{\bar{T}} \to \bar{\mathcal{T}} \to A \to 1$ that may or may not be split. Dividing out by $\widehat{Z}$ we obtain an extension $1 \to \widehat{T} \to \mathcal{T} \to A \to 1$. We emphasize that no relation is assumed between $\tilde{T}$ and $\mathcal{T}$. We assume that after taking F-points the sequence $1 \to T(F) \to \tilde{T}(F) \to A^\Gamma \to 1$ is still exact, and after taking Γ-invariants the sequence $1 \to \widehat{T}^\Gamma \to \mathcal{T}^\Gamma \to A^\Gamma \to 1$ remains exact. Let $[\bar{\mathcal{T}}]^+$ be the preimage of $\mathcal{T}^\Gamma$ in $\bar{\mathcal{T}}$.

Let $1 \to \mathbb{C}^\times \to \mathcal{E}^0_{[\phi]} \to A^{[\phi]} \to 1$ be the push-out of $1 \to T(F) \to \tilde{T}(F)^{[\phi]} \to A^{[\phi]} \to 1$ along $[\phi] : T(F) \to \mathbb{C}^\times$. Let $1 \to \mathbb{C}^\times \to \mathcal{E}^0_{[z]} \to A^{[z]} \to 1$ be the push-out of $1 \to [\hat{\bar{T}}]^+ \to [\bar{\mathcal{T}}]^{+,[z]} \to A^{[z]} \to 1$ along $[z] : [\hat{\bar{T}}]^+ \to \mathbb{C}^\times$.

We now consider the inner form $\tilde{T}_z$. We have $\tilde{T}_z(F) = \{\tilde{t} \in \tilde{T}(\bar{F}) | \forall \sigma \in \Gamma : \tilde{t} = \mathrm{Ad}(\bar{z}(\sigma))\sigma(\tilde{t})\}$, where again $\bar{z} \in Z^1(\Gamma, \bar{T})$ is the image of z, and we are using that the conjugation action of T on $\tilde{T}$ factors through $\bar{T}$ because Z is pointwise fixed by A. The assumption that $\tilde{T}(F) \to A^\Gamma$ is surjective implies that $\tilde{T}_z(F) \to A^{[z]}$ is surjective, where again $A^{[z]}$ is the stabilizer in A of the class $[z] \in H^1(u \to W, Z \to T)$. Thus we have the extension

$$1 \to T(F) \to \tilde{T}_z(F) \to A^{[z]} \to 1.$$

We pull back along the inclusion $A^{[z],[\phi]} \to A^{[z]}$ and push out along $[\phi] : T(F) \to \mathbb{C}^\times$ to obtain an extension $1 \to \mathbb{C}^\times \to \mathcal{E}^z_{[\phi]} \to A^{[z],[\phi]} \to 1$.

Dually we consider the centralizer $S_\phi = \mathcal{T}^{\phi(W_F)}$ of ϕ in $\mathcal{T}$. Again the assumption that $\mathcal{T}^\Gamma \to A^\Gamma$ is surjective implies that $S_\phi \to A^{[\phi]}$ is surjective. Let S^+_ϕ be the preimage of S_ϕ in $\bar{\mathcal{T}}$, so that we have the extension

$$1 \to [\hat{\bar{T}}]^+ \to S^+_\phi \to A^{[\phi]} \to 1.$$

We pull back along the inclusion $A^{[z],[\phi]} \to A^{[\phi]}$ and push out along $[z] : [\hat{\bar{T}}]^+ \to \mathbb{C}^\times$ to obtain an extension $1 \to \mathbb{C}^\times \to \mathcal{E}^\phi_{[z]} \to A^{[z],[\phi]} \to 1$.

Proposition 8.2. *Let $\mathcal{E}^{0,[z]}_{[\phi]}$ and $\mathcal{E}^{0,[\phi]}_{[z]}$ be the pull-backs of $\mathcal{E}^0_{[\phi]}$ and $\mathcal{E}^0_{[z]}$ along the inclusions of $A^{[z],[\phi]}$ into $A^{[\phi]}$ and $A^{[z]}$. An isomorphism of extensions $\zeta : \mathcal{E}^{0,[z]}_{[\phi]} \to \mathcal{E}^{0,[\phi]}_{[z]}$ determines an isomorphism of extensions $\xi : \mathcal{E}^z_{[\phi]} \to \mathcal{E}^\phi_{[z]}$. If ζ is multiplied by a character $A^{[z],[\phi]} \to \mathbb{C}^\times$, then ξ is multiplied by the same character.*

Proof. To lighten notation, we replace A by its subgroup $A^{[z],[\phi]}$. For each $a \in A$ choose lifts $\theta_a \in \tilde{T}(F)$ and $\tau_a \in [\bar{\mathcal{T}}]^+$, as well as elements $t_a \in T(\bar{F})$ such that $t_a\theta_a \in \tilde{T}_z(F)$ and $\dot{s}_a \in \hat{\bar{T}}$ such that $\dot{s}_a\tau_a \in S^+_\phi$.

The section $a \mapsto t_a\theta_a$ realizes the extension $\mathcal{E}^z_{[\phi]}$ as the twisted product $\mathbb{C}^\times \boxtimes_{\bar{\alpha}'} A$, where $\alpha' \in Z^2(A, T(F))$ is defined as $\alpha'(a,b) = t_a\theta_a t_b\theta_b(t_{ab}\theta_{ab})^{-1}$ and $\bar{\alpha}' = [\phi] \circ \alpha' \in Z^2(A, \mathbb{C}^\times)$. The section $a \mapsto \dot{s}_a\tau_a$ realizes the extension $\mathcal{E}^\phi_{[z]}$ as the twisted product $\mathbb{C}^\times \boxtimes_{\bar{\beta}'} A$, where $\beta' \in Z^2(A, [\hat{\bar{T}}]^+)$ is defined as $\beta'(a,b) = \dot{s}_a\tau_a\dot{s}_b\tau_b(\dot{s}_{ab}\tau_{ab})^{-1}$ and $\bar{\beta}' = [z] \circ \beta' \in Z^2(A, \mathbb{C}^\times)$.

We have $\alpha'(a,b) = \alpha(a,b) \cdot \alpha_0(a,b)$ with $\alpha(a,b) = t_a \cdot {}^a t_b \cdot t_{ab}^{-1}$ and $\alpha_0(a,b) = \theta_a\theta_b\theta_{ab}^{-1}$. The element $\alpha_0 \in Z^2(A, T(F))$ is the 2-cocycle corresponding to the section $a \mapsto \theta_a$, which then identifies $\mathcal{E}^{0,[z]}_{[\phi]}$ with $\mathbb{C}^\times \boxtimes_{\bar\alpha_0} A$. Analogously we have $\beta'(a,b) = \beta(a,b)\beta_0(a,b)$ with $\beta(a,b) = \dot s_a{}^a \dot s_b \dot s_{ab}^{-1}$ and $\beta_0(a,b) = \tau_a\tau_b\tau_{ab}^{-1}$. The element $\beta_0 \in Z^2(A, \hat{\bar{T}})$ is the 2-cocycle corresponding to the section $a \mapsto \tau_a$, which then identifies $\mathcal{E}^{0,[\phi]}_{[z]}$ with $\mathbb{C}^\times \boxtimes_{\bar\beta_0} A$.

Let $\zeta : \mathcal{E}^{0,[z]}_{[\phi]} \to \mathcal{E}^{0,[\phi]}_{[z]}$ be an isomorphism of extensions. The composition $\mathbb{C}^\times \boxtimes_{\bar\alpha_0} A \to \mathcal{E}^{0,[z]}_{[\phi]} \to \mathcal{E}^{0,[\phi]}_{[z]} \to \mathbb{C}^\times \boxtimes_{\bar\beta_0} A$ is given by $x \boxtimes a \mapsto x\zeta_0(a) \boxtimes a$, where $\zeta_0 : A \to \mathbb{C}^\times$ is defined as $\zeta_0(a) = \zeta(\bar\theta_a)\bar\tau_a^{-1}$ and satisfies $\zeta_0(a)\zeta_0(b)\zeta_0(ab)^{-1} = \bar\alpha_0(a,b)^{-1}\bar\beta_0(a,b)$. Here $\bar\theta_a \in \mathcal{E}^{0,[z]}_{[\phi]}$ and $\bar\tau_a \in \mathcal{E}^{0,[\phi]}_{[z]}$ are the images of θ_a and τ_a respectively.

Let $h : A \to \mathbb{C}^\times$ be defined as in Subsection 8.2. We claim that

$$\mathbb{C}^\times \boxtimes_{\bar\alpha\bar\alpha_0} A \to \mathbb{C}^\times \boxtimes_{\bar\beta\bar\beta_0} A, \qquad x \boxtimes a \mapsto h(a)\zeta_0(a) \boxtimes a$$

is an isomorphism of extensions and the composition

$$\xi : \mathcal{E}^z_{[\phi]} \to \mathbb{C}^\times \boxtimes_{\bar\alpha\bar\alpha_0} A \to \mathbb{C}^\times \boxtimes_{\bar\beta\bar\beta_0} A \to \mathcal{E}^\phi_{[z]}$$

depends only on ζ, and not on the choices of θ_a, τ_a, t_a, or $\dot s_a$.

The first part of the claim is equivalent to $h(a)h(b)h(ab)^{-1} = \bar\alpha(a,b)^{-1}\bar\beta(a,b)$, which was the content of the proof of Proposition 8.1. This proof remains valid verbatim in the current situation. For the second claim, the independence of the choices of t_a and $\dot s_a$ was already addressed in the proof of Proposition 8.1. Now say we replace θ_a by $x_a\theta_a$ and τ_a by $\dot y_a\tau_a$, for $x_a \in T(F)$ and $\dot y_a \in [\hat{\bar{T}}]^+$. Since we already have independence of t_a and $\dot s_a$, we are free to replace t_a by $x_a^{-1}t_a$ and $\dot s_a$ by $\dot y_a^{-1}\dot s_a$. This has the effect of keeping $\bar\alpha\bar\alpha_0$ and $\bar\beta\bar\beta_0$, as well as the first and third arrows in the last displayed sequence, unchanged. At the same time, $h(a)$ is replaced by $h(a)\langle[\phi], x_a^{-1}\rangle\langle[z], \dot y_a\rangle$, while $\zeta_0(a)$ is replaced by $\zeta_0(a)\langle[\phi], x_a\rangle\langle[z], \dot y_a^{-1}\rangle$, so the middle arrow is unchanged as well.

Finally, if ζ is replaced by $\delta\zeta$, then ζ_0 is replaced by $(\delta\zeta)_0$ specified by $(\delta\zeta)_0(a) = (\delta\zeta)(\bar\theta_0)\bar\tau_a^{-1} = \delta(a)\zeta(\bar\theta_0)\bar\tau_a^{-1} = \delta(a)\zeta_0(a)$. If follows that the isomorphism $\mathbb{C}^\times \boxtimes_{\bar\alpha\bar\alpha_0} A \to \mathbb{C}^\times \boxtimes_{\bar\beta\bar\beta_0} A$ is multiplied by δ, and the same is then true for ξ. □

8.4 Computing the right-hand side of (4.4)

In this section we will compute the virtual character

$$\Theta^{\tilde s}_\phi := \sum_\rho \mathrm{tr}\rho(\tilde s) \cdot \Theta_{\pi_\rho}$$

where ρ runs over the set $\mathrm{Irr}(\pi_0(\tilde{S}_\phi^{[z]}), [z])$. We recall from Subsections 8.1 and 8.2 that we have the following diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & T(F) & \longrightarrow & (T(\bar{F}) \rtimes A)^{\tilde{\bar{z}}(\Gamma)} & \longrightarrow & A^{[z]} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & T(F) & \longrightarrow & (T(\bar{F}) \rtimes A^{[\phi]})^{\tilde{\bar{z}}(\Gamma)} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle [\phi]} & & \downarrow{\scriptstyle F} & & \| & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{E}^z_{[\phi]} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle H} & & \| & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{E}^\phi_{[z]} & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1 \\
 & & {\scriptstyle [z]}\uparrow & & {\scriptstyle G}\uparrow & & \| & & \\
1 & \longrightarrow & \pi_0([\hat{\bar{T}}]^+) & \longrightarrow & \pi_0(\tilde{S}^+_\phi) & \longrightarrow & A^{[z],[\phi]} & \longrightarrow & 1
\end{array}
$$

We can be more explicit about the maps F, G, and H. Recall from Subsection 8.2 that we have fixed elements $t_a \in T(\bar{F})$ and $\dot{s}_a \in \hat{\bar{T}}$ such that $a \mapsto t_a \rtimes a$ and $a \mapsto \dot{s}_a \rtimes a$ are sections of the top and bottom extensions. The corresponding 2-cocycles are

$$\alpha(a,b) = t_a \cdot {}^a t_b \cdot t_{ab}^{-1} \qquad \text{and} \qquad \beta(a,b) = \dot{s}_a \cdot {}^a\dot{s}_b \cdot \dot{s}_{ab}^{-1}.$$

and setting $\bar{\alpha} = [\phi] \circ \alpha$ and $\bar{\beta} = [z] \circ \beta$ allows us to make the identifications

$$\mathcal{E}^z_{[\phi]} = \mathbb{C}^\times \boxtimes_{\bar{\alpha}} A^{[\phi],[z]} \qquad \text{and} \qquad \mathcal{E}^\phi_{[z]} = \mathbb{C}^\times \boxtimes_{\bar{\beta}} A^{[\phi],[z]}.$$

The maps F, G, and H are now explicitly given by

$$F(t \cdot t_a \rtimes a) = [\phi](t) \boxtimes a, \qquad G(\dot{s} \cdot \dot{s}_b \rtimes b) = [z](s) \boxtimes b, \qquad H(x \boxtimes a) = xh(a) \boxtimes a.$$

We now fix $t \cdot t_a \rtimes a \in \tilde{T}_z(F)$ and $\dot{s} \cdot \dot{s}_b \rtimes b \in \tilde{S}_\phi^{[z]}$ and set out to compute

$$\Theta_\phi^{s \cdot s_b \rtimes b}(t \cdot t_a \rtimes a).$$

Since this is a virtual character of $(T(\bar{F}) \rtimes A)^z$ which is induced from a virtual character of $(T(\bar{F}) \rtimes A^{[\phi]})^z$, there is no loss of generality if we assume $t \cdot t_a \rtimes a \in (T(\bar{F}) \rtimes A^{[\phi]})^z$, which simply means $a \in A^{[\phi],[z]}$. Then, by construction, we have

$$
\begin{aligned}
O_\phi^{s \cdot s_b \rtimes b}(t \cdot t_a \rtimes a) &= \sum_{\tau \in \mathrm{Irr}(\mathbb{C}^\times \boxtimes_{\bar{\beta}} A^{[\phi],[z]}, \mathrm{id})} \chi_\tau(G(\dot{s} \cdot \dot{s}_b \rtimes b)) \qquad (8.3) \\
&\cdot \quad |A^{[\phi],[z]}|^{-1} \sum_{\substack{c \in A^{[z]} \\ cac^{-1} \in A^{[\phi],[z]}}} \chi_\tau(HF((t_c \rtimes c)(t \cdot t_a \rtimes a)(t_c \rtimes c)^{-1}))
\end{aligned}
$$

{eq:os1}

where χ_τ denotes the character of the finite dimensional representation τ and the second line is the Frobenius formula for the character of the representation on $\tilde{T}_z(F)$ induced from $\tau \circ HF$.

We compute

$$(t_c \rtimes c)(t \cdot t_a \rtimes a)(t_c \rtimes c)^{-1} = {}^c t \cdot \zeta(c,a) \cdot t_{cac^{-1}} \rtimes cac^{-1},$$

where

$$\zeta(c,a) := {}^c t_a t_c {}^{(cac)^{-1}} t_c^{-1} t_{cac^{-1}}^{-1} \in T(F).$$

With this we have

$$\Theta_\phi^{\dot{s}\cdot\dot{s}_b \rtimes b}(t \cdot t_a \rtimes a) = [z](\dot{s}) \sum_c [\phi]({}^c t \zeta(c,a)) h(cac^{-1}) |A^{[\phi],[z]}|^{-1} \sum_\tau \chi_{\bar{\tau}}(b) \chi_{\bar{\tau}}(cac^{-1}).$$

We now apply Lemma C.1 to the sum over τ and conclude that if cac^{-1} is not conjugate to b^{-1} then the corresponding summand is zero. It is more convenient to apply this information not to the expression we just obtained, but to the original expression we started with, namely (8.3). This allows us rewrite that expression as

$$|A^{[\phi],[z]}|^{-1} |Z_{A^{[\phi],[z]}}(b^{-1})|^{-1} \sum_\tau \chi_\tau(G(\dot{s} \cdot \dot{s}_b \rtimes b))$$
$$\sum_{\substack{y \in A^{[\phi],[z]} \\ c \in A^{[z]} \\ cac^{-1} = yb^{-1}y^{-1}}} \chi_\tau(HF((t_c \rtimes c)(t \cdot t_a \rtimes a)(t_c \rtimes c)^{-1}))$$

and making the substitution $c \mapsto yc$ this equals

$$|A^{[\phi],[z]}|^{-1} |Z_{A^{[\phi],[z]}}(b^{-1})|^{-1} \sum_\tau \chi_\tau(G(\dot{s} \cdot \dot{s}_b \rtimes b))$$
$$\sum_{\substack{y \in A^{[\phi],[z]} \\ c \in A^{[z]} \\ cac^{-1} = b^{-1}}} \chi_\tau(HF((t_{yc} \rtimes yc)(t \cdot t_a \rtimes a)(t_{yc} \rtimes yc)^{-1}))$$

Since the images of $t_c \rtimes c \in \tilde{T}_z(F)$ and $t_{yc} \rtimes yc \in \tilde{T}_z(F)$ in the quotient

$$(T(\bar{F}) \rtimes A^{[\phi]})^z \setminus (T(\bar{F}) \rtimes A)^z \cong A^{[\phi],[z]} \setminus A^{[z]}$$

are equal, the character of $\tau \circ HF$ will remain unchanged if we replace yc by y. Doing this leads to the expression

$$\begin{aligned} \Theta_\phi^{\dot{s}\cdot\dot{s}_b \rtimes b}(t \cdot t_a \rtimes a) &= |Z_{A^{[\phi],[z]}}(b^{-1})|^{-1} \sum_\tau \chi_\tau(G(\dot{s} \cdot \dot{s}_b \rtimes b)) \\ &\quad \sum_{\substack{c \in A^{[z]} \\ cac^{-1} = b^{-1}}} \chi_\tau(HF((t_c \rtimes c)(t \cdot t_a \rtimes a)(t_c \rtimes c)^{-1})). \end{aligned}$$

The same analysis as for (8.3) now leads to

$$\Theta_\phi^{\dot s\cdot\dot s_b\rtimes b}(t\cdot t_a\rtimes a)=[z](\dot s)\sum_c[\phi](^ct\zeta(c,a))h(cac^{-1})|Z_{A^{[\phi],[z]}}(b^{-1})|^{-1}\sum_\tau\chi_{\bar\tau}(b)\chi_{\bar\tau}(cac^{-1}).$$

Since now $cac^{-1}=b^{-1}$ we can apply Lemma C.1 and, recalling the definition of $h(b)$ from Subsection 8.2, we obtain

$$\begin{aligned}\Theta_\phi^{\dot s\cdot\dot s_b\rtimes b}(t\cdot t_a\rtimes a) &= [z](\dot s)h(b^{-1})\bar\beta(b,b^{-1})\sum_c[\phi](^ct\zeta(c,a))\\ &= [z](\dot s)h(b)^{-1}\bar\alpha(b,b^{-1})\sum_c[\phi](^ct\zeta(c,a))\\ &= [z](\dot s)\left\langle(z^{-1},t_{b^{-1}}),(\phi^{-1},\dot s_b)\right\rangle^{-1}\sum_c[\phi](^ct\zeta(c,a))\end{aligned}$$

8.5 Computing the left-hand side of (4.4)

We will now compute the lift to $\tilde T_z(F)$ of the virtual character $S\Theta_{\phi^{\mathfrak e}}$. Recall that we have fixed an element $\dot{\tilde s}=\dot s\cdot\dot s_b\rtimes b\in\tilde S_\phi^{[z]}$ and $\mathfrak e$ is the endoscopic triple for the twisted group (G,b^{-1}) corresponding to $\tilde s$ and ϕ and augmented by a choice of an L-embedding $\xi^{\mathfrak e}:{}^LG^{\mathfrak e}\to{}^LG$ whose image contains the image of ϕ. In our special case of $G=T$, we have $G^{\mathfrak e}=T_{b^{-1}}$ and we can choose $\xi^{\mathfrak e}:[\widehat T]^{b,\circ}\rtimes W_F\to\widehat T\rtimes W_F$ to be given by $(t,w)\mapsto t\cdot\phi(w)$. With this choice, $\phi^{\mathfrak e}(w)=1\rtimes w$ and hence $S\Theta_{\phi^{\mathfrak e}}$ is the trivial character of $T_{b^{-1}}(F)$. On the other hand, the function $f^{\dot{\mathfrak e}}$ from Lemma 4.6 is given by

$$\begin{aligned}f^{\dot{\mathfrak e}}(\gamma^{\mathfrak e}) &= \sum_{\tilde\delta\in\tilde T_z(F)/\tilde T_z(F)-conj}\Delta(\gamma^{\mathfrak e},\tilde\delta)\int_{\tilde x\in\tilde T_z(F)/\tilde T_z(F)_{\tilde\delta}}f(\tilde x\tilde\delta\tilde x^{-1})d\tilde x\\ &= \sum_{\tilde\delta\in\tilde T_z(F)/T_z(F)-conj}\Delta(\gamma^{\mathfrak e},\tilde\delta)\int_{x\in T_z(F)/T_z(F)_{\tilde\delta}}f(\tilde x\tilde\delta\tilde x^{-1})d\tilde x,\end{aligned}$$

where Δ is the transfer factor determined by $\dot{\mathfrak e}$ and the fixed L-embedding (there is no Whittaker datum since we are dealing with tori). Thus the lift of $S\Theta_{\phi^{\mathfrak e}}$ evaluated at f is equal to

$$\begin{aligned}&\int_{\gamma\in T_{b^{-1}}(F)}f^{\dot{\mathfrak e}}(\gamma)d\gamma\\ =&\int_\gamma\sum_{\tilde\delta\in\tilde T_z(F)/T_z(F)-conj}\Delta(\gamma,\tilde\delta)\int_{\tilde x\in T_z(F)/T_z(F)_\delta}f(\tilde x\tilde\delta\tilde x^{-1})d\tilde x\\ =&\int_\gamma\sum_{\tilde\delta\in\tilde T_z(F)/T_z(F)-conj}\sum_{c\in\tilde T_z(F)/T_z(F)}\Delta_{KS}(\gamma,c\tilde\delta c^{-1})\int_{\tilde x\in T_z(F)/T_z(F)_{\tilde\delta}}f(\tilde x\tilde\delta\tilde x^{-1})d\tilde x\end{aligned}$$

Recall that Δ_{KS} is supported in the variable $\tilde{\delta}$ on the coset $(G \rtimes b^{-1})_z(F)$. We obtain

$$= \int_\gamma \sum_{a\in A^{[z]}} \sum_{\tilde{\delta}\in[T\rtimes a]_z(F)/T_z(F)-conj} \sum_{\substack{c\in\tilde{T}_z(F)/T_z(F)\\ cac^{-1}=b^{-1}}} \Delta_{KS}(\gamma, c\tilde{\delta}c^{-1}) \int_{\tilde{x}\in T_z(F)/T_z(F)_{\tilde{\delta}}} f(\tilde{x}\tilde{\delta}\tilde{x}^{-1})d\tilde{x}$$

We interchange the integral over γ with the sums over a and c. Moreover, as γ runs over $T_{b^{-1}}(F)$, $c^{-1}\gamma c$ runs over $T_a(F)$. We make the substitution $\gamma \mapsto c^{-1}\gamma c$ and arrive at

$$= \sum_{a\in A^{[z]}} \int_{\gamma\in T_a(F)} \sum_{\tilde{\delta}\in[T\rtimes a]_z(F)/T_z(F)-conj} \sum_{\substack{c\in\tilde{T}_z(F)/T_z(F)\\ cac^{-1}=b^{-1}}} \Delta_{KS}(c\gamma c^{-1}, c\tilde{\delta}c^{-1}) \int_{\tilde{x}\in T_z(F)/T_z(F)_{\tilde{\delta}}} f(\tilde{x}\tilde{\delta}\tilde{x}^{-1})d\tilde{x}$$

Now $\Delta_{KS}(\gamma, \tilde{\delta})$, in our special case of tori, is non-zero if and only if $\tilde{\delta} = \delta \rtimes b^{-1}$ and the image of δ in $T_b(F)$ equals γ. Thus the function

$$\Phi(\tilde{\delta}) = \sum_{\substack{c\in\tilde{T}_z(F)/T_z(F)\\ cac^{-1}=b^{-1}}} \Delta_{KS}(c\gamma c^{-1}, c\tilde{\delta}c^{-1})$$

depends only on $\tilde{\delta}$, as γ can be recovered from $\tilde{\delta}$. We arrive at the formula

$$\sum_{a\in A^{[z]}} \int_{\gamma\in T_a(F)} \sum_{\substack{\tilde{\delta}\in[T\rtimes a]_z(F)/T_z(F)-conj\\ \delta\mapsto\gamma}} \Phi(\tilde{\delta}) \int_{\tilde{x}\in T_z(F)/T_z(F)_\delta} f(\tilde{x}\delta\tilde{x}^{-1})d\tilde{x}$$

Since $\Phi(\tilde{\delta})$ is conjugation-invariant under $T_z(F)$, we obtain

$$\sum_{a\in A^{[z]}} \int_{\gamma\in T_a(F)} \sum_{\substack{\tilde{\delta}\in[T\rtimes a]_z(F)/T_z(F)-conj\\ \delta\mapsto\gamma}} \int_{\tilde{x}\in T_z(F)/T_z(F)_\delta} \Phi(\tilde{x}\tilde{\delta}\tilde{x}^{-1}) f(\tilde{x}\tilde{\delta}\tilde{x}^{-1})d\tilde{x}$$

A simple integration formula now shows that this is equal to

$$\int_{\tilde{\delta}\in\tilde{T}_z(F)} \Phi(\tilde{\delta})f(\tilde{\delta}).$$

We conclude that the lift of $S\Theta_{\phi^\mathfrak{c}}$ to $\tilde{T}_z(F)$ is represented by the function Φ. We have

$$\begin{aligned}
\Phi(t \cdot t_a \rtimes a) &= \sum_{\substack{c \in \tilde{T}_z(F)/T_z(F) \\ cac^{-1}=b^{-1}}} \Delta_{KS}(c\gamma c^{-1}, c(t \cdot t_a \rtimes a)c^{-1}) \\
&= \sum_{\substack{c \in \tilde{T}_z(F)/T_z(F) \\ cac^{-1}=b^{-1}}} \Delta_{KS}(c\gamma c^{-1}, (t_c \rtimes c)(t \cdot t_a \rtimes a)(t_c \rtimes c)^{-1}) \\
&= \sum_{\substack{c \in \tilde{T}_z(F)/T_z(F) \\ cac^{-1}=b^{-1}}} \Delta_{KS}(c\gamma c^{-1}, {}^c t\zeta(c,a)t_{cac^{-1}} \rtimes cac^{-1}) \\
&= \sum_{\substack{c \in A^{[z]} \\ cac^{-1}=b^{-1}}} \langle (z^{-1}, {}^c t\zeta(c,a)t_{b^{-1}}), (\phi_0^{-1}, \dot{s}\dot{s}_b)\rangle^{-1} \\
&= \langle (z^{-1}, t_{b^{-1}}), (\phi_0^{-1}, \dot{s}\dot{s}_b)\rangle^{-1} \sum_{\substack{c \in A^{[z]} \\ cac^{-1}=b^{-1}}} [\phi]({}^c t\zeta(c,a)).
\end{aligned}$$

The final expression is equal to the formula for $\Theta_\phi^{\dot{s}\cdot\dot{s}_b \rtimes b}(t \cdot t_a \rtimes a)$ obtained in the previous section. The proof of Conjecture 4.7 in the case of tori is now complete.

Appendix

A FUNCTORIALITY OF THE LOCAL CORRESPONDENCE FOR CONNECTED GROUPS

{app:func}

Let $\phi : L_F \to {}^L G$ be a tempered Langlands parameter. As in [Kal16b, §5.4] and [Kal18, §4.1] we expect to have a compound L-packet Π_ϕ and a commutative diagram

$$\begin{array}{ccc}
\Pi_\phi & \longrightarrow & \mathrm{Irr}(S_\phi^+) \\
\downarrow & & \downarrow \\
H^1(u \to W, Z(G) \to G) & \longrightarrow & \pi_0(Z(\widehat{\overline{G}})^+)^*
\end{array}$$

Recall here that Π_ϕ is a subset of the set Π_{temp} of tempered representations of rigid inner twists, that consists of tuples (G_z, ξ, z, π), where $\xi : G \to G_z$ is an inner twist, $z \in Z^1(u \to W, Z(G) \to G)$ is such that $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(\bar{z}(\sigma))$, where $\bar{z} \in Z^1(\Gamma, G_{\text{ad}})$ is the image of z under the natural projection $G \to G_{\text{ad}}$, and π is an irreducible tempered representation of $G_z(F)$.

The group A acts on $Z^1(u \to W, Z(G) \to G)$ by $a(z)(w) = a(z(w))$. Given rigid inner twists $(\xi_i, z_i) : G \to G_i$ for $i = 1, 2$ and $a \in A$ such that $z_2 = a(z_1)$ one checks that the isomorphism $b := \xi_2 \circ a \circ \xi_1^{-1} : G_1 \to G_2$ is defined over

F. More generally, if $a(z_1)$ and z_2 are cohomologous and one chooses $h \in G$ with $z(w) = h^{-1}a(z(w))\sigma_w(h)$, then $b := \xi_2 \circ \mathrm{Ad}(h) \circ a \circ \xi^{-1}$ is defined over F. A different choice of h will change b only by an inner automorphism coming from $G_1(F)$.

Seen from a slightly different perspective, this can be formulated as an action of A on the category of rigid inner twists of G, namely $a(\xi, z) = (\xi \circ a^{-1}, a(z))$. This action can be upgraded to an action of A on the set Π_{temp} by $a(G_z, \xi, z, \pi) = (G_z, \xi \circ a^{-1}, a(z), \pi)$.

Consider now the dual side. Given a tempered Langlands parameter $\phi : L_F \to {}^LG$ and $\rho \in \mathrm{Irr}(S_\phi^+)$ we obtain $a\phi := a \circ \phi : L_F \to {}^LG$ and $a\rho := \rho \circ a^{-1} \in \mathrm{Irr}(S_{a\phi}^+)$. Thus A acts on the space of refined Langlands parameters.

It is reasonable to expect that the above commutative diagram is natural with respect to this action. More precisely:

{cnj:func}

Conjecture A.1. *If $\dot\pi \in \Pi_{temp}$ corresponds to (ϕ, ρ), then $a\dot\pi$ corresponds to $(a\phi, a\rho)$. Formulated equivalently, if (G_1, ξ_1, z_1, π_1) and (G_2, ξ_2, z_2, π_2) correspnd to (ϕ, ρ) and $(a\phi, a\rho)$ respectively, then the isomorphism $b : G_1 \to G_2$ constructed above identifies π_1 with π_2.*

In the special case of a rigid inner twist (G_z, ξ, z) for which the cohomology class of z is fixed by a, in particular in the case $z = 1$ where $G_z = G$, this amounts to a compatibility with automorphims of the refined local Langlands correspondence for the group G_z. However, the above statement applies even to inner forms of G which do not admit a as an automorphism defined over F.

B Automorphisms of Weil-restricted groups

{app:weil}

Let E/F be a finite extension, $\Delta = \mathrm{Gal}(\bar F/E) \subset \Gamma = \mathrm{Gal}(\bar F/F)$. Let G be an absolutely simple connected reductive E-group. Let a be an automorphism of $H = \mathrm{Res}_{E/F}G$. Recall the natural identification $H(\bar F) = \mathrm{Ind}_\Delta^\Gamma G(\bar F)$. For every $\sigma \in \Gamma$ let ${}^\sigma E$ be the subfield $\sigma(E)$ of $\bar F$ and let G^σ be the ${}^\sigma E$-group obtained by twisting the rational structure, i.e. $G^\sigma = G \times_{\mathrm{Spec}(E)} \mathrm{Spec}({}^\sigma E)$, where we have used the map $\sigma : E \to {}^\sigma E$.

{lem:weilauto}

Lemma B.1. *There exists $\sigma_0 \in N_\Gamma(\Delta)$ and an isomorphism $a' : G \to G^{\sigma_0}$ such that*

$$a(f)(\sigma) = a'(f(\sigma_0^{-1}\sigma)), \qquad \forall f \in H(\bar F) = \mathit{Ind}_\Delta^\Gamma G(\bar F),\ \ \forall \sigma \in \Gamma.$$

The Δ-coset of σ_0 is unique and a' is uniquely determined by the choice of σ_0 within its Δ-coset. If σ_0 is replaced by $\tau\sigma_0$ with $\tau \in \Delta$ then a' is replaced by $\tau \circ a'$.

Proof. Choose a set of representatives $\sigma_1, \dots, \sigma_n$ for $\Delta \setminus \Gamma$ and arrange $\sigma_1 = 1$. Then $f \mapsto (f(\sigma_1), \dots, f(\sigma_n))$ is an isomorphism $H(\bar F) \to \prod_{i=1}^n G(\bar F)$ of algebraic groups. It translates the automorphism a to an automorphism of $\prod_{i=1}^n G(\bar F)$. Such an automorphism must map each factor in the product to another factor. In this way we obtain a permutation p of the set $\Delta \setminus \Gamma$ which has the property that if $f \in H(\bar F)$ is a function supported on the coset $\Delta\sigma$, then

$a(f)$ is a function supported on the coset $\Delta p(\sigma)$. Since a is an F-automorphism, the permutation p is Γ-equivariant, i.e. $p(\sigma\gamma) = p(\sigma)\gamma$. It follows that there exists $\sigma_0 \in N_\Gamma(\Delta)$ such that $p(\gamma) = \sigma_0\gamma$, and the Δ-coset of σ_0 is unique.

Given $g \in G(\bar{F})$ and $\sigma \in \Gamma$ let $g^{\delta_\sigma} \in H(\bar{F})$ be the unique function supported on $\Delta\sigma$ and with value g at σ. Define the $\bar{F}$-automorphism a' of G by $a'(g) = a(g^{\delta_1})(\sigma_0)$. One checks immediately that $a'(\tau g) = \sigma_0\tau\sigma_0^{-1}a'(g)$, so that a' is in fact an isomorphism of E-groups $G \to G^{\sigma_0}$. The equality $a(f)(\sigma) = a'(f(\sigma_0^{-1}\sigma))$ can be checked on functions f of the form g^{δ_γ} for arbitrary $g \in G(\bar{F})$ and $\gamma \in \Gamma$. We compute that $a(g^{\delta_\gamma})(\sigma)$ equals

$$a(\gamma^{-1}(g^{\delta_1}))(\sigma) = a(g^{\delta_1})(\sigma\gamma^{-1}) = \sigma\gamma^{-1}\sigma_0^{-1}a'(g) = a'(\sigma_0^{-1}\gamma^{-1}\sigma g) = a'(g^{\delta_\gamma}(\sigma_0^{-1}\sigma)),$$

provided $\sigma_0^{-1}\gamma^{-1}\sigma \in \Delta$, and that $a(g^{\delta_\gamma}) = 1 = a'(g^{\delta_\gamma}(\sigma_0^{-1}\sigma))$ otherwise. $\square$

C ORTHOGONALITY RELATIONS FOR PROJECTIVE CHARACTERS

{app:projchar}

We have now constructed the bijection (8.1). Our next goal is to show that with this bijection the character identities (4.4) hold. In this section we will prove a lemma that will be needed for the evaluation of the right hand side of (4.4). It is a refinement of the orthogonality relation

$$\sum_{\tau \in \mathrm{Irr}(A)} \overline{\chi_\tau(a)}\chi_\tau(b) = \begin{cases} |Z_A(a)|, & b \in C_A(a) \\ 0, & \text{else} \end{cases}$$

for the characters of the irreducible representations of a finite group A. Here $Z_A(a)$ and $C_A(a)$ are the centralizer and the conjugacy class of a in A.

The refinement we need is the following. Consider a central extension

$$1 \to Z \to E \to A \to 1.$$

We assume that A is finite. For $e \in E$ we will write $\bar{e}$ for its image in A. Let $\psi : Z \to \mathbb{C}^\times$ be a character, and let $\mathrm{Irr}(E, \psi)$ be the set of all irreducible representations of E whose central character restricted to Z equals ψ. This is a finite set and each element of it is finite dimensional. For each $\tau \in \mathrm{Irr}(E, \psi)$ its character χ_τ is a class function on E that satisfies $\chi_\tau(ze) = \psi(z)\chi_\tau(e)$. This implies that if the images of $e, e' \in E$ in A commute, so that $ee'e^{-1}e'^{-1} \in Z$, but $\psi(ee'e^{-1}e'^{-1}) \neq 1$, then $\chi_\tau(e') = 0$. We will say that $e' \in E$ is ψ-centralizing, if for all $e \in E$ such that $ee'e^{-1}e'^{-1} \in Z$ we have $\psi(ee'e^{-1}e'^{-1}) = 1$. Note that the notion if being ψ-centralizing is invariant under conjugation as well as under translation by Z. The refinement of the orthogonality relations we need is the following.

{lem:orth}

Lemma C.1. *Assume that $e \in E$ is ψ-centralizing. Then*

$$\sum_{\tau \in Irr(E,\psi)} \chi_\tau(e)\chi_\tau(e') = \begin{cases} |Z_A(e)|\psi(ee'), & ee' \in Z \\ 0, & \bar{e}^{-1} \notin C_A(\bar{e}') \end{cases}$$

Proof. We will make use of the character theory of projective representations of finite groups, for an exposition of which we refer the reader to [Che]. We first form the push-out

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & E_\psi & \longrightarrow & A & \longrightarrow & 1 \end{array}$$

Inflation provides a bijection $\mathrm{Irr}(E_\psi, \mathrm{id}) \cong \mathrm{Irr}(E, \psi)$ that preserves characters. Moreover, $e' \in E$ is ψ-centralizing if and only if its image in E_ψ is id-centralizing. This reduces the problem to $Z = \mathbb{C}^\times$ and $\psi = \mathrm{id}$. Next we fix a set-theoretic splitting $s : A \to E$ such that all values of the corresponding 2-cocycle $\alpha(a, b) = s(a)s(b)s(ab)^{-1}$ are complex roots of unity, see [Che, Lemma 3.1]. For each $\tau \in \mathrm{Irr}(E_\psi, \mathrm{id})$ set $\bar{\tau} = \tau \circ s$. Then $\bar{\tau}$ is a projective representation of A with cocycle α and the map $\tau \mapsto \tau \circ s$ is a bijection between $\mathrm{Irr}(E_\psi, \mathrm{id})$ and the isomorphism classes of projective representations of A with cocycle α. Let f be the α-class function [Che, Definition 3.13] on A supported on the A-conjugacy class of $\bar{e}^{-1}$ and having the property with $f(\bar{e}^{-1}) = 1$. This class function exists because $\bar{e}^{-1}$ is an α-element. According to [Che, Theorem 3.15], we have

$$f = \sum_{\bar{\tau}} \langle f, \chi_{\bar{\tau}} \rangle \chi_{\bar{\tau}}.$$

Since both f and $\chi_{\bar{\tau}}$ are α-class functions and α is unitary, the product $f \cdot \overline{\chi_{\bar{\tau}}}$ is a 1-class function (i.e. an honest class function) and one sees

$$\langle f, \chi_{\bar{\tau}} \rangle = |A|^{-1} \sum_{a \in A} f(a) \overline{\chi_{\bar{\tau}}(a)} = |Z_A(\bar{e}^{-1})|^{-1} \overline{\chi_{\bar{\tau}}(\bar{e}^{-1})}.$$

We thus obtain

$$|Z_A(\bar{e})| f(\bar{e}') = \sum_{\tau} \overline{\chi_{\bar{\tau}}(\bar{e}^{-1})} \chi_{\bar{\tau}}(\bar{e}')$$

and then further

$$\begin{aligned} \sum_{\tau} \chi_\tau(e) \chi_\tau(e') &= \sum_{\tau} \mathrm{tr}(\tau(e^{-1})^{-1}) \mathrm{tr}(\tau(e')) \\ &= \sum_{\tau} \mathrm{tr}((z_{e^{-1}} \bar{\tau}(\bar{e}^{-1}))^{-1}) \mathrm{tr}(\tau(e')) \\ &= z_{e^{-1}}^{-1} z_{e'} \sum_{\tau} \overline{\chi_{\bar{\tau}}(\bar{e}^{-1})} \chi_{\bar{\tau}}(\bar{e}') \\ &= ee' \sum_{\tau} \overline{\chi_{\bar{\tau}}(\bar{e}^{-1})} \chi_{\bar{\tau}}(\bar{e}') \\ &= ee' |Z_A(\bar{e})| f(\bar{e}') \end{aligned}$$

We have used in this computation that the projective representation $\bar{\tau}$ is unitarizable, which is a consequence of our choice of s. The lemma follows. □

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