Graded multiplicities in the Macdonald kernel, Part I

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1. Introduction

Each complex semisimple Lie group G has a Macdonald kernel associated to it. This kernel is a bigraded virtual G-character depending on parameters q and t, and may be used to define the scalar product relative to which the Macdonald polynomials form an orthogonal basis. In the special cases $(q,t)=(0,t)$ and $(q,t)=(-q,q^2)$, it is the graded character of the symmetric and exterior algebras of the adjoint representation of G.

The subject of this paper is the decomposition of the Macdonald kernel into irreducible characters. In particular, we will be concerned with recurrences, explicit formulas, and practical algorithms for computing the decomposition. In a sequel to this paper, we will develop some connections between this decomposition and a bi-graded module associated to the Weyl group, generalizing some classical results in the invariant theory of Lie groups.

In order to more fully explain the motivations for studying this decomposition, as well as to summarize the results we obtain, we first need to introduce...

A. Some notation.

Let Φ be a finite crystallographic root system of rank n embedded in a real Euclidean space $\mathbf{E}$ with inner product $\langle\ ,\ \rangle$. We let $\alpha_1,\ldots,\alpha_n\in\Phi$ denote a fixed choice of simple roots and Φ^+ the corresponding set of positive roots. For each root $\alpha\in\Phi$, we let $s_\alpha:\mathbf{E}\to\mathbf{E}$ denote reflection through the hyperplane orthogonal to α. Thus $s_\alpha(\lambda)=\lambda-\langle\lambda,\alpha^\vee\rangle\alpha$, where $\alpha^\vee:=2\alpha/\langle\alpha,\alpha\rangle$ denotes the co-root corresponding to α.

The Weyl group W is the group of orthogonal transformations of $\mathbf{E}$ generated by the reflections $\{s_\alpha:\alpha\in\Phi\}$, or equivalently, by the (simple) reflections $s_1,\ldots,s_n$ corresponding to the simple roots $\alpha_1,\ldots,\alpha_n$. For $\lambda\in\mathbf{E}$, let W_λ denote the stabilizer of λ in W; if λ is dominant (i.e., $\langle\lambda,\alpha_i\rangle\geqslant 0$ for $i=1,\ldots,n$), then W_λ is the Weyl group of the parabolic root subsystem of Φ whose simple roots are $\{\alpha_i:\langle\lambda,\alpha_i\rangle=0\}$.

The unique dominant member of the W-orbit of λ is denoted λ^+.

Given $w\in W$, let $\ell(w)$ denote the minimum length among all expressions $w=s_{i_1}\cdots s_{i_l}$. It is well-known that this is also the inversion number; i.e., $\ell(w)=|\Phi(w)|$, where

$$\Phi(w):=\{\alpha\in\Phi^+:-w\alpha\in\Phi^+\}=\{\alpha\in\Phi^+:\ell(ws_\alpha)<\ell(w)\}.$$

For each $X\subseteq W$, we let

$$X(t)=\sum_{w\in X}t^{\ell(w)}=\sum_{w\in X}t^{|\Phi(w)|}$$

denote the Poincaré series for X.

The root lattice is $\mathcal{Q}=\mathbf{Z}\Phi$; it is a sublattice of the weight lattice

$$\mathcal{P}:=\{\mu\in\operatorname{Span}\Phi:\langle\mu,\alpha^\vee\rangle\in\mathbf{Z}\text{ for all }\alpha\in\Phi\}.$$

Similarly, the co-root system $\Phi^\vee=\{\alpha^\vee:\alpha\in\Phi\}$ affords a co-root lattice $\mathcal{Q}^\vee=\mathbf{Z}\Phi^\vee$ and a co-weight lattice $\mathcal{P}^\vee=\{\mu\in\operatorname{Span}\Phi:\langle\mu,\alpha\rangle\in\mathbf{Z}\text{ for all }\alpha\in\Phi\}$.

We let $\mathcal{Q}^+$ and $\mathcal{P}^+$ denote the dominant parts of $\mathcal{Q}$ and $\mathcal{P}$. There is a standard partial ordering of $\mathcal{P}$ in which $\mu \leqslant \nu$ if $\nu - \mu$ is in the nonnegative integer span of the simple roots. It is well-known that the restriction of this ordering to $\mathcal{P}^+$ has no infinite descending chains, and that 0 is the unique minimum element of $(\mathcal{Q}^+, \leqslant)$. Another important property of $(\mathcal{P}^+, \leqslant)$ is related to the saturation of weight strings; viz.,

$$0 \leqslant i \leqslant \langle \lambda, \alpha^\vee \rangle \ \Rightarrow \ (\lambda - i\alpha)^+ \leqslant \lambda^+ \qquad (\lambda \in \mathcal{P}, \alpha \in \Phi). \tag{1.1}$$

See Exercise VI.1.23 of [**B**] or Section 1.2 of [**St2**].

The Weyl group permutes $\mathcal{P}$, so this induces a W-action on $\mathbf{Z}\langle\mathcal{P}\rangle := \mathbf{Z}\{e^\lambda : \lambda \in \mathcal{P}\}$, the group ring of $\mathcal{P}$. We let $J(f) := \sum_{w\in W} \operatorname{sgn}(w) w.f$ denote the anti-symmetrization operator in $\mathbf{Z}\langle\mathcal{P}\rangle$. The Weyl denominator formula (e.g., Section VI.3.3 of [**B**]) may be expressed as the identity

$$J(e^\rho) = e^\rho \cdot \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha}) = \prod_{\alpha\in\Phi^+} (e^{\alpha/2} - e^{-\alpha/2}), \tag{1.2}$$

where $\rho \in \mathcal{P}$ denotes half the sum of the positive roots. The Weyl characters

$$\chi(\lambda) := J(e^{\lambda+\rho})/J(e^\rho) \qquad (\lambda \in \mathcal{P}^+), \tag{1.3}$$

form a free $\mathbf{Z}$-basis for the W-invariants in $\mathbf{Z}\langle\mathcal{P}\rangle$, and similarly, $\{\chi(\lambda) : \lambda \in \mathcal{Q}^+\}$ is a free $\mathbf{Z}$-basis for the W-invariants in $\mathbf{Z}\langle\mathcal{Q}\rangle$.

We define the *Macdonald kernel* associated to Φ to be the formal series

$$\Delta(q,t) := \prod_{i\geqslant 0} \left(\frac{1-q^{i+1}}{1-tq^i}\right)^n \cdot \prod_{i\geqslant 0}\prod_{\alpha\in\Phi} \frac{1-q^{i+1}e^\alpha}{1-tq^ie^\alpha}.$$

More precisely, we interpret $\Delta(q,t)$ as a member of the formal power series ring $\mathbf{Z}\langle\mathcal{Q}\rangle[[q,t]]$. It is clearly W-invariant, and thus has a Weyl character expansion

$$\Delta(q,t) = \sum_{\lambda\in\mathcal{Q}^+} C_\lambda(q,t)\chi(\lambda),$$

for certain formal series $C_\lambda(q,t) \in \mathbf{Z}[[q,t]]$. In fact, one sees from (1.3) that

$$C_\lambda(q,t) := [e^{\lambda+\rho}] J(e^\rho)\Delta(q,t) \tag{1.4}$$

where $[\,\cdot\,]$ denotes the coefficient operator (i.e., $[e^\mu]\sum a_\nu e^\nu = a_\mu$).

The graded multiplicities $C_\lambda(q,t)$ are the central subject of this paper.

B. Motivation.

We list five reasons for studying the decomposition of the Macdonald kernel.

1. Let G be a complex semisimple Lie group with root system Φ. In the case $q = 0$, the Macdonald kernel specializes to the graded character of $S(\mathfrak{g})$, the symmetric algebra of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Thus, $C_\lambda(0,t)$ is the graded multiplicity of the irreducible G-representation of highest weight λ in $S(\mathfrak{g})$. By a classical theorem of Chevalley, the G-invariant subalgebra $S(\mathfrak{g})^G$ (i.e., the case $\lambda = 0$) is a polynomial ring with Poincaré series

$$C_0(0,t) = \prod_{j=1}^{n}(1 - t^{e_j+1})^{-1},$$

where $e_1, \ldots, e_n$ denote the exponents of Φ. Furthermore, by a theorem of Kostant [**K1**], one knows that $S(\mathfrak{g})$ is free as a module over $S(\mathfrak{g})^G$, and thus $C_\lambda(0,t)/C_0(0,t)$ is a polynomial with nonnegative integer coefficients; these are the generating functions for the "generalized exponents" of G. Explicit combinatorial descriptions of these generalized exponents are available in type $\mathcal{A}$, but the general case seems to be an open problem.

2. In the case $(q,t) = (-q, q^2)$, the Macdonald kernel specializes to the graded character of $\Lambda(\mathfrak{g})$, the exterior algebra of the adjoint representation of G. The irreducible components of $\Lambda(\mathfrak{g})$ are not as well-understood as are those in the symmetric algebra case, but have been studied extensively by Kostant [**K2**], Reeder [**R**], and others. For example, by a classical theorem of invariant theory (equivalent to determining the Betti numbers of compact Lie groups), one knows that the invariant subalgebra $\Lambda(\mathfrak{g})^G$ is an exterior algebra with generators in degrees $2e_1 + 1, \ldots, 2e_n + 1$, and thus

$$C_0(-q, q^2) = \prod_{j=1}^{n}(1 + q^{2e_j+1}).$$

In addition, Bazlov [**Ba**] has a similar formula for the graded multiplicity of the adjoint representation in $\Lambda(\mathfrak{g})$, and Reeder [**R**] has multiplicity formulas for irreducible representations whose highest weights are sufficiently close to 2ρ, the highest weight of $\Lambda(\mathfrak{g})$.

3. In 1982, Macdonald [**M2**] formulated a constant term conjecture for root systems that may be translated into the present context as a conjectured formula for the multiplicity of the trivial character in $\Delta(q,t)$; namely,

$$C_0(q,t) = \prod_{i\geqslant 0}\prod_{j=1}^{n}\frac{1 - q^{i+1}t^{e_j}}{1 - q^i t^{e_j+1}}. \tag{1.5}$$

This conjecture was proved in 1995 by Cherednik [**C**].

4. In 1986, Hanlon [**H**] formulated a homology conjecture (recently proved by Fishel, Grojnowski, and Teleman [**FGT**]) for the Lie algebra $\mathfrak{g}[z]/(z^k)$ that also implies (1.5).

In this context, the series $C_\lambda(q, q^{k+1})$ may be interpreted as a graded multiplicity in the (virtual) G-character of the Euler characteristic of Hanlon's complex.

5. Let $(\,\cdot\,,\cdot\,)$ denote the usual symmetric pairing on the group ring of $\mathcal{P}$ relative to which the Weyl characters are orthonormal; viz.,

$$(f,g) = \frac{1}{|W|}[e^0] f g^* J(e^\rho) J(e^{-\rho}) \qquad (f, g \in \mathbf{Z}\langle\mathcal{P}\rangle),$$

where $g \mapsto g^*$ is the automorphism defined by $e^\mu \mapsto e^{-\mu}$ ($\mu \in \mathcal{P}$). In 1988, Macdonald [**M3**] proved that there is a unique $\mathbf{Q}(q,t)$-basis $M(\lambda) = M(\lambda; q, t)$ ($\lambda \in \mathcal{P}^+$) of the W-invariant part of the group ring $\mathbf{Q}(q,t)\langle\mathcal{P}\rangle$ with the defining properties

(i) $(\Delta(q,t)M(\mu), M(\lambda)) = 0$ for $\lambda \neq \mu$,
(ii) $M(\lambda) - \chi(\lambda)$ is in the $\mathbf{Q}(q,t)$-span of $\{\chi(\mu) : \mu < \lambda\}$.

These are the *Macdonald polynomials*, and property (i) explains why we refer to $\Delta(q,t)$ as the Macdonald kernel. Note that $M(0) = \chi(0) = 1$, so (1.5) may also be interpreted as the evaluation of $(\Delta(q,t)M(0), M(0))$. We remark that Macdonald extended (1.5) to a conjectured evaluation of $(\Delta(q,t)M(\lambda), M(\lambda))$, also proved by Cherednik [**C**].

In this context, (i) and (ii) show that the Macdonald polynomials may be computed by applying the Gram-Schmidt algorithm to the basis of Weyl characters. However, there is a significant difficulty to overcome in that one needs to be able to evaluate pairings of the form $(\Delta(q,t)f, g)$ in order to be able to use this algorithm effectively. Given the algorithms for computing $C_\lambda(q,t)$ we present here (see Section 5), together with any of the standard algorithms for tensor product multiplicity, it is possible to compute the decomposition of $\Delta(q,t)\chi(\mu)$ into Weyl characters, and thereby evaluate $(\Delta(q,t)\chi(\mu), \chi(\lambda))$. For example, we had no difficulty computing $C_\lambda(q,t)$ for the highest weights of the 25 smallest representations of the Lie group $E_8(\mathbf{C})$, and these multiplicities contain enough information to enable the calculation of the first 6 Macdonald polynomials for the root system $\mathcal{E}_8$.

C. Outline.

The remainder of the paper is structured as follows.

At $q = 0$, the Macdonald polynomials may be interpreted as zonal spherical functions for p-adic groups (see [**M3**] or [**M4**]); in type $\mathcal{A}$, they are the Hall-Littlewood functions. In Section 2, we introduce a new family of W-invariant polynomials that include these spherical functions as special cases, but have an extra parameter that varies over parabolic subgroups of the Weyl group. It will develop that we will have uses only for those polynomials whose parabolic parameter is the stabilizer of a minuscule or quasi-minuscule co-weight, but the results in this section make no such assumptions.

In Section 3, we derive a pair of functional equations satisfied by the Macdonald kernel for each minuscule or quasi-minuscule dominant co-weight (see Lemmas 3.1 and 3.2). These functional equations provide triangular linear systems for computing the multiplicities $C_\lambda(q,t)$ (Theorems 3.3 and 3.4), and the coefficients of these systems involve the Weyl character decompositions of the degenerate Macdonald polynomials of Section 2. It is interesting to note that these results generalize methods specific to type $\mathcal{A}$ that we developed twenty years ago (see Section 5 of [**St1**]). We should also point out that Ion [**I**] has recently developed recursive methods for computing the coefficients of e^μ (for $\mu \in \mathcal{Q}$) in the closely related "Cherednik kernel," and this may be used to compute the multiplicities $C_\lambda(q,t)$, at least for the highest weights of sufficiently small G-representations.

In Section 4, we use the recursive formulas of the previous section to derive explicit formulas for the graded multiplicities in the Macdonald kernel indexed by dominant roots, generalizing formulas due to Kostant in the symmetric algebra case [**K1**], and the previously mentioned formula of Bazlov in the exterior algebra case [**Ba**].

In Section 5, we discuss the practical issues involved in implementing the recurrences of Section 3. We also develop a second family of recursive formulas for computing $C_\lambda(q,t)$, one for each minuscule or quasi-minuscule dominant co-weight (Propositions 5.1 and 5.2). This second family has lower complexity if λ has a sufficiently large stabilizer.

We show (Corollary 3.5) that the renormalized multiplicities

$$\bar{C}_\lambda(q,t) := C_\lambda(q,t)/C_0(q,t),$$

are rational functions of q and t, and the final section is devoted to a detailed analysis of the denominators of these rational functions. We use Cherednik's evaluation of $C_0(q,t)$ to deduce that $\bar{C}_\lambda(q,t)$ has a denominator expressible as a product of factors $1 - q^a t^e$, where $a \geqslant 1$ and e is an exponent of Φ (Theorem 6.1). Combining this with the results of Section 3, we deduce an explicit formula for the denominator of $\bar{C}_\lambda(q,t)$ whose form is controlled by chains in a Boolean algebra naturally associated to Φ (Theorem 6.6). Empirical evidence suggests that these denominators are close to minimal. This leads to Conjecture 6.12, a unified way (if true) to describe explicit denominators for each of the normalized multiplicities, based on a mysterious relationship between exponents and sums of roots in parabolic complements (see Empirical Fact 6.11). In some cases (including $\mathcal{E}_8$), the conjecture follows from Theorem 6.6.

2. Degenerate Macdonald polynomials

In order to study the graded multiplicities in the Macdonald kernel, it will be convenient to first introduce a family of W-invariant elements in the group ring of $\mathcal{P}$ over $\mathbf{Z}[t]$.

Given $\lambda \in \mathcal{P}$ and a dominant vector $\omega \in \mathbf{E}$, we define

$$P_\omega(\lambda) = P_\omega(\lambda\,; t) := J\Big(e^{\lambda+\rho} \cdot \textstyle\prod_{\alpha\in\Psi^+}(1 - te^{-\alpha})\Big)\Big/ J(e^\rho), \tag{2.1}$$

where $\Psi^+ = \{\alpha \in \Phi : \langle \omega, \alpha\rangle > 0\}$. It is clear that $P_\omega(\lambda)$ is W-invariant, so we may define polynomials $A^\omega_{\lambda\mu}(t) \in \mathbf{Z}[t]$ via the expansion

$$P_\omega(\lambda) = \sum_{\mu\in\mathcal{P}^+} A^\omega_{\lambda\mu}(t)\chi(\mu).$$

Note that $\Phi^+ - \Psi^+$ is the positive part of a parabolic root subsystem of Φ, and $P_\omega(\lambda)$ depends on ω only through Ψ^+. Thus in effect, we have a family of $\mathcal{P}$-indexed W-invariants in the group ring $\mathbf{Z}[t]\langle\mathcal{P}\rangle$ for each parabolic subgroup of W.

Note that $P_0(\lambda) = \chi(\lambda)$. At the opposite extreme, one may choose $\omega = \rho$; this is the generic case in the sense that $\Psi^+ = \Phi^+$. Furthermore, if λ is dominant, then

$$P(\lambda) = P(\lambda\,;t) := \frac{1}{W_\lambda(t)} P_\rho(\lambda\,;t) \tag{2.2}$$

is the Macdonald polynomial $M(\lambda; 0, t)$ (e.g., see [**M3**] or [**M4**]), and it is in this sense that we regard $P_\omega(\lambda)$ as a degenerate Macdonald polynomial.

The following result provides another connection between the Macdonald polynomials at $q = 0$ and their degenerate variations. It is superior to (2.2) in that it shows clearly that $P(\lambda)$ has polynomial coefficients. The proof will be given after the proof of Proposition 2.2.

PROPOSITION 2.1. *If λ is dominant, then $P(\lambda\,;t) = P_\lambda(\lambda\,;t)$.*

A. The decomposition matrix.

Our first objective is to analyze the matrix of polynomials $A^\omega_{\lambda\mu}(t)$ ($\lambda \in \mathcal{P}$, $\mu \in \mathcal{P}^+$).

PROPOSITION 2.2. *For all $\lambda \in \mathcal{P}$, we have*

(a) $A^\omega_{\lambda\mu}(t) = [e^{\lambda+\rho}]J(e^{\mu+\rho}) \cdot \prod_{\alpha\in\Psi^+}(1 - te^\alpha)$.

(b) $P_\omega(\lambda) = \sum_{\mu\leqslant\lambda^+} A^\omega_{\lambda\mu}(t)\chi(\mu)$ *(i.e., $A^\omega_{\lambda\mu}(t) = 0$ unless $\mu \leqslant \lambda^+$).*

(c) $A^\omega_{\lambda,\lambda^+}(t) = X^\omega_\lambda(t)$, *where* $X^\omega_\lambda := \{w \in W : w\lambda = \lambda^+$ *and* $\Phi(w) \subseteq \Psi^+\}$.

As an immediate corollary, note that if λ is dominant, then $1 \in X^\omega_\lambda$ and $A^\omega_{\lambda\lambda}(t) \neq 0$. Hence the matrix $[A^\omega_{\lambda\mu}(t)]_{\lambda,\mu\in\mathcal{P}^+}$ is lower triangular with respect to $\leqslant$, and invertible over the ring $\mathbf{Q}(t)$, so $\{P_\omega(\lambda) : \lambda \in \mathcal{P}^+\}$ is a $\mathbf{Q}(t)$-basis of the W-invariant part of $\mathbf{Q}(t)\langle\mathcal{P}\rangle$.

Recall that $\Phi(w) = \{\alpha \in \Phi^+ : w\alpha < 0\}$ is the inversion set of $w \in W$.

LEMMA 2.3. *For all $w \in W$ and $\Theta \subseteq \Phi^+$, we have*

$$w(\rho - \sigma(\Theta)) = \rho \ \ \textit{if and only if} \ \ \Theta = \Phi(w),$$

where $\sigma(\Theta)$ denotes the sum of the members of Θ.

Proof. Since s_i permutes $\Phi^+ - \{\alpha_i\}$, it follows that $s_i\rho = \rho - \alpha_i$ and

$$w(\rho - \sigma(\Theta)) = ws_i(\rho - \alpha_i - \sigma(s_i\Theta)) = ws_i(\rho - \sigma(\Theta')),$$

where

$$\Theta' = \begin{cases} s_i(\Theta - \{\alpha_i\}) & \text{if } \alpha_i \in \Theta, \\ s_i\Theta \cup \{\alpha_i\} & \text{if } \alpha_i \notin \Theta. \end{cases}$$

On the other hand, similar reasoning shows that

$$\Phi(ws_i) = \begin{cases} s_i(\Phi(w) - \{\alpha_i\}) & \text{if } \alpha_i \in \Phi(w), \\ s_i\Phi(w) \cup \{\alpha_i\} & \text{if } \alpha_i \notin \Phi(w), \end{cases}$$

and hence the result holds for w if and only if it holds for ws_i. However, the case $w = 1$ is trivial, so it holds for all $w \in W$. $\square$

Proof of Proposition 2.2. (a) By the Weyl character formula (i.e., (1.3)), we have

$$\begin{aligned} A^{\omega}_{\lambda\mu}(t) &= [e^{\mu+\rho}]J\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})\Big) \\ &= \sum_{w\in W}\operatorname{sgn}(w)[e^{\mu+\rho}]e^{w(\lambda+\rho)}\cdot\prod_{\alpha\in\Psi^+}(1-te^{-w\alpha}) \\ &= \sum_{w\in W}\operatorname{sgn}(w)[e^{-w^{-1}(\mu+\rho)}]e^{-\lambda-\rho}\cdot\prod_{\alpha\in\Psi^+}(1-te^{\alpha}) \\ &= \sum_{w\in W}\operatorname{sgn}(w)[e^{\lambda+\rho}]e^{w^{-1}(\mu+\rho)}\cdot\prod_{\alpha\in\Psi^+}(1-te^{\alpha}) \qquad (2.3) \\ &= [e^{\lambda+\rho}]J(e^{\mu+\rho})\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{\alpha}). \end{aligned}$$

(b) To get a nonzero contribution to $A^{\omega}_{\lambda\mu}(t)$ from a summand in (2.3), there must be an element $w\in W$ and a subset $\Theta\subseteq\Psi^+$ such that $\lambda+\rho-\sigma(\Theta)=w^{-1}(\mu+\rho)$, where σ denotes the sum operator (as in Lemma 2.3). However, the set

$$\{\rho-\sigma(\Pi):\Pi\subseteq\Phi^+\}=\big\{\textstyle\sum_{\alpha>0}\delta(\alpha)\alpha:\delta(\alpha)=\pm1/2\big\}$$

is clearly W-stable, so $\mu+\rho=w\lambda+\rho-\sigma(\Pi)$ for some $\Pi\subseteq\Phi^+$, whence

$$\lambda^+\geqslant w\lambda=\mu+\sigma(\Pi)\geqslant\mu. \qquad (2.4)$$

(c) If $\mu=\lambda^+$, then equality occurs in (2.4); hence, each nonzero contribution requires a pair (w,Θ) such that $w\lambda=\lambda^+$ and $w(\rho-\sigma(\Theta))=\rho$. By Lemma 2.3, the latter occurs if and only if $\Theta=\Phi(w)$, and hence $w\in X^{\omega}_{\lambda}$. The net contribution of each $w\in X^{\omega}_{\lambda}$ to (2.3) is therefore $\operatorname{sgn}(w)(-t)^{|\Phi(w)|}=t^{\ell(w)}$, so $A^{\omega}_{\lambda,\lambda^+}(t)$ is the Poincaré series for X^{ω}_{λ}. ☐

Proof of Proposition 2.1. Taking $\lambda=\omega=0$ in Proposition 2.2, we see that $W(t)$ is the leading coefficient in the Weyl character expansion of $P_0(0)$. However, 0 is the minimum element of $(\mathcal{Q}^+,\leqslant)$, so the leading term is the only term; i.e., $P_0(0)=W(t)\chi(0)$. Hence,

$$J\Big(\textstyle\prod_{\alpha>0}(e^{\alpha/2}-te^{-\alpha/2})\Big)=J\Big(e^{\rho}\cdot\prod_{\alpha>0}(1-te^{-\alpha})\Big)=W(t)J(e^{\rho}). \qquad (2.5)$$

Given $\lambda\in\mathcal{P}^+$, let $\Psi^+=\{\alpha\in\Phi:\langle\lambda,\alpha\rangle>0\}$, so that $\Phi^+-\Psi^+$ is the set of positive roots associated to W_λ. Since W_λ fixes $\prod_{\alpha\in\Psi^+}(e^{\alpha/2}-te^{-\alpha/2})$, the W_λ-version of (2.5) implies

$$\begin{aligned} J_\lambda\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha>0}(1-te^{-\alpha})\Big) &= e^{\lambda}\cdot J_\lambda\Big(\textstyle\prod_{\alpha>0}(e^{\alpha/2}-te^{-\alpha/2})\Big) \\ &= W_\lambda(t)J_\lambda\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})\Big), \qquad (2.6) \end{aligned}$$

where J_λ denotes the W_λ-version of the anti-symmetrization operator J.

Now choose a set of coset representatives X for W/W_λ; we have

$$\begin{aligned}
J(e^\rho)P_\rho(\lambda) &= J\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha>0}(1-te^{-\alpha})\Big)\\
&= \sum_{x\in X}\operatorname{sgn}(x)\,x\Big[J_\lambda\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha>0}(1-te^{-\alpha})\Big)\Big]\\
&= W_\lambda(t)\sum_{x\in X}\operatorname{sgn}(x)\,x\Big[J_\lambda\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})\Big)\Big]\\
&= W_\lambda(t)J\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})\Big) = W_\lambda(t)J(e^\rho)P_\lambda(\lambda),
\end{aligned}$$

using (2.6) for the third equality. Thus, $P_\rho(\lambda) = W_\lambda(t)P_\lambda(\lambda)$. $\square$

B. Leading terms.

Proposition 2.2 suggests that $\chi(\lambda^+)$ is the highest Weyl character that occurs in the expansion of $P_\omega(\lambda)$; however, the coefficient of $\chi(\lambda^+)$ may vanish if λ is not dominant.

LEMMA 2.4. *For all dominant ω and all $\lambda\in\mathcal{P}$, the following are equivalent:*

(a) $A^\omega_{\lambda,\lambda^+}(t)\neq 0$.

(b) *λ is dominant with respect to the action of W_ω (i.e., $\langle\omega,\alpha_i\rangle = 0 \Rightarrow \langle\lambda,\alpha_i\rangle \geqslant 0$).*

(c) *X^ω_λ includes the shortest element $x\in W$ such that $x\lambda=\lambda^+$.*

It follows that for all $\lambda\in\mathcal{P}^+$, the number of choices for $\nu\in W\lambda$ such that $A^\omega_{\nu\lambda}(t)\neq 0$ is the number of W_ω-orbits in $W\lambda$; i.e., the number of (W_ω, W_λ)-double cosets.

Proof. (a)$\Rightarrow$(b). If $A^\omega_{\lambda,\lambda^+}(t)$ is nonzero, then X^ω_λ must be nonempty (Proposition 2.2(c)), so we may choose $w\in X^\omega_\lambda$. If λ failed to be dominant with respect to W_ω, then there would exist a simple root α_i such that $\langle\omega,\alpha_i\rangle = 0$ and $\langle\lambda^+, w\alpha_i\rangle = \langle\lambda,\alpha_i\rangle < 0$. The latter forces $w\alpha_i < 0$ and hence $\alpha_i\in\Phi(w)$, whereas the former implies $\alpha_i\notin\Psi^+$, contradicting the membership of w in X^ω_λ.

(b)$\Rightarrow$(c). Let x be the shortest element of W such that $x\lambda=\lambda^+$. If $x\notin X^\omega_\lambda$, then $\Phi(x)\not\subseteq\Psi^+$, so there must be a simple root α_i such that $\langle\omega,\alpha_i\rangle = 0$ and $x\alpha_i<0$, whence $\ell(xs_i)<\ell(x)$. Furthermore, $x\alpha_i<0$ also implies $\langle\lambda,\alpha_i\rangle = \langle\lambda^+, x\alpha_i\rangle \leqslant 0$, and if equality occurred, then s_i would fix λ, and xs_i would be a shorter element mapping λ to λ^+, contradicting the choice of x. The remaining possibility is that $\langle\lambda,\alpha_i\rangle<0$, which contradicts (b).

(c)$\Rightarrow$(a). This is immediate from Proposition 2.2(c). $\square$

Proposition 2.5. *Let $\lambda \in \mathcal{P}$ and let x be the shortest member of W such that $x\lambda = \lambda^+$. If λ is dominant with respect to W_ω, and $\theta := x\omega$, then*

(a) $\ell(x) = d(\lambda) := |\{\alpha \in \Phi^+ : \langle \lambda, \alpha \rangle < 0\}|$,

(b) $W_{\lambda^+} \cap W_\theta$ *is a parabolic subgroup of* W_{λ^+},

(c) $Y^\omega_\lambda := X^\omega_\lambda \cdot x^{-1} = \{w \in W_{\lambda^+} : \Phi(wx) \subseteq \Psi^+\}$ *is the set of minimum-length coset representatives for* $W_{\lambda^+}/(W_{\lambda^+} \cap W_\theta)$, *and*

(d) $A^\omega_{\lambda,\lambda^+}(t) = t^{\ell(x)} Y^\omega_\lambda(t) = t^{d(\lambda)} W_{\lambda^+}(t)/(W_{\lambda^+} \cap W_\theta)(t)$.

Proof. (a) If α is a positive root and $\langle \lambda, \alpha \rangle < 0$, then $\langle \lambda^+, x\alpha \rangle < 0$ and hence $\alpha \in \Phi(x)$. Conversely, if $\alpha \in \Phi(x)$, then $\ell(xs_\alpha) < \ell(x)$ and $\langle \lambda, \alpha \rangle = \langle \lambda^+, x\alpha \rangle \leqslant 0$. If equality occurs, then we contradict the minimality of x, so we have $\Phi(x) = \{\alpha \in \Phi^+ : \langle \lambda, \alpha \rangle < 0\}$ and hence $\ell(x) = |\Phi(x)| = d(\lambda)$.

(b) Parabolic subgroups are stabilizers of dominant points, so it suffices to show that θ is dominant with respect to the action of W_{λ^+}. If this failed, there would be a simple root α_i such that $\langle \lambda, x^{-1}\alpha_i \rangle = \langle \lambda^+, \alpha_i \rangle = 0$ and $\langle \omega, x^{-1}\alpha_i \rangle = \langle \theta, \alpha_i \rangle < 0$. The latter implies $x^{-1}\alpha_i < 0$ and hence $\ell(s_i x) < \ell(x)$, whereas the former implies that $s_i x(\lambda) = \lambda^+$, contradicting the minimality of x.

(c) First, we argue that if $w \in W_{\lambda^+}$ is the shortest member of the coset $w(W_{\lambda^+} \cap W_\theta)$, then $w \in Y^\omega_\lambda$. If not, then $\Phi(wx) \not\subseteq \Psi^+$, so there is a simple root α_i such that $\langle \omega, \alpha_i \rangle = 0$ and $wx\alpha_i < 0$. By Lemma 2.4, we have $x \in X^\omega_\lambda$, so $\Phi(x) \subseteq \Psi^+$ and hence $\alpha := x\alpha_i > 0$. However, $ws_\alpha\theta = ws_\alpha x\omega = wxs_i\omega = w\theta$, so ws_α and w are in the same coset, yet $w\alpha < 0$ implies $\ell(ws_\alpha) < \ell(w)$, a contradiction. (Note that $s_\alpha \in W_{\lambda^+}$, since every inversion in a parabolic subgroup must be a root in the corresponding root subsystem.)

Conversely, if $w \in Y^\omega_\lambda$ is not the shortest member of the coset $w(W_{\lambda^+} \cap W_\theta)$, then there is some simple reflection $s_i \in W_{\lambda^+} \cap W_\theta$ such that $\ell(ws_i) < \ell(w)$. If $\alpha := x^{-1}\alpha_i$ is a negative root, then $-\alpha \in \Phi(x) \subseteq \Psi^+$, and hence $\langle \theta, \alpha_i \rangle = \langle \omega, \alpha \rangle < 0$, contradicting the fact that s_i fixes θ. Otherwise, α is a positive root such that $wx\alpha = w\alpha_i < 0$, whence $\alpha \in \Phi(wx) \subseteq \Psi^+$ and $\langle \theta, \alpha_i \rangle = \langle \omega, \alpha \rangle > 0$, again a contradiction.

(d) One knows that if W' is a parabolic subgroup of W and v is the shortest member of $W'v$, then $\ell(uv) = \ell(u) + \ell(v)$ for all $u \in W'$ (e.g., see Exercise IV.1.3 of [**B**]). Hence part (a) and Proposition 2.2(c) imply $A^\omega_{\lambda,\lambda^+}(t) = t^{d(\lambda)} Y^\omega_\lambda(t)$, and part (c) implies that the Poincaré series for Y^ω_λ is $W_{\lambda^+}(t)/(W_{\lambda^+} \cap W_\theta)(t)$. $\square$

REMARK 2.6. For all $\lambda \in \mathcal{P}^+$, Proposition 2.5(d) shows that $W_\lambda(t)/(W_\lambda \cap W_\omega)(t)$ is the leading coefficient of $P_\omega(\lambda)$. In the diagonal case $\lambda = \omega$, it follows that $P(\lambda) = P_\lambda(\lambda)$ (recall Proposition 2.1) has leading coefficient 1, and the transition matrix expressing $P(\lambda)$ in terms of Weyl characters may be inverted over $\mathbf{Z}[t]$; i.e.,

$$\chi(\lambda) = \sum_{\mu \leqslant \lambda} K_{\lambda\mu}(t) P(\mu), \qquad K_{\lambda\mu}(t) \in \mathbf{Z}[t].$$

By a theorem of Kato [**Ka**] (see also [**Gu**]), one knows that $K_{\lambda\mu}(t)$ is a Kazhdan-Lusztig polynomial for the associated affine Weyl group, and therefore has nonnegative coefficients. In type $\mathcal{A}$, these are also known as the Koskta-Foulkes polynomials.

COROLLARY 2.7. *For all $\lambda \in \mathcal{P}$, we have*

$$A^\omega_{\lambda,\lambda^+}(t) = t^{2d(\lambda)+N_0(\lambda,\omega)} A^\omega_{\lambda,\lambda^+}(t^{-1}),$$

where $N_0(\lambda,\omega) := |\{\alpha \in \Phi : \langle \lambda, \alpha \rangle = 0, \langle \omega, \alpha \rangle > 0\}|$.

Proof. If λ is not dominant with respect to W_ω, then $A^\omega_{\lambda,\lambda^+}(t) = 0$ and there is nothing to prove. Otherwise, Proposition 2.5 implies

$$t^{2d(\lambda)} A^\omega_{\lambda,\lambda^+}(t^{-1}) = t^{d(\lambda)} W_{\lambda^+}(t^{-1})/(W_{\lambda^+} \cap W_\theta)(t^{-1}).$$

Now since multiplication by the longest element of a reflection group complements the inversion set, one knows that $W(t) = t^{|\Phi^+|} W(t^{-1})$ (Exercise IV.1.26 of [**B**]), and hence

$$\begin{aligned} t^{N+2d(\lambda)} A^\omega_{\lambda,\lambda^+}(t^{-1}) &= t^{N+d(\lambda)} W_{\lambda^+}(t^{-1})/(W_{\lambda^+} \cap W_\theta)(t^{-1}) \\ &= t^{d(\lambda)} W_{\lambda^+}(t)/(W_{\lambda^+} \cap W_\theta)(t) = A^\omega_{\lambda,\lambda^+}(t), \end{aligned}$$

where N is the difference between the numbers of reflections in W_{λ^+} and $W_{\lambda^+} \cap W_\theta$; i.e.,

$$\begin{aligned} N &= |\{\alpha \in \Phi^+ : \langle \lambda^+, \alpha \rangle = 0\}| - |\{\alpha \in \Phi^+ : \langle \lambda^+, \alpha \rangle = \langle \theta, \alpha \rangle = 0\}| \\ &= |\{\alpha \in \Phi : \langle \lambda^+, \alpha \rangle = 0, \langle \theta, \alpha \rangle > 0\}| = N_0(\lambda^+, \theta), \end{aligned}$$

since θ is dominant with respect to W_{λ^+} (see the proof of Proposition 2.5(b)). However, the ordered pairs (λ^+, θ) and (λ, ω) are in the same W-orbit, and it is clear as defined that $N_0(\ ,\ )$ is constant on W-orbits, so $N_0(\lambda,\omega) = N_0(\lambda^+,\theta) = N$. $\square$

C. Inversion.

Let w_0 denote the longest element of W; one knows that $-w_0$ permutes the simple roots and thus determines an order-preserving involution $\lambda \mapsto \lambda^*$ on $\mathbf{E}$, $\mathcal{P}$, $\mathcal{P}^+$, and $\mathcal{Q}^+$.

PROPOSITION 2.8. *If ω is dominant and x_0 is the shortest member of w_0W_ω, then*

$$t^{\ell(x_0)}P_\omega(\lambda\,;t^{-1}) = P_{\omega^*}(x_0\lambda\,;t) \qquad (\lambda\in\mathcal{P}).$$

Proof. Setting $\Psi^+=\{\alpha\in\Phi:\langle\omega,\alpha\rangle>0\}$, we have

$$\begin{aligned}P_\omega(\lambda\,;t^{-1}) &= J\Big(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-t^{-1}e^{-\alpha})\Big)\Big/J(e^\rho),\\ &= (-t)^{-|\Psi^+|}J\Big(e^{\lambda+\rho-\sigma(\Psi^+)}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{\alpha})\Big)\Big/J(e^\rho).\end{aligned}$$

Since x_0 is the shortest member of its W_ω-coset, we have $\Phi(x_0)\subseteq\Psi^+$. On the other hand, there cannot be a root $\alpha\in\Psi^+$ in the inversion set of $w_0x_0\in W_\omega$. Thus $w_0x_0\alpha>0$ and hence $x_0\alpha<0$ for all $\alpha\in\Psi^+$, so we conclude that $\Phi(x_0)=\Psi^+$, $\mathrm{sgn}(x_0)=(-1)^{|\Psi^+|}$, and $x_0(\rho-\sigma(\Psi^+))=\rho$ (Lemma 2.3). Bearing in mind that $J(x_0f)=\mathrm{sgn}(x_0)J(f)$ for any element f of the group ring, we obtain

$$\begin{aligned}P_\omega(\lambda\,;t^{-1}) &= t^{-\ell(x_0)}J\Big(e^{x_0\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{x_0\alpha})\Big)\Big/J(e^\rho),\\ &= t^{-\ell(x_0)}J\Big(e^{x_0\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Theta^+}(1-te^{-\alpha})\Big)\Big/J(e^\rho) = t^{-\ell(x_0)}P_{\omega^*}(x_0\lambda\,;t),\end{aligned}$$

where $\Theta^+=-x_0\Psi^+=\{\beta\in\Phi:\langle x_0\omega,\beta\rangle<0\}=\{\beta\in\Phi:\langle\omega^*,\beta\rangle>0\}$. □

COROLLARY 2.9. *If β is a dominant root and $\omega=\beta^\vee$, then*

$$t^mP_\omega(\lambda\,;t^{-1}) = P_\omega(s_\beta\lambda\,;t) \qquad (\lambda\in\mathcal{P}),$$

where $m=\ell(s_\beta)=|\{\alpha\in\Phi:\langle\omega,\alpha\rangle>0\}|$.

Proof. Since $\omega^*=-w_0\omega=w_0s_\beta(\omega)$ is dominant and in the same orbit as ω, it follows that $\omega^*=\omega$ and s_β and w_0 are in the same W_ω-coset. If some member of this coset were shorter than s_β, then we would have $s_i(\omega)=\omega$ and $\ell(s_\beta s_i)<\ell(s_\beta)$ for some i. However, the former implies $\langle\beta,\alpha_i\rangle=0$ and $s_\beta(\alpha_i)=\alpha_i>0$, a contradiction. Hence, s_β is the shortest member of w_0W_ω, and the result follows from Proposition 2.8. (In particular, the claimed expression for $\ell(s_\beta)$ follows from the fact that for dominant β, we have $\alpha\in\Phi(s_\beta)$ if and only if $\langle\beta,\alpha\rangle>0$, an easy exercise involving root systems of rank 2.) □

D. Principal specialization.

Let $\rho^\vee$ denote half the sum of the positive co-roots (the co-root analogue of ρ). One knows that $\langle\rho^\vee,\alpha_i\rangle=1$ for all i, so $\mathrm{ht}(\alpha):=\langle\rho^\vee,\alpha\rangle$ is the height function on $\mathrm{Span}\,\Phi$; i.e., the sum of the simple root coordinates.

The *principal specialization map* $f\mapsto\mathrm{Pr}(f)$ is a $\mathbf{Z}[t]$-linear homomorphism on the group ring of $\mathcal{P}$, taking values in $\mathbf{Z}[t^{\pm1/2}]$, and defined by

$$\mathrm{Pr}(e^\lambda)=t^{-\langle\lambda,\rho^\vee\rangle} \qquad (\lambda\in\mathcal{P}).$$

It is more common to use $\rho^\vee$ in place of $-\rho^\vee$ here; however, the variation we have chosen will be more convenient in what follows.

There is a well-known formula for the principal specialization of any Weyl character that follows easily from the Weyl denominator formula. Indeed, given $\lambda \in \mathcal{P}$, we have

$$\Pr(J(e^{\lambda+\rho})) = \sum_{w \in W} \operatorname{sgn}(w) t^{-\langle w(\lambda+\rho), \rho^\vee \rangle} = \sum_{w \in W} \operatorname{sgn}(w) t^{-\langle w\rho^\vee, \lambda+\rho \rangle}.$$

This may be recognized as the image of $J(e^{\rho^\vee})$ under the linear map $e^\mu \mapsto t^{-\langle \mu, \lambda+\rho \rangle}$, so by the $\Phi^\vee$-version of the denominator formula (1.2), we have

$$\Pr(J(e^{\lambda+\rho})) = t^{-\langle \lambda+\rho, \rho^\vee \rangle} \cdot \prod_{\alpha \in \Phi^+} \bigl(1 - t^{\langle \lambda+\rho, \alpha^\vee \rangle}\bigr), \tag{2.7}$$

$$\Pr(\chi(\lambda)) = \Pr(J(e^{\lambda+\rho})) / \Pr(J(e^\rho)) = t^{-\langle \lambda, \rho^\vee \rangle} \cdot \prod_{\alpha \in \Phi^+} \frac{1 - t^{\langle \lambda+\rho, \alpha^\vee \rangle}}{1 - t^{\langle \rho, \alpha^\vee \rangle}}. \tag{2.8}$$

We remark that $\Pr(J(e^\rho))$ may be computed in two ways—either by taking $\lambda = 0$ in (2.7) or via the denominator formula for Φ. This yields the identity

$$\prod_{\alpha \in \Phi^+} (1 - t^{\langle \rho, \alpha^\vee \rangle}) = \prod_{\alpha \in \Phi^+} (1 - t^{\langle \rho^\vee, \alpha \rangle}), \tag{2.9}$$

and implies in particular that the heights of roots and co-roots (relative to the height functions of their respective root systems) have the same distribution.

The following result generalizes a well-known formula for the principal specialization of Macdonald polynomials at $q = 0$ (e.g., see p. 39 of [**M3**]).

Theorem 2.10. *For all $\lambda \in \mathcal{P}$ and all dominant ω, we have*

$$\Pr(P_\omega(\lambda)) = t^{-\langle \lambda, \rho^\vee \rangle} \cdot \prod_{\alpha \in \Phi^+} \frac{1 - t^{\langle \rho, \alpha^\vee \rangle + \varepsilon(\alpha)}}{1 - t^{\langle \rho, \alpha^\vee \rangle}}, \ \text{where } \varepsilon(\alpha) := \begin{cases} \langle \lambda, \alpha^\vee \rangle & \text{if } \langle \omega, \alpha \rangle = 0, \\ 1 & \text{if } \langle \omega, \alpha \rangle > 0. \end{cases}$$

Proof. Let $\Psi^+ = \{\alpha \in \Phi : \langle \omega, \alpha \rangle > 0\}$. Given $w \in W$, write $w = xy$, where $y \in W_\omega$ and x is the shortest element of the coset wW_ω. Since W_ω permutes Ψ^+, it follows that

$$\Pr\Bigl(e^{w(\lambda+\rho)} \cdot \textstyle\prod_{\alpha \in \Psi^+} (1 - te^{-w\alpha})\Bigr) = t^{-\langle xy(\lambda+\rho), \rho^\vee \rangle} \cdot \textstyle\prod_{\alpha \in \Psi^+} (1 - t^{1 + \langle x\alpha, \rho^\vee \rangle}).$$

Now if $x \neq 1$, then $x^{-1}\alpha_i < 0$ for some i, and hence $\beta = -x^{-1}\alpha_i$ is a positive root such that $x\beta < 0$, whence $\beta \in \Psi^+$ (or we contradict the fact that x is the shortest member of its coset). However, $1 + \langle x\beta, \rho^\vee\rangle = 1 - \langle \alpha_i, \rho^\vee\rangle = 0$, so the above expression vanishes unless $w = y \in W_\omega$, and hence

$$\begin{aligned}\Pr\Bigl(J\Bigl(e^{\lambda+\rho}\cdot\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})\Bigr)\Bigr) &= \sum_{y\in W_\omega}\operatorname{sgn}(y)t^{-\langle y(\lambda+\rho),\rho^\vee\rangle}\cdot\prod_{\alpha\in\Psi^+}\bigl(1-t^{\langle\rho^\vee,\,\alpha\rangle+1}\bigr)\\ &= \Pr(J_\omega(e^{\lambda+\rho}))\cdot\textstyle\prod_{\alpha\in\Psi^+}\bigl(1-t^{\langle\rho^\vee,\,\alpha\rangle+1}\bigr)\\ &= \Pr(J_\omega(e^{\lambda+\rho}))\cdot\textstyle\prod_{\alpha\in\Psi^+}\bigl(1-t^{\langle\rho,\alpha^\vee\rangle+1}\bigr),\end{aligned}$$

where J_ω denotes anti-symmetrization relative to W_ω, as in the proof of Proposition 2.1. Note that the last equality is justified by the fact that the heights of roots and co-roots are equally distributed on any root system (recall (2.9)), and hence also on any parabolic complement such as Ψ^+. Applying the W_ω-analogue of (2.7), we obtain

$$\Pr\Bigl(J\Bigl(e^{\lambda+\rho}\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})\Bigr)\Bigr) = t^{-\langle\lambda+\rho,\rho^\vee\rangle}\cdot\textstyle\prod_{\alpha\in\Phi^+}\bigl(1-t^{\langle\rho,\alpha^\vee\rangle+\varepsilon(\alpha)}\bigr).$$

To complete the proof, use (2.7) at $\lambda = 0$ and (2.1). □

REMARK 2.11. If $\lambda\in\mathcal{P}^+$, then the leading coefficient of $P_\omega(\lambda)$ is $W_\lambda(t)/(W_\lambda\cap W_\omega)(t)$, by Proposition 2.5. If $\lambda = 0$, then the leading term is the only term (Proposition 2.2(b)), whence $P_\omega(0) = (W(t)/W_\omega(t))\chi(0)$. Thus a very special case of Theorem 2.10 yields a formula for the Poincaré series of a parabolic quotient; viz.,

$$\frac{W(t)}{W_\omega(t)} = \prod_{\alpha\in\Psi^+}\frac{1-t^{\langle\rho,\alpha^\vee\rangle+1}}{1-t^{\langle\rho,\alpha^\vee\rangle}} = \prod_{\alpha\in\Psi^+}\frac{1-t^{\mathrm{ht}(\alpha)+1}}{1-t^{\mathrm{ht}(\alpha)}}, \tag{2.10}$$

again making use of the equidistribution of heights and co-heights in Ψ^+. This is equivalent to Macdonald's formula for the Poincaré series of a finite Weyl group [**M1**].

3. Recurrences

In the following, it will be convenient to assume that $t = q^k$ for some (arbitrary) positive integer k. This imposes no real loss of generality, since any formal power series identity in q and t is valid if and only if it is valid at $t = q^k$ for infinitely many positive integers k. The advantage of this specialization is that the Macdonald kernel simplifies to

$$\Delta(q,t) = \prod_{i=1}^{k-1}(1-q^i)^n\cdot\prod_{i=1}^{k-1}\prod_{\alpha\in\Phi}(1-q^ie^\alpha),$$

and thus may be viewed as an element of the group ring of $\mathcal{Q}$ over $\mathbf{Z}[q]$.

A. Functional equations.

Fix a dominant co-weight $\omega \in \mathcal{P}^\vee$.

Since $\mathcal{P}^\vee$ is dual to $\mathcal{Q}$, it follows that there is a positive integer d no larger than the index of $\mathcal{Q}$ in $\mathcal{P}$ such that the functional $\lambda \mapsto \langle \lambda, \omega \rangle$ on $\mathcal{P}$ takes values in $(1/d)\mathbf{Z}$. Given a Laurent monomial $u = q^a t^b = q^{a+kb}$ $(a, b \in \mathbf{Z})$, let $\varphi_{\omega,u}$ denote the unique $\mathbf{Z}[q^{\pm 1/d}]$-linear automorphism of the group ring $\mathbf{Z}[q^{\pm 1/d}]\langle \mathcal{P} \rangle$ such that

$$\varphi_{\omega,u}(e^\lambda) = u^{\langle \lambda, \omega \rangle} e^\lambda \qquad (\lambda \in \mathcal{P}).$$

In case ω is *minuscule* (i.e., $\langle \omega, \alpha \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Phi$), these automorphisms may be used to construct functional equations satisfied by the Macdonald kernel.

LEMMA 3.1. *If $\omega \in \mathcal{P}^\vee$ is dominant and minuscule, then we have*

(a) $\varphi_{\omega,q}\Big(J(e^\rho)\Delta(q,t) \cdot \prod_{\alpha \in \Psi^+}(1 - t^{-1}e^\alpha)\Big) = q^{m/2}t^{-m} J(e^\rho)\Delta(q,t) \cdot \prod_{\alpha \in \Psi^+}(1 - te^\alpha)$,

(b) $\varphi_{\omega,q}\Big(J(e^\rho)\Delta(q,t) \cdot \varphi_{\omega,t^{-1}}(J(e^{-\rho}))\Big) = J(e^\rho)\Delta(q,t) \cdot \varphi_{\omega,t}(J(e^{-\rho}))$,

where $\Psi^+ = \{\alpha \in \Phi : \langle \omega, \alpha \rangle = 1\}$ and $m = |\Psi^+| = \langle 2\rho, \omega \rangle$.

Proof. We have $\varphi_{\omega,q}(e^\alpha) = q^{\pm 1}e^\alpha$ if $\pm\alpha \in \Psi^+$, and $\varphi_{\omega,q}(e^\alpha) = e^\alpha$ for all other roots α. It follows that

$$\begin{aligned} \frac{\varphi_{\omega,q}(\Delta(q,t))}{\Delta(q,t)} &= \prod_{\alpha \in \Psi^+} \frac{1 - te^\alpha}{1 - qe^\alpha} \cdot \frac{1 - e^{-\alpha}}{1 - tq^{-1}e^{-\alpha}} \\ &= t^{-m} \prod_{\alpha \in \Psi^+} \frac{1 - te^\alpha}{1 - q^{-1}e^{-\alpha}} \cdot \frac{1 - e^{-\alpha}}{1 - qt^{-1}e^\alpha}. \end{aligned}$$

Similarly, by the Weyl denominator formula (i.e., (1.2)), we have

$$\frac{\varphi_{\omega,q}(J(e^\rho))}{J(e^\rho)} = q^{m/2} \prod_{\alpha \in \Psi^+} \frac{1 - q^{-1}e^{-\alpha}}{1 - e^{-\alpha}},$$

and hence

$$\frac{\varphi_{\omega,q}(\Delta(q,t))}{\Delta(q,t)} \cdot \frac{\varphi_{\omega,q}(J(e^\rho))}{J(e^\rho)} = q^{m/2}t^{-m} \prod_{\alpha \in \Psi^+} \frac{1 - te^\alpha}{\varphi_{\omega,q}(1 - t^{-1}e^\alpha)} = \frac{\varphi_{\omega,t}(J(e^{-\rho}))}{\varphi_{\omega,qt^{-1}}(J(e^{-\rho}))}.$$

These two equalities are equivalent to (a) and (b). $\square$

Although some root systems have no nonzero minuscule co-weights (the irreducible examples being $\mathcal{G}_2$, $\mathcal{F}_4$ and $\mathcal{E}_8$), every root system has a long dominant root $\widetilde{\alpha}$ (the highest root, if Φ is irreducible), and the corresponding co-root $\omega = \widetilde{\alpha}^\vee \in \mathcal{P}^\vee$ is *quasi-minuscule* in the sense that $\langle \omega, \alpha \rangle \leqslant 2$ for all roots α, with equality if and only if $\alpha = \widetilde{\alpha}$.

In these cases (i.e., when $\omega = \widetilde{\alpha}^\vee$), it will be convenient to introduce

$$\begin{aligned}\Omega_1(q,t) &:= (1 - qte^{\widetilde{\alpha}}) \cdot \prod_{\alpha\in\Psi^+}(1 - te^{\alpha}),\\ \Omega_2(q,t) &:= \frac{(1 - qte^{\widetilde{\alpha}})(1 - te^{\widetilde{\alpha}})}{1 - t^2e^{\widetilde{\alpha}}} \cdot \varphi_{\omega,t}(J(e^{-\rho})),\end{aligned}\tag{3.1}$$

where $\Psi^+ = \{\alpha \in \Phi : \langle \omega, \alpha\rangle \geqslant 1\}$.

LEMMA 3.2. *If $\widetilde{\alpha}$ is a long dominant root and $\omega = \widetilde{\alpha}^\vee$, then we have*

(a) $\varphi_{\omega,q}\big(J(e^\rho)\Delta(q,t) \cdot \Omega_1(q^{-1},t^{-1})\big) = q^{m/2}t^{-m}J(e^\rho)\Delta(q,t) \cdot \Omega_1(q,t)$,

(b) $\varphi_{\omega,q}\big(J(e^\rho)\Delta(q,t) \cdot \Omega_2(q^{-1},t^{-1})\big) = J(e^\rho)\Delta(q,t) \cdot \Omega_2(q,t)$,

where $m = \langle 2\rho, \omega\rangle = |\Psi^+| + 1$.

Proof. This calculation is similar to the minuscule case; we have

$$\begin{aligned}\frac{\varphi_{\omega,q}(\Delta(q,t))}{\Delta(q,t)} &= \frac{1 - tqe^{\widetilde{\alpha}}}{1 - q^2e^{\widetilde{\alpha}}} \cdot \frac{1 - q^{-1}e^{-\widetilde{\alpha}}}{1 - tq^{-2}e^{-\widetilde{\alpha}}} \cdot \prod_{\alpha\in\Psi^+} \frac{1 - te^{\alpha}}{1 - qe^{\alpha}} \cdot \frac{1 - e^{-\alpha}}{1 - tq^{-1}e^{-\alpha}}\\ &= t^{-m}\frac{1 - qte^{\widetilde{\alpha}}}{1 - q^{-2}e^{-\widetilde{\alpha}}} \cdot \frac{1 - q^{-1}e^{-\widetilde{\alpha}}}{1 - q^2t^{-1}e^{\widetilde{\alpha}}} \cdot \prod_{\alpha\in\Psi^+} \frac{1 - te^{\alpha}}{1 - q^{-1}e^{-\alpha}} \cdot \frac{1 - e^{-\alpha}}{1 - qt^{-1}e^{\alpha}},\\ \frac{\varphi_{\omega,q}(J(e^\rho))}{J(e^\rho)} &= q^{m/2}\frac{1 - q^{-2}e^{-\widetilde{\alpha}}}{1 - q^{-1}e^{-\widetilde{\alpha}}} \cdot \prod_{\alpha\in\Psi^+} \frac{1 - q^{-1}e^{-\alpha}}{1 - e^{-\alpha}},\\ \frac{\varphi_{\omega,q}(\Omega_1(q^{-1},t^{-1}))}{\Omega_1(q,t)} &= \frac{1 - q^2t^{-1}e^{\widetilde{\alpha}}}{1 - qte^{\widetilde{\alpha}}} \cdot \prod_{\alpha\in\Psi^+} \frac{1 - qt^{-1}e^{\alpha}}{1 - te^{\alpha}},\end{aligned}$$

and the combination of these three identities yields

$$\frac{\varphi_{\omega,q}(\Delta(q,t))}{\Delta(q,t)} \cdot \frac{\varphi_{\omega,q}(J(e^\rho))}{J(e^\rho)} \cdot \frac{\varphi_{\omega,q}(\Omega_1(q^{-1},t^{-1}))}{\Omega_1(q,t)} = q^{m/2}t^{-m},$$

which is equivalent to (a). Setting $z = q^{-m/2}t^m$, we also have

$$\begin{aligned}\frac{\varphi_{\omega,q}(\Omega_2(q^{-1},t^{-1}))}{\Omega_2(q,t)} &= \frac{(1 - qt^{-1}e^{\widetilde{\alpha}})(1 - q^2t^{-1}e^{\widetilde{\alpha}})/(1 - q^2t^{-2}e^{\widetilde{\alpha}})}{(1 - qte^{\widetilde{\alpha}})(1 - te^{\widetilde{\alpha}})/(1 - t^2e^{\widetilde{\alpha}})} \cdot \frac{\varphi_{\omega,qt^{-1}}(J(e^{-\rho}))}{\varphi_{\omega,t}(J(e^{-\rho}))}\\ &= z \cdot \frac{(1 - qt^{-1}e^{\widetilde{\alpha}})(1 - q^2t^{-1}e^{\widetilde{\alpha}})(1 - t^2e^{\widetilde{\alpha}})}{(1 - qte^{\widetilde{\alpha}})(1 - te^{\widetilde{\alpha}})(1 - q^2t^{-2}e^{\widetilde{\alpha}})} \cdot \prod_{\alpha\in\Psi^+} \frac{\varphi_{\omega,qt^{-1}}(1 - e^{\alpha})}{\varphi_{\omega,t}(1 - e^{\alpha})}\\ &= z \cdot \frac{1 - q^2t^{-1}e^{\widetilde{\alpha}}}{1 - qte^{\widetilde{\alpha}}} \cdot \prod_{\alpha\in\Psi^+} \frac{1 - qt^{-1}e^{\alpha}}{1 - te^{\alpha}} = z \cdot \frac{\varphi_{\omega,q}(\Omega_1(q^{-1},t^{-1}))}{\Omega_1(q,t)}.\end{aligned}$$

Combining this with the previous identity yields (b). $\square$

B. The minuscule recurrence.

In case Φ is irreducible and has a minuscule co-weight ω, the following result allows the graded multiplicities $C_\lambda(q,t)$ to be computed recursively, starting from knowledge of $C_0(q,t)$ and the transition matrix $A^\omega_{\lambda\mu}(t)$.

Recall from Section 2 that $\sigma(\Theta)$ denotes the sum of the members of Θ.

THEOREM 3.3. *If $\omega \in \mathcal{P}^\vee$ is dominant and minuscule, then for all $\lambda \in \mathcal{Q}$, we have*

$$\sum_{\mu \leqslant \lambda^+} B^\omega_{\lambda\mu}(q,t) C_\mu(q,t) = 0,$$

where $B^\omega_{\lambda\mu}(q,t) := A^\omega_{\lambda\mu}(t) - q^{\langle\lambda,\omega\rangle} t^{\langle 2\rho,\omega\rangle} A^\omega_{\lambda\mu}(t^{-1})$. Furthermore,

$$B^\omega_{\lambda,\lambda^+}(q,t) = \big(1 - q^{\langle\lambda,\omega\rangle} t^{\langle\eta(\lambda),\omega\rangle}\big) A^\omega_{\lambda,\lambda^+}(t), \tag{3.2}$$

where $\eta(\lambda) := \sigma(\{\alpha \in \Phi : \langle\lambda,\alpha\rangle > 0\})$, and if λ is dominant, then

$$B^\omega_{\lambda\lambda}(q,t) = \big(1 - q^{\langle\lambda,\omega\rangle} t^{\langle\eta(\lambda),\omega\rangle}\big) W_\lambda(t)/(W_\lambda \cap W_\omega)(t).$$

Proof. Let $\Psi^+ = \{\alpha \in \Phi : \langle\omega,\alpha\rangle = 1\}$. For any element f of the group ring of $\mathcal{P}$, we have $[e^\mu]\varphi_{\omega,q}(f) = q^{\langle\mu,\omega\rangle}[e^\mu]f$, so

$$\begin{aligned}
&[e^{\lambda+\rho}]\varphi_{\omega,q}\Big(J(e^\rho)\Delta(q,t) \cdot \prod_{\alpha\in\Psi^+}(1 - t^{-1}e^\alpha)\Big) \\
&\qquad = q^{\langle\lambda+\rho,\omega\rangle}[e^{\lambda+\rho}]J(e^\rho)\Delta(q,t) \cdot \prod_{\alpha\in\Psi^+}(1 - t^{-1}e^\alpha) \\
&\qquad = q^{\langle\lambda+\rho,\omega\rangle} \sum_{\mu\in\mathcal{Q}^+} [e^{\lambda+\rho}] C_\mu(q,t) J(e^{\mu+\rho}) \cdot \prod_{\alpha\in\Psi^+}(1 - t^{-1}e^\alpha) \\
&\qquad = q^{\langle\lambda+\rho,\omega\rangle} \sum_{\mu\leqslant\lambda^+} A^\omega_{\lambda\mu}(t^{-1}) C_\mu(q,t),
\end{aligned} \tag{3.3}$$

the final equality being a consequence of Proposition 2.2(a). Similarly,

$$\begin{aligned}
&[e^{\lambda+\rho}] q^{\langle\rho,\omega\rangle} t^{-\langle 2\rho,\omega\rangle} J(e^\rho)\Delta(q,t) \cdot \prod_{\alpha\in\Psi^+}(1 - te^\alpha), \\
&\qquad = q^{\langle\rho,\omega\rangle} t^{-\langle 2\rho,\omega\rangle} \sum_{\mu\in\mathcal{Q}^+} [e^{\lambda+\rho}] C_\mu(q,t) J(e^{\mu+\rho}) \cdot \prod_{\alpha\in\Psi^+}(1 - te^\alpha) \\
&\qquad = q^{\langle\rho,\omega\rangle} t^{-\langle 2\rho,\omega\rangle} \sum_{\mu\leqslant\lambda^+} A^\omega_{\lambda\mu}(t) C_\mu(q,t),
\end{aligned} \tag{3.4}$$

again by Proposition 2.2(a). The claimed relation now follows by equating (3.3) and (3.4), which is justified by Lemma 3.1(a). To prove (3.2), note that Corollary 2.7 implies

$$B^\omega_{\lambda,\lambda^+}(q,t) = \big(1 - q^{\langle\lambda,\omega\rangle} t^{N(\lambda,\omega)}\big) A^\omega_{\lambda,\lambda^+}(t),$$

where $N(\lambda,\omega) := \langle 2\rho,\omega\rangle - 2d(\lambda) - N_0(\lambda,\omega)$. Also, we may assume that λ is dominant with respect to W_ω, otherwise $A^\omega_{\lambda,\lambda^+}(t) = 0$ by Lemma 2.4. Hence,

$$d(\lambda) = |\{\alpha \in \Phi^+ : \langle\lambda,\alpha\rangle < 0, \langle\omega,\alpha\rangle > 0\}|.$$

The fact that ω is minuscule implies that for every $\Theta \subseteq \Phi^+$, the number of roots $\alpha \in \Theta$ such that $\langle\omega,\alpha\rangle > 0$ is $\langle\sigma(\Theta),\omega\rangle$. In particular,

$$\begin{aligned} d(\lambda) &= \langle\sigma(\{\alpha \in \Phi^+ : \langle\lambda,\alpha\rangle < 0\}),\omega\rangle, \\ N_0(\lambda,\omega) &= \langle\sigma(\{\alpha \in \Phi^+ : \langle\lambda,\alpha\rangle = 0\}),\omega\rangle. \end{aligned}$$

Bearing in mind that $\langle 2\rho,\omega\rangle = \langle\sigma(\Phi^+),\omega\rangle$, we obtain

$$\begin{aligned} N(\lambda,\omega) &= \langle\sigma(\{\alpha \in \Phi^+ : \langle\lambda,\alpha\rangle > 0\}),\omega\rangle - \langle\sigma(\{\alpha \in \Phi^+ : \langle\lambda,\alpha\rangle < 0\}),\omega\rangle \\ &= \langle\sigma(\{\alpha \in \Phi^+ : \langle\lambda,\alpha\rangle > 0\}),\omega\rangle + \langle\sigma(\{\alpha \in \Phi^- : \langle\lambda,\alpha\rangle > 0\}),\omega\rangle = \langle\eta(\lambda),\omega\rangle, \end{aligned}$$

and thus (3.2) follows. In the dominant case, recall that $A^\omega_{\lambda\lambda}(t) = W_\lambda(t)/(W_\lambda \cap W_\omega)(t)$ by Proposition 2.5(d). $\square$

Assuming Φ is irreducible, every simple root occurs in the support of every nonzero dominant co-weight ω, so we have $\langle\lambda,\omega\rangle \geqslant 1$ for all $\lambda \in \mathcal{Q}^+$ except $\lambda = 0$. Theorem 3.3 thus shows that the matrix $B^\omega := [B^\omega_{\lambda\mu}(q,t)]_{\lambda,\mu\in\mathcal{Q}^+}$ is lower triangular and has a nonzero diagonal except for the $(0,0)$-entry, so the null space of B^ω is one-dimensional. It follows that the relations in Theorem 3.3 uniquely determine the multiplicities $C_\lambda(q,t)$ up to an overall scalar factor. Equivalently, if we renormalize the multiplicities by setting

$$\bar{C}_\lambda(q,t) := C_\lambda(q,t)/C_0(q,t),$$

then (trivially) $\bar{C}_0(q,t) = 1$, and Theorem 3.3 provides a triangular linear system that uniquely determines all remaining normalized multiplicities $\bar{C}_\lambda(q,t)$ for $\lambda \in \mathcal{Q}^+$. Given Cherednik's evaluation of $C_0(q,t)$ (see (1.5)), one may then recover $C_\lambda(q,t)$.

C. The quasi-minuscule recurrence.

There is a variation of Theorem 3.3 that may be developed from the choice of a long dominant root $\widetilde{\alpha}$, and thus is available for every root system, including those that have no (nonzero) minuscule co-weights. Letting $\omega = \widetilde{\alpha}^\vee$, recall that ω is a quasi-minuscule co-weight in the sense that for all roots β, we have $\langle\omega,\beta\rangle \in \{0,\pm1,\pm2\}$, and $\langle\omega,\beta\rangle = 2$ occurs only when $\beta = \widetilde{\alpha}$.

In this setting (i.e., $\omega = \widetilde{\alpha}^\vee$), it will be convenient to introduce the matrix

$$A_{\lambda\mu}(q,t) := A^\omega_{\lambda\mu}(t) - qtA^\omega_{\lambda-\widetilde{\alpha},\mu}(t).$$

This may be interpreted as the coefficient of $\chi(\mu)$ in $P_\omega(\lambda) - qtP_\omega(\lambda - \widetilde{\alpha})$.

THEOREM 3.4. *Assume $\widetilde{\alpha}$ is a long dominant root and $\omega = \widetilde{\alpha}^\vee$.*

(a) *We have $\sum_{\mu\in\mathcal{Q}^+} B_{\lambda\mu}(q,t)C_\mu(q,t) = 0$ for all $\lambda \in \mathcal{Q}$, where*

$$B_{\lambda\mu}(q,t) = B^\omega_{\lambda\mu}(q,t) := A_{\lambda\mu}(q,t) - q^{\langle\lambda,\widetilde{\alpha}^\vee\rangle}t^{\langle 2\rho,\widetilde{\alpha}^\vee\rangle}A_{\lambda\mu}(q^{-1},t^{-1}).$$

(b) *If $\langle\lambda,\widetilde{\alpha}^\vee\rangle \geqslant 1$, then $B_{\lambda\mu}(q,t) = 0$ unless $\mu \leqslant \lambda^+$.*

(c) *If $\langle\lambda,\widetilde{\alpha}^\vee\rangle = 1$, then $B_{\lambda\mu}(q,t) = 2\big(A^\omega_{\lambda\mu}(t) - qt^{\langle 2\rho,\widetilde{\alpha}^\vee\rangle}A^\omega_{\lambda\mu}(t^{-1})\big)$.*

(d) *If $\langle\lambda,\widetilde{\alpha}^\vee\rangle \geqslant 2$ and $c = 1$, or $\langle\lambda,\widetilde{\alpha}^\vee\rangle = 1$ and $c = 2$, then*

$$B_{\lambda,\lambda^+}(q,t) = c\big(1 - q^{\langle\lambda,\widetilde{\alpha}^\vee\rangle}t^{\langle\eta(\lambda),\widetilde{\alpha}^\vee\rangle}\big)A^\omega_{\lambda,\lambda^+}(t),$$

where $\eta(\lambda)$ is defined as in Theorem 3.3. If λ is also dominant, then

$$B_{\lambda\lambda}(q,t) = c\big(1 - q^{\langle\lambda,\widetilde{\alpha}^\vee\rangle}t^{\langle\eta(\lambda),\widetilde{\alpha}^\vee\rangle}\big)W_\lambda(t)/(W_\lambda \cap W_\omega)(t).$$

Proof. (a) Setting $\Psi^+ = \{\alpha \in \Phi : \langle\omega,\alpha\rangle \geqslant 1\}$, a calculation similar to (3.3) yields

$$\begin{aligned}[e^{\lambda+\rho}]\varphi_{\omega,q}\Big(J(e^\rho)\Delta(q,t)\cdot(1-q^{-1}t^{-1}e^{\widetilde{\alpha}})\cdot\prod_{\alpha\in\Psi^+}(1-t^{-1}e^\alpha)\Big)\\ = q^{\langle\lambda+\rho,\widetilde{\alpha}^\vee\rangle}\sum_{\mu\in\mathcal{Q}^+}\big(A^\omega_{\lambda\mu}(t^{-1}) - q^{-1}t^{-1}A^\omega_{\lambda-\widetilde{\alpha},\mu}(t^{-1})\big)C_\mu(q,t), \qquad (3.5)\end{aligned}$$

using Proposition 2.2(a). Similarly, the calculation analogous to (3.4) yields

$$\begin{aligned}[e^{\lambda+\rho}]q^{\langle\rho,\widetilde{\alpha}^\vee\rangle}t^{-\langle 2\rho,\widetilde{\alpha}^\vee\rangle}J(e^\rho)\Delta(q,t)(1-qte^{\widetilde{\alpha}})\cdot\prod_{\alpha\in\Psi^+}(1-te^\alpha),\\ = q^{\langle\rho,\widetilde{\alpha}^\vee\rangle}t^{-\langle 2\rho,\widetilde{\alpha}^\vee\rangle}\sum_{\mu\in\mathcal{Q}^+}\big(A^\omega_{\lambda\mu}(t) - qtA^\omega_{\lambda-\widetilde{\alpha},\mu}(t)\big)C_\mu(q,t), \qquad (3.6)\end{aligned}$$

again by Proposition 2.2(a). The claimed relation now follows by equating (3.5) and (3.6), which is justified by Lemma 3.2(a).

(b) Whenever $B_{\lambda\mu}(q,t) \neq 0$, we must have $A_{\lambda\mu}(q,t) = A^\omega_{\lambda\mu}(t) - qtA^\omega_{\lambda-\widetilde{\alpha},\mu}(t) \neq 0$, and hence $\mu \leqslant \lambda^+$ or $\mu \leqslant (\lambda-\widetilde{\alpha})^+$ by Proposition 2.2(b). However, given that $\langle\lambda,\widetilde{\alpha}^\vee\rangle \geqslant 1$, it follows via (1.1) that $(\lambda-\widetilde{\alpha})^+ \leqslant \lambda^+$.

(c) If $\langle\lambda,\widetilde{\alpha}^\vee\rangle = 1$, then $s_{\widetilde{\alpha}}(\lambda) = \lambda - \widetilde{\alpha}$ and $t^{m-1}A^\omega_{\lambda,\mu}(t^{-1}) = A^\omega_{\lambda-\widetilde{\alpha},\mu}(t)$ (Corollary 2.9), where $m = |\Psi^+| + 1 = \langle 2\rho,\widetilde{\alpha}^\vee\rangle$. Hence,

$$A_{\lambda\mu}(q,t) = A^\omega_{\lambda\mu}(t) - qt^{\langle 2\rho,\widetilde{\alpha}^\vee\rangle}A^\omega_{\lambda\mu}(t^{-1}) = -qt^{\langle 2\rho,\widetilde{\alpha}^\vee\rangle}A_{\lambda\mu}(q^{-1},t^{-1}),$$

and this implies the claimed formula.

(d) If $\langle \lambda, \widetilde{\alpha}^\vee \rangle \geqslant 2$, then $\lambda - \widetilde{\alpha}$ is shorter than λ, and therefore $(\lambda - \widetilde{\alpha})^+ < \lambda^+$ by (1.1). It follows that $A^\omega_{\lambda-\widetilde{\alpha},\lambda^+}(t) = 0$ by Proposition 2.2(b), and hence

$$B_{\lambda,\lambda^+}(q,t) = A^\omega_{\lambda,\lambda^+}(t) - q^{\langle \lambda, \widetilde{\alpha}^\vee \rangle} t^{\langle 2\rho, \widetilde{\alpha}^\vee \rangle} A^\omega_{\lambda,\lambda^+}(t^{-1}) = (1 - q^{\langle \lambda, \widetilde{\alpha}^\vee \rangle} t^{N(\lambda, \widetilde{\alpha}^\vee)}) A^\omega_{\lambda,\lambda^+}(t)$$

by Corollary 2.7, where $N(\lambda, \widetilde{\alpha}^\vee) = \langle 2\rho, \widetilde{\alpha}^\vee \rangle - 2d(\lambda) - N_0(\lambda, \widetilde{\alpha}^\vee)$. Part (c) shows that the same formula holds in the case $\langle \lambda, \widetilde{\alpha}^\vee \rangle = 1$, except that there is now an extra factor of 2. The remainder of the proof now follows by the same reasoning used to prove Theorem 3.3. The only adjustment involves the verification that $N(\lambda, \widetilde{\alpha}^\vee) = \langle \eta(\lambda), \widetilde{\alpha}^\vee \rangle$: in expressions of the form $\langle \sigma(\Theta), \widetilde{\alpha}^\vee \rangle$ ($\Theta \subseteq \Phi^+$), the contribution of $\widetilde{\alpha} \in \Theta$ is doubled. □

Assuming Φ is irreducible, the long dominant root $\widetilde{\alpha}$ is unique and $\langle \lambda, \widetilde{\alpha}^\vee \rangle \geqslant 1$ for all dominant λ except $\lambda = 0$. Thus Theorem 3.4 shows that the matrix $B = [B_{\lambda\mu}(q,t)]_{\lambda,\mu \in \mathcal{Q}^+}$ is lower triangular except for the row corresponding to $\lambda = 0$ (where there are entries in the columns $\leqslant \widetilde{\alpha}$), and has a nonzero diagonal (the $(0,0)$-entry is exceptional, but still easily shown to be nonzero), so the null space of B is at most one-dimensional.

On the other hand, the multiplicities $C_\lambda(q,t)$ provide a nontrivial point annihilated by B, so we conclude that the null space is exactly one-dimensional, and that the relations in Theorem 3.4 uniquely determine the multiplicities $C_\lambda(q,t)$ up to an overall scalar factor. Equivalently, if we use the normalized multiplicities $\bar{C}_\lambda(q,t)$ and eliminate $\bar{C}_0(q,t) = 1$, Theorem 3.4 provides a triangular linear system that uniquely determines all remaining multiplicities $\bar{C}_\lambda(q,t)$ for $\lambda \in \mathcal{Q}^+$.

A more careful analysis of Theorem 3.4 leads to the following:

Corollary 3.5. *If Φ is irreducible with highest root $\widetilde{\alpha}$, then for all $\lambda \in \mathcal{Q}^+$, the multiplicity $\bar{C}_\lambda(q,t)$ is a rational function of q and t such that $\bar{C}_\lambda(q,t) = \bar{C}_\lambda(q^{-1},t^{-1})$. Furthermore, $\bar{C}_\lambda(q,t)$ has a denominator consisting of a product of terms of the form $1 - q^a t^b$ and $1 - t^{c+1}$, where $a > 0$, $1 \leqslant b \leqslant \langle 2\rho, \widetilde{\alpha}^\vee \rangle$, and c is an exponent of Φ.*

Proof. Assume that λ is dominant and nonzero. It is clear from the definition that $A^\omega_{\lambda\mu}(t)$ is a polynomial of degree $\leqslant |\{\alpha \in \Phi : \langle \alpha, \widetilde{\alpha}^\vee \rangle \geqslant 0\}| = \langle 2\rho, \widetilde{\alpha}^\vee \rangle - 1$, so $A_{\lambda\mu}(q,t)$ is of degree $\leqslant 1$ with respect to q and degree $\leqslant \langle 2\rho, \widetilde{\alpha}^\vee \rangle$ with respect to t. Hence $B_{\lambda\mu}(q,t) \in \mathbf{Z}[q,t]$ and parts (a) and (b) of Theorem 3.4 imply that

$$\sum_{\mu \leqslant \lambda} B_{\lambda\mu}(q,t) \bar{C}_\mu(q,t) = 0, \tag{3.7}$$

so by induction, if $\bar{C}_\mu(q,t) \in \mathbf{Q}(q,t)$ for all dominant $\mu < \lambda$, then the same must be true for $\bar{C}_\lambda(q,t)$. Moreover, it is clear from the definition that

$$B_{\lambda\mu}(q,t) = -q^{\langle \lambda, \widetilde{\alpha}^\vee \rangle} t^{\langle 2\rho, \widetilde{\alpha}^\vee \rangle} B_{\lambda\mu}(q^{-1},t^{-1})$$

so again by induction, if $\bar{C}_\mu(q,t) = \bar{C}_\mu(q^{-1},t^{-1})$ for all dominant $\mu < \lambda$, then (3.7) implies that the same must be true for $\bar{C}_\lambda(q,t)$.

Finally, note that (3.7) shows that the least denominator for $\bar{C}_\lambda(q,t)$ is a divisor the product of $B_{\lambda\lambda}(q,t)$ and the denominators for $\bar{C}_\mu(q,t)$ for $\mu < \lambda$. By Theorem 3.4(d), the former divides $(1-q^a t^b)W(t)$, where $a = \langle \lambda, \widetilde{\alpha}^\vee \rangle > 0$ and $b = \langle \eta(\lambda), \widetilde{\alpha}^\vee \rangle \leqslant \langle 2\rho, \widetilde{\alpha}^\vee \rangle$. Note also that $\widetilde{\alpha}$ is a summand of $\eta(\lambda)$, and hence $b \geqslant 2$. (If $\langle \lambda, \widetilde{\alpha}^\vee \rangle = 1$, there is an extra factor of 2 in $B_{\lambda\lambda}(q,t)$, but part (c) shows that this extra factor may be cancelled from (3.7).) It is well-known that $W(t)$ is a divisor of factors of the form $1-t^{c+1}$, where c ranges over the exponents, so $\bar{C}_\lambda(q,t)$ has a denominator of the claimed form by induction. $\square$

Much sharper constraints on the denominator of $\bar{C}_\lambda(q,t)$ will be provided in Section 6.

REMARK 3.6. As noted in the introduction, the Macdonald kernel at $q = 0$ is the graded character of the symmetric algebra $S(\mathfrak{g})$, and the normalized multiplicities $\bar{C}_\lambda(0,t)$ are the generating functions for Kostant's "generalized exponents." In this special case, the recurrences of Theorems 3.3 and 3.4 (for dominant λ) uniformly simplify to

$$\sum_{\mu \leqslant \lambda} A^\omega_{\lambda\mu}(t) \bar{C}_\mu(0,t) = 0 \qquad (\lambda \neq 0)$$

for both minuscule and quasi-minuscule co-weights ω. It is interesting to note that this specialized recurrence might have been difficult to find directly—many intermediate steps in the derivation are invalid when $q = 0$.

4. Explicit formulas

Throughout this section, we assume that Φ is irreducible with highest root $\widetilde{\alpha}$. In case there are two orbits of roots, then there is a second dominant root, denoted $\bar{\alpha}$. These two dominant roots may be distinguished by the fact that $\widetilde{\alpha}$ is long and $\bar{\alpha}$ is short. In case there is only one root orbit, we adopt the convention that all roots are regarded as being both long and short, and set $\bar{\alpha} = \widetilde{\alpha}$.

The exponents of Φ will be of fundamental importance in what follows. They may be defined in various ways, but the most appropriate one for our purpose involves the distribution of heights of positive roots. More specifically, if the exponents of Φ are $e_1, \ldots, e_n$, then the positive roots may be partitioned into n blocks so that the jth block has roots whose heights are $1, 2, \ldots, e_j$; i.e.,

$$\sum_{\alpha \in \Phi^+} t^{\mathrm{ht}(\alpha)} = \sum_{j=1}^{n} t + t^2 + \cdots + t^{e_j} = \frac{t(n - K(t))}{1-t}, \tag{4.1}$$

where $K(t) := \sum_j t^{e_j}$ denotes the exponent generating function. Implicit in this definition is that the number of positive roots of height k is a non-increasing function of k. This can be easily established in many ways (e.g., via representation theory), but we do not know of a completely elementary (and uniform) explanation. On the other hand, one could use the left and right extremes of (4.1) as the definition of the polynomial $K(t)$; the fact that it has nonnegative coefficients will not matter here.

Φ	h	h_s	$h_s^\vee$	e_j	$\bar{e}_j$	$\bar{e}_j^\vee$
$\mathcal{A}_n$	$n+1$			$1, 2, \ldots, n$		
$\mathcal{B}_n$	$2n$	$n+1$	$2n-1$	$1, 3, \ldots, 2n-1$	n	$2, 4, \ldots, 2n-2$
$\mathcal{C}_n$	$2n$	$2n-1$	$n+1$	$1, 3, \ldots, 2n-1$	$2, 4, \ldots, 2n-2$	n
$\mathcal{D}_n$	$2n-2$			$1, 3, \ldots, 2n-3, n-1$		
$\mathcal{E}_6$	12			$1, 4, 5, 7, 8, 11$		
$\mathcal{E}_7$	18			$1, 5, 7, 9, 11, 13, 17$		
$\mathcal{E}_8$	30			$1, 7, 11, 13, 17, 19, 23, 29$		
$\mathcal{F}_4$	12	9	9	$1, 5, 7, 11$	$4, 8$	$4, 8$
$\mathcal{G}_2$	6	4	4	$1, 5$	3	3

TABLE 4.1. Exponent data.

Similarly, the height distribution on the set Φ_s^+ of short positive roots is non-increasing, so there are unique positive integers $\bar{e}_1, \bar{e}_2, \ldots$ such that

$$\sum_{\alpha \in \Phi_s^+} t^{\mathrm{ht}(\alpha)} = \sum_{j=1}^{n_s} t + t^2 + \cdots + t^{\bar{e}_j} = \frac{t(n_s - K_s(t))}{1-t}, \tag{4.2}$$

where $K_s(t) := \sum_j t^{\bar{e}_j}$, and n_s is the number of short simple roots. We refer to $\bar{e}_1, \bar{e}_2, \ldots$ as the *short exponents* of Φ. Again, if one refuses to accept the monotonicity of heights of short roots, one may take (4.2) as the definition of a polynomial $K_s(t)$ with possibly negative coefficients.

Recalling that (2.9) shows that the distribution of heights in Φ and $\Phi^\vee$ are the same, it follows that Φ and $\Phi^\vee$ have the same exponents. On the other hand, the distribution of heights of short roots in Φ and $\Phi^\vee$ need not be the same; we let $\bar{e}_1^\vee, \bar{e}_2^\vee, \ldots$ denote the short exponents of $\Phi^\vee$, and $K_s^\vee(t)$ the corresponding generating function. Thus,

$$\sum_{\alpha \in \Phi_l^+} t^{\langle \rho, \alpha^\vee \rangle} = \frac{t(n_l - K_s^\vee(t))}{1-t}, \tag{4.3}$$

where $\Phi_l^+ \subset \Phi$ is the set of long positive roots and n_l is the number of long simple roots.

Finally, we need to introduce the parameters

$$h := 1 + \langle \rho^\vee, \widetilde{\alpha} \rangle = 1 + \langle \rho, \bar{\alpha}^\vee \rangle, \quad h_s := 1 + \langle \rho^\vee, \bar{\alpha} \rangle, \quad h_s^\vee := 1 + \langle \rho, \widetilde{\alpha}^\vee \rangle.$$

Note that h is the Coxeter number, and $h-1$ is the height of the highest root of Φ (or $\Phi^\vee$), or equivalently, the largest exponent of Φ (or $\Phi^\vee$). Similarly, $h_s - 1$ is the largest short exponent of Φ, and $h_s^\vee - 1$ is the largest short exponent of $\Phi^\vee$. We remark that the quantity $2h_s^\vee - 2 = \langle 2\rho, \widetilde{\alpha}^\vee \rangle$ has appeared in Lemma 3.2, Theorem 3.4, and Corollary 3.5.

A list of the above parameters, along with the exponents, short exponents, and co-short exponents for each irreducible root system is provided in Table 4.1. Note that if Φ has only one root orbit, then $K(t) = K_s(t) = K_s^\vee(t)$ and $h = h_s = h_s^\vee$.

The main result of this section is the following pair of formulas for normalized multiplicities in the Macdonald kernel. These generalize results of Bazlov [**Ba**] for graded multiplicities in $\Lambda(\mathfrak{g})$ (i.e., the specialization $(q,t)=(-q,q^2)$), although it should be noted that Bazlov's formula in the $\bar\alpha$ case is not expressed in terms of short exponents.

THEOREM 4.1. *If Φ is irreducible with highest root $\widetilde\alpha$ and short dominant root $\bar\alpha$, then*

$$\bar C_{\widetilde\alpha}(q,t)=\frac{1-qt^{-1}}{1-qt^{h-1}}K(t), \tag{4.4}$$

$$\bar C_{\bar\alpha}(q,t)=\frac{1-qt^{-1}}{1-qt^{h-1}}K_s(t). \tag{4.5}$$

REMARK 4.2. The fact that $\bar C_{\widetilde\alpha}(0,t)=K(t)$ means that the "classical" exponents are the generalized exponents for the adjoint representation, a result due to Kostant [**K1**]. Similarly, the short exponents are the generalized exponents for the irreducible representation whose highest weight is $\bar\alpha$.

LEMMA 4.3. *We have*

(a) $K_s(t)=t^hK_s(t^{-1})$,
(b) $K(t)=t^hK(t^{-1})$,
(c) $W(t)/W_{\bar\alpha}(t)=t^{h_s-h-1}K_s(t)(1-t^h)/(1-t)$,
(d) $W(t)/W_{\widetilde\alpha}(t)=t^{h_s^\vee-h-1}K_s^\vee(t)(1-t^h)/(1-t)$,
(e) $\Pr(\chi(\bar\alpha))=t^{-h}K_s(t)(1-t^{h+1})/(1-t)$,
(f) $\Pr(\chi(\widetilde\alpha))=t^{-h}K(t)(1-t^{h+1})/(1-t)$.

Proof. One may recognize $W(t)/W_{\bar\alpha}(t)$ as the Poincaré series for the set of shortest coset representatives for $W/W_{\bar\alpha}$. There is one such coset for each short root β. Furthermore, short roots are quasi-minuscule, so $s_i\beta\in\{\beta,\beta\pm\alpha_i\}$ for all i, except in the case $\beta=\pm\alpha_i$. It follows that the length of the shortest member of the coset corresponding to β is a function of height. More precisely, it is $\mathrm{ht}(\bar\alpha)-\mathrm{ht}(\beta)$ if β is positive, and $\mathrm{ht}(\bar\alpha)-\mathrm{ht}(\beta)-1$ if β is negative, hence

$$\begin{aligned} W(t)/W_{\bar\alpha}(t) &= \sum_{\beta\in\Phi_s^+} t^{\mathrm{ht}(\bar\alpha)-\mathrm{ht}(\beta)}+t^{\mathrm{ht}(\bar\alpha)+\mathrm{ht}(\beta)-1}\\ &= t^{h_s-1}\frac{t^{-1}(n_s-K_s(t^{-1}))}{1-t^{-1}}+t^{h_s-2}\frac{t(n_s-K_s(t))}{1-t}\\ &= t^{h_s-1}\frac{K_s(t^{-1})-K_s(t)}{1-t}, \end{aligned} \tag{4.6}$$

the second equality being a consequence of (4.2). Furthermore, via (2.8) we have

$$\Pr(\chi(\bar\alpha))=t^{-\langle\rho^\vee,\,\bar\alpha\rangle}\cdot\prod_{\alpha\in\Phi^+}\frac{1-t^{\langle\bar\alpha+\rho,\alpha^\vee\rangle}}{1-t^{\langle\rho,\alpha^\vee\rangle}},$$

and $\bar{\alpha}$ is quasi-minuscule, so as α varies over the positive roots, either $\langle\bar{\alpha},\alpha^\vee\rangle$ is 2 (if $\alpha=\bar{\alpha}$) or 0 or 1 (otherwise). Therefore,

$$\mathrm{Pr}(\chi(\bar{\alpha})) = t^{-(h_s-1)}\cdot\frac{1-t^{h+1}}{1-t^h}\cdot\prod_{\alpha\in\Psi^+}\frac{1-t^{\langle\rho,\alpha^\vee\rangle+1}}{1-t^{\langle\rho,\alpha^\vee\rangle}},$$

where $\Psi^+=\{\alpha\in\Phi:\langle\bar{\alpha},\alpha^\vee\rangle>0\}$. Comparing this with (2.10), we see that the third factor is $W(t)/W_{\bar{\alpha}}(t)$, so (4.6) implies

$$\mathrm{Pr}(\chi(\bar{\alpha})) = \frac{1-t^{h+1}}{1-t^h}\cdot\frac{K_s(t^{-1})-K_s(t)}{1-t}. \tag{4.7}$$

On the other hand, one may compute the principal specialization of $\chi(\bar{\alpha})$ a second way, by recognizing that there is a simple description of the monomial expansion of $\chi(\bar{\alpha})$. Indeed, if $\beta\in\Phi_s$ is not dominant, we must have $\langle\beta,\alpha_i^\vee\rangle<0$ for some i, so either $\langle\beta,\alpha_i^\vee\rangle=-1$ or $\beta=-\alpha_i$ and α_i is short. In the former case, s_i fixes $\beta+\rho$ (hence $J(e^{\beta+\rho})=0$), and in the latter case, $s_i(\beta+\rho)=\rho$ (hence $J(e^{\beta+\rho})=-J(e^\rho)$). Therefore,

$$\sum_{\beta\in\Phi_s}e^\beta J(e^\rho)=\sum_{\beta\in\Phi_s}J(e^{\beta+\rho})=J(e^{\bar{\alpha}+\rho})-n_sJ(e^\rho),$$

and hence

$$\chi(\bar{\alpha})=n_se^0+\sum_{\beta\in\Phi_s}e^\beta.$$

Taking the principal specialization and applying (4.2), we obtain

$$\mathrm{Pr}(\chi(\bar{\alpha}))=n_s+\frac{t^{-1}(n_s-K_s(t^{-1}))}{1-t^{-1}}+\frac{t(n_s-K_s(t))}{1-t}=\frac{K_s(t^{-1})-tK_s(t)}{1-t}.$$

Comparing this with (4.7) yields (a). Using (a) to replace $K_s(t^{-1})$ with $t^{-h}K_s(t)$ in (4.6) and (4.7) yields (c) and (e). Also, (b) is a well-known property of the exponents (e.g., see Section V.6.2 of [**B**]), and (d) is obtained by applying (c) to $\Phi^\vee$. Finally, for (f), note that $\chi(\widetilde{\alpha})$ is the character of the adjoint representation, so we have

$$\chi(\widetilde{\alpha})=ne^0+\sum_{\beta\in\Phi}e^\beta,$$

and hence (4.1) implies

$$\mathrm{Pr}(\chi(\widetilde{\alpha}))=n+\frac{t^{-1}(n-K(t^{-1}))}{1-t^{-1}}+\frac{t(n-K(t))}{1-t}=\frac{K(t^{-1})-tK(t)}{1-t}.$$

Now use (b) to replace $K(t^{-1})$ with $t^{-h}K(t)$. $\square$

Proof of (4.5). Set $\omega = \widetilde{\alpha}^\vee$. We may assume that $\bar{\alpha} \neq \widetilde{\alpha}$, since the remaining instances of (4.5) are subsumed by (4.4). Bearing in mind that every simple root must occur in the support of $\widetilde{\alpha}$ and $\bar{\alpha}$ is dominant, it follows that $\langle \bar{\alpha}, \widetilde{\alpha} \rangle > 0$, and hence $\langle \bar{\alpha}, \widetilde{\alpha}^\vee \rangle = 1$, since $\bar{\alpha}$ is quasi-minuscule and (in this case) distinct from $\widetilde{\alpha}$. Thus by Theorem 3.4(c),

$$B_{\bar{\alpha},\mu}(q,t) = 2\big(A^{\omega}_{\bar{\alpha},\mu}(t) - qt^{2h_s^\vee - 2} A^{\omega}_{\bar{\alpha},\mu}(t^{-1})\big) \qquad (\mu \in \mathcal{Q}^+). \tag{4.8}$$

Furthermore, it is easy to show that every nonzero $\mu \in \mathcal{Q}^+$ is $\geqslant \bar{\alpha}$ (e.g., Proposition 2.1 of [**St2**]), so 0 and $\bar{\alpha}$ are the only dominant weights $\leqslant \bar{\alpha}$. Hence Theorem 3.4 implies

$$B_{\bar{\alpha},\bar{\alpha}}(q,t)\bar{C}_{\bar{\alpha}}(q,t) + B_{\bar{\alpha},0}(q,t) = 0. \tag{4.9}$$

Since $\Delta(q,t)$ collapses to 1 when $q = t$, it follows that $\bar{C}_\mu(t,t) = 0$ if and only if $\mu \neq 0$, and hence Theorem 3.4(a) also implies

$$B_{\lambda,0}(t,t) = 0 \qquad (\lambda \in \mathcal{Q}). \tag{4.10}$$

In the special case $\lambda = \bar{\alpha}$, we obtain $A^{\omega}_{\bar{\alpha},0}(t) = t^{2h_s^\vee - 1} A^{\omega}_{\bar{\alpha},0}(t^{-1})$ via (4.8), and hence

$$B_{\bar{\alpha},0}(q,t) = 2(1 - qt^{-1}) A^{\omega}_{\bar{\alpha},0}(t).$$

Theorem 3.4(d) also describes the diagonal terms of the matrix B explicitly, including

$$B_{\bar{\alpha},\bar{\alpha}}(q,t) = 2\big(1 - qt^{\langle \eta(\bar{\alpha}), \widetilde{\alpha}^\vee \rangle}\big) W_{\bar{\alpha}}(t)/(W_{\bar{\alpha}} \cap W_\omega)(t) = 2\big(1 - qt^{\langle \eta(\bar{\alpha}), \widetilde{\alpha}^\vee \rangle}\big) A^{\omega}_{\bar{\alpha},\bar{\alpha}}(t),$$

so we deduce via (4.9) that

$$\bar{C}_{\bar{\alpha}}(q,t) = -\frac{1 - qt^{-1}}{1 - qt^{\langle \eta(\bar{\alpha}), \widetilde{\alpha}^\vee \rangle}} \cdot \frac{A^{\omega}_{\bar{\alpha},0}(t)}{A^{\omega}_{\bar{\alpha},\bar{\alpha}}(t)}. \tag{4.11}$$

Our next step is to determine $A^{\omega}_{\bar{\alpha},0}(t)/A^{\omega}_{\bar{\alpha},\bar{\alpha}}(t)$ by evaluating $\Pr(P_\omega(\bar{\alpha}))$ in two ways.

First, it is easy to check that in the quasi-minuscule case $\lambda = \bar{\alpha}$, the parameter $\varepsilon(\alpha)$ in Theorem 2.10 is 0,1-valued, with $\varepsilon(\alpha) = 1$ if and only if $\alpha \in \Theta^+$, where

$$\Theta^+ = \{\alpha \in \Phi : \langle \bar{\alpha}, \alpha^\vee \rangle > 0 \text{ or } \langle \omega, \alpha \rangle > 0\}.$$

Thus Theorem 2.10 implies

$$\Pr(P_\omega(\bar{\alpha})) = t^{-(h_s - 1)} \cdot \prod_{\alpha \in \Theta^+} \frac{1 - t^{\langle \rho, \alpha^\vee \rangle + 1}}{1 - t^{\langle \rho, \alpha^\vee \rangle}}.$$

Aside from powers of t, this is the Poincaré series for $W/(W_{\bar{\alpha}} \cap W_\omega)$ (see (2.10)), so

$$\begin{aligned}\Pr(P_\omega(\bar{\alpha})) &= t^{-(h_s - 1)} W(t)/(W_{\bar{\alpha}} \cap W_\omega)(t) \\ &= t^{-(h_s - 1)} A^{\omega}_{\bar{\alpha},\bar{\alpha}}(t) W(t)/W_{\bar{\alpha}}(t) = t^{-h} \frac{1 - t^h}{1 - t} K_s(t) A^{\omega}_{\bar{\alpha},\bar{\alpha}}(t),\end{aligned}$$

the last equality being a consequence of Lemma 4.3(c).

On the other hand, we have $P_\omega(\bar\alpha) = A^\omega_{\bar\alpha,\bar\alpha}(t)\chi(\bar\alpha) + A^\omega_{\bar\alpha,0}(t)\chi(0)$, and the principal specialization of this identity yields

$$t^{-h}\frac{1-t^h}{1-t}K_s(t)A^\omega_{\bar\alpha,\bar\alpha}(t) = t^{-h}\frac{1-t^{h+1}}{1-t}K_s(t)A^\omega_{\bar\alpha,\bar\alpha}(t) + A^\omega_{\bar\alpha,0}(t)$$

by Lemma 4.3(e), and therefore,

$$A^\omega_{\bar\alpha,0}(t) = -K_s(t)A^\omega_{\bar\alpha,\bar\alpha}(t).$$

Combining this with (4.11), we obtain

$$\bar C_{\bar\alpha}(q,t) = \frac{1-qt^{-1}}{1-qt^{\langle\eta(\bar\alpha),\widetilde\alpha^\vee\rangle}}K_s(t).$$

To finish, recall that $K_s(t) = t^hK_s(t^{-1})$ by Lemma 4.3(a) and $\bar C_{\bar\alpha}(q,t) = \bar C_{\bar\alpha}(q^{-1},t^{-1})$ by Corollary 3.5; reconciling this with the above formula forces $\langle\eta(\bar\alpha),\widetilde\alpha^\vee\rangle = h-1$, even though it is not obvious from the definition of $\eta(\cdot)$ in Theorem 3.3. $\square$

Lemma 4.4. *If $\omega = \widetilde\alpha^\vee$, then* $\deg(A^\omega_{\widetilde\alpha,0}(t)) \leqslant h^\vee_s - 1$.

Proof. Let $\Psi^+ = \{\alpha\in\Phi : \langle\omega,\alpha\rangle > 0\}$. Since ω is quasi-minuscule, we have $\langle\omega,\beta\rangle = 1$ for all $\beta\in\Psi^+ - \{\widetilde\alpha\}$, and hence $s_{\widetilde\alpha}(-\beta) = \widetilde\alpha - \beta$ is also a member of $\Psi^+ - \{\widetilde\alpha\}$. Thus $\Psi^+ - \{\widetilde\alpha\}$ may be partitioned into

$$(|\Psi^+| - 1)/2 = \langle\rho,\omega\rangle - 1 = h^\vee_s - 2$$

pairs of roots of the form $\{\beta,\widetilde\alpha-\beta\}$.

By Proposition 2.2(a) (more specifically, see (2.3)), we know that $A^\omega_{\widetilde\alpha,0}(t)$ is expressible as a sum of terms $\pm t^{|\Theta|}$, where Θ ranges over subsets of Ψ^+ such that $\widetilde\alpha+\rho-\sigma(\Theta) = w^{-1}\rho$ for some $w\in W$. If Θ includes at most one member from each pair $\beta,\widetilde\alpha-\beta$ (and possibly $\widetilde\alpha$ as well), then $|\Theta| \leqslant h^\vee_s - 1$, which fits the claimed degree bound.

Otherwise, Θ contains at least one of the pairs $\{\beta,\widetilde\alpha-\beta\}$. Setting $\Theta' = \Theta - \{\beta,\widetilde\alpha-\beta\}$, we have $\rho - \sigma(\Theta') = w^{-1}\rho$, and thus Θ' must be the inversion set $\Phi(w)$, by Lemma 2.3. Hence $w\beta > 0$ and $w(\widetilde\alpha-\beta) > 0$, so $w\widetilde\alpha > 0$ and $\widetilde\alpha\notin\Theta'$. There cannot be a second pair $\{\alpha,\widetilde\alpha-\alpha\}$ in Θ (and hence also in $\Theta' = \Phi(w)$); otherwise, we would have $w\alpha < 0$, $w(\widetilde\alpha-\alpha) < 0$, and $w\widetilde\alpha < 0$, a contradiction. Hence $|\Theta'| \leqslant h^\vee_s - 3$ and $|\Theta| \leqslant h^\vee_s - 1$. $\square$

Lemma 4.5. *We have* $B_{\widetilde\alpha,0}(q,t) = -t^{h^\vee_s - h}(1-qt^{-1})(1+qt^{h-1})K^\vee_s(t)$.

Proof. Setting $\omega = \widetilde\alpha^\vee$, Proposition 2.5(d) and Lemma 4.3(d) imply

$$A^\omega_{00}(t) = W(t)/W_\omega(t) = t^{h^\vee_s-h-1}\frac{1-t^h}{1-t}K^\vee_s(t).$$

Starting from the definition of $B_{\lambda\mu}(q,t)$ in Theorem 3.4(a), we thereby obtain

$$\begin{aligned} B_{\widetilde{\alpha},0}(q,t) &= A^{\omega}_{\widetilde{\alpha},0}(t) - q^2t^{2h_s^\vee-2}A^{\omega}_{\widetilde{\alpha},0}(t^{-1}) - qtA^{\omega}_{00}(t) + qt^{2h_s^\vee-3}A^{\omega}_{00}(t^{-1}) \\ &= A^{\omega}_{\widetilde{\alpha},0}(t) - q^2t^{2h_s^\vee-2}A^{\omega}_{\widetilde{\alpha},0}(t^{-1}) + qt^{h_s^\vee-h-1}(1-t^h)K_s^\vee(t), \end{aligned} \tag{4.12}$$

using Lemma 4.3(a) (applied to $\Phi^\vee$ and $K_s^\vee(t)$) for the second equality. Bearing in mind that $B_{\widetilde{\alpha},0}(q,t)$ vanishes at $q=t$ by (4.10), we deduce that

$$A^{\omega}_{\widetilde{\alpha},0}(t) - t^{2h_s^\vee}A^{\omega}_{\widetilde{\alpha},0}(t^{-1}) = -t^{h_s^\vee-h}(1-t^h)K_s^\vee(t).$$

By Lemma 4.4, the degree of $A^{\omega}_{\widetilde{\alpha},0}(t)$ is $\leqslant h_s^\vee-1$, and the least term of $t^{2h_s^\vee}A^{\omega}_{\widetilde{\alpha},0}(t^{-1})$ has degree $\geqslant h_s^\vee+1$. Similarly, $t^{h_s^\vee-h}K_s^\vee(t)$ has degree $(h_s^\vee-h)+(h_s^\vee-1)\leqslant h_s^\vee-1$ and the least term of $t^{h_s^\vee}K_s^\vee(t)$ has degree $h_s^\vee+(h-(h_s^\vee-1))\geqslant h_s^\vee+1$. This forces

$$A^{\omega}_{\widetilde{\alpha},0}(t) = -t^{h_s^\vee-h}K_s^\vee(t), \quad t^{2h_s^\vee}A^{\omega}_{\widetilde{\alpha},0}(t^{-1}) = -t^{h_s^\vee}K_s^\vee(t), \tag{4.13}$$

and substituting these into (4.12) yields the claimed formula. $\square$

Proof of (4.4). Again setting $\omega=\widetilde{\alpha}^\vee$, Theorem 3.4(d) implies

$$B_{\widetilde{\alpha},\widetilde{\alpha}}(q,t) = 1-q^2t^{\langle\eta(\widetilde{\alpha}),\widetilde{\alpha}^\vee\rangle} = 1-q^2t^{\langle 2\rho,\widetilde{\alpha}^\vee\rangle} = 1-q^2t^{2h_s^\vee-2}.$$

If it happens that $\widetilde{\alpha}=\bar{\alpha}$, then (4.9) is still valid and Lemma 4.5 implies

$$\bar{C}_{\widetilde{\alpha}}(q,t) = -B_{\widetilde{\alpha},0}(q,t)/B_{\widetilde{\alpha},\widetilde{\alpha}}(q,t) = \frac{1-qt^{-1}}{1-qt^{h-1}}K(t),$$

since $h=h_s^\vee$ and $K(t)=K_s^\vee(t)$ in this case.

Otherwise, if $\widetilde{\alpha}\neq\bar{\alpha}$, then there are three dominant weights $\leqslant\widetilde{\alpha}$; namely, 0, $\bar{\alpha}$, and $\widetilde{\alpha}$ (this is equivalent to the fact that the nonzero weights of the adjoint representation are roots), so Theorem 3.4 implies that the analogue of (4.9) is

$$B_{\widetilde{\alpha},\widetilde{\alpha}}(q,t)\bar{C}_{\widetilde{\alpha}}(q,t) + B_{\widetilde{\alpha},\bar{\alpha}}(q,t)\bar{C}_{\bar{\alpha}}(q,t) = -B_{\widetilde{\alpha},0}(q,t).$$

From the definition, we have $B_{\widetilde{\alpha},\bar{\alpha}}(q,t) = A^{\omega}_{\widetilde{\alpha},\bar{\alpha}}(t) - q^2t^{2h_s^\vee-2}A^{\omega}_{\widetilde{\alpha},\bar{\alpha}}(t^{-1})$ (in particular, note that $A^{\omega}_{0,\bar{\alpha}}(t)=0$). Thus, to prove $\bar{C}_{\widetilde{\alpha}}(q,t)=(1-qt^{-1})K(t)/(1-qt^{h-1})$, it suffices to show

$$\begin{aligned} (1-q^2t^{2h_s^\vee-2})\frac{1-qt^{-1}}{1-qt^{h-1}}K(t) + \big(A^{\omega}_{\widetilde{\alpha},\bar{\alpha}}(t) - q^2t^{2h_s^\vee-2}A^{\omega}_{\widetilde{\alpha},\bar{\alpha}}(t^{-1})\big)\frac{1-qt^{-1}}{1-qt^{h-1}}K_s(t) \\ = t^{h_s^\vee-h}(1-qt^{-1})(1+qt^{h-1})K_s^\vee(t), \end{aligned}$$

by (4.5) and Lemma 4.5. Equivalently, we seek to prove that

$$(1-q^2t^{2h_s^\vee-2})K(t)+\big(A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)-q^2t^{2h_s^\vee-2}A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t^{-1})\big)K_s(t)=t^{h_s^\vee-h}(1-q^2t^{2h-2})K_s^\vee(t).$$

Eliminating q, we may reduce this to the pair of identities

$$\begin{aligned} K(t)+A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)K_s(t)&=t^{h_s^\vee-h}K_s^\vee(t), \\ t^{2h_s^\vee-2}K(t)+t^{2h_s^\vee-2}A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t^{-1})K_s(t)&=t^{h_s^\vee+h-2}K_s^\vee(t). \end{aligned} \tag{4.14}$$

However, these are equivalent via Lemma 4.3(a) and (b), so it suffices to prove the first.

Our strategy is to compute the principal specialization of $P_\omega(\widetilde{\alpha})$ in two ways. First, by Theorem 2.10 and (2.10), we have

$$\Pr(P_\omega(\widetilde{\alpha}))=t^{-(h-1)}\prod_{\alpha\in\Psi^+}\frac{1-t^{\langle\rho,\alpha^\vee\rangle+1}}{1-t^{\langle\rho,\alpha^\vee\rangle}}=t^{-(h-1)}W(t)/W_\omega(t),$$

where $\Psi^+=\{\alpha\in\Phi:\langle\omega,\alpha\rangle>0\}$. On the other hand, we have

$$P_\omega(\widetilde{\alpha})=A^\omega_{\widetilde{\alpha},\widetilde{\alpha}}(t)\chi(\widetilde{\alpha})+A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)\chi(\bar{\alpha})+A^\omega_{\widetilde{\alpha},0}(t)\chi(0),$$

and hence,

$$t^{-(h-1)}W(t)/W_\omega(t)=\Pr(\chi(\widetilde{\alpha}))+A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)\Pr(\chi(\bar{\alpha}))+A^\omega_{\widetilde{\alpha},0}(t).$$

Furthermore, we have evaluated $W(t)/W_\omega(t)$, $\Pr(\chi(\widetilde{\alpha}))$, and $\Pr(\chi(\bar{\alpha}))$ in Lemma 4.3, and $A^\omega_{\widetilde{\alpha},0}(t)$ is given in (4.13), yielding

$$t^{h_s^\vee-2h}\frac{1-t^h}{1-t}K_s^\vee(t)=t^{-h}\frac{1-t^{h+1}}{1-t}K(t)+t^{-h}\frac{1-t^{h+1}}{1-t}A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)K_s(t)-t^{h_s^\vee-h}K_s^\vee(t).$$

This is equivalent to (4.14) and completes the proof of Theorem 4.1. $\square$

Remark 4.6. Since $\widetilde{\alpha}$ is proportional to $\omega=\widetilde{\alpha}^\vee$, Proposition 2.1 implies

$$P(\widetilde{\alpha})=P_\omega(\widetilde{\alpha})=\chi(\widetilde{\alpha})+A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)\chi(\bar{\alpha})+A^\omega_{\widetilde{\alpha},0}(t)\chi(0).$$

Similarly, $P(\bar{\alpha})-\chi(\bar{\alpha})$ is a multiple of $P(0)=\chi(0)$, so $\chi(\widetilde{\alpha})-P(\widetilde{\alpha})+A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)P(\bar{\alpha})$ is also a multiple of $P(0)$, and hence $-A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)=K_{\widetilde{\alpha},\bar{\alpha}}(t)$ is a Kazhdan-Lusztig polynomial with nonnegative coefficients (see Remark 2.6). Setting $t=1$ in (4.14), we obtain $A^\omega_{\widetilde{\alpha},\bar{\alpha}}(1)=-1$, so $-A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)$ must be a power of t. Since $K(t)$ has degree $h-1$, it follows that (4.14) forces $A^\omega_{\widetilde{\alpha},\bar{\alpha}}(t)=-t^{h-h_s}$, and thereby yields a simple identity connecting the classical, short, and co-short exponents; namely,

$$K(t)=t^{h-h_s}K_s(t)+t^{h_s^\vee-h}K_s^\vee(t).$$

Note that this is valid only when $\widetilde{\alpha}\neq\bar{\alpha}$.

5. Computational issues

In principle, the recursive formulas in Section 3 make it possible to compute graded multiplicities in the Macdonald kernel; however, there are significant obstacles that need to be overcome in order to make the recurrences practical for explicit computation.

Throughout this section, we assume that Φ is irreducible.

A. The exterior character strategy.

For simplicity, let us initially consider the cases in which there is a (nonzero, dominant) minuscule co-weight ω available. Starting from Theorem 3.3, we have a triangular system of linear equations that determine the normalized multiplicities $\bar{C}_\lambda(q,t)$ for each $\lambda \in \mathcal{Q}^+$, and the problem of explicitly generating the equation that determines $\bar{C}_\lambda(q,t)$ easily reduces to the problem of computing the coefficients $A^\omega_{\lambda\mu}(t)$ in the Weyl character expansion of the corresponding degenerate Macdonald polynomial; i.e., $P_\omega(\lambda) = \sum_{\mu\leqslant\lambda} A^\omega_{\lambda\mu}(t)\chi(\mu)$.

Setting $\Psi^+ = \{\alpha \in \Phi : \langle \omega, \alpha\rangle > 0\}$, let us consider the expansion

$$\prod_{\alpha\in\Psi^+} (1 - te^{-\alpha}) = \sum_{\nu\in\mathcal{Q}} F^\omega_\nu(t)e^\nu, \qquad F^\omega_\nu(t) \in \mathbf{Z}[t]. \tag{5.1}$$

Of course only finitely many of the coefficients $F^\omega_\nu(t)$ are nonzero. Directly from (2.1), one sees that the Weyl character decomposition of $P_\omega(\lambda)$ may be expressed in the form

$$P_\omega(\lambda) = \sum_{\nu\in\mathcal{Q}} F^\omega_\nu(t)\chi(\lambda + \nu). \tag{5.2}$$

However, hidden in this expansion is the fact that $\lambda + \nu$ need not be dominant. For each such term, one needs to perform the operation of ρ-shifted *orbit-sorting*; i.e., replacing $\chi(\lambda + \nu)$ with $\mathrm{sgn}(w)\chi(\mu)$ or 0, the former occurring if μ is the unique dominant member of $\mathcal{Q}$ such that $\mu + \rho = w(\lambda + \nu + \rho)$ for some $w \in W$, and the latter occurring if there is no such μ (or equivalently, if $\lambda + \nu + \rho$ has a nontrivial stabilizer).

The main difficulty of this approach is that of generating the polynomials $F^\omega_\nu(t)$. Indeed, the expansion of (5.1) has 2^m (not necessarily distinct) terms, where $m = |\Psi^+| = \langle 2\rho, \omega\rangle$. Certainly for the classical root systems of low rank this expansion is easy. For example, for the minuscule co-weights corresponding to the end nodes of the root system $\mathcal{A}_n$, there are 2^n terms, which is quite small compared to $(n + 1)! = |W|$. At the other extreme, there are 2^{27} terms (about 130 million) in the case of the unique minuscule co-weight for $\mathcal{E}_7$.

The following three critical observations make this calculation feasible even for $\mathcal{E}_7$, and allow one to use (5.2) and Theorem 3.3 as the basis for a good algorithm for computing graded multiplicities in the Macdonald kernel.

1. Although there is no cancellation of terms in (5.1), it happens that the number of *distinct* weights ν such that $F^\omega_\nu(t) \neq 0$ is generally much smaller than 2^m (except for the root systems $\mathcal{A}_n$, where it is already quite small). For example, in $\mathcal{E}_7$ it turns out that there are 722,080 such weights; thus for each $\lambda \in \mathcal{Q}^+$, the decomposition of $P_\omega(\lambda)$

Φ	$\lvert W\rvert$	m	$\mathrm{NT}(\omega)$	$\mathrm{NT}(\omega)/\sim$
$\mathcal{A}_n$	$(n+1)!$	n	2^n	$n+1$
$\mathcal{B}_n$	$2^n n!$	$2n-1$	$(2n+4)3^{n-2}$	$n(n+1)$
$\mathcal{D}_n$	$2^{n-1}n!$	$2n-2$	$(n+2)3^{n-2}$	$n(n+1)/2+1$
$\mathcal{E}_6$	51840	16	10169	93
$\mathcal{E}_7$	2903040	27	722080	486

TABLE 5.1. The number of terms and W_ω-orbits of terms in (5.1).

may be reduced to the application of 722,080 orbit-sorts, not 2^{27}. In Table 5.1, we list the number of distinct terms $\mathrm{NT}(\omega)$ appearing in (5.1) for various root systems. In cases where there is more than one minuscule co-weight, ω has been chosen to minimize $\mathrm{NT}(\omega)$. We remark that a consequence of the fact that ω is minuscule is that $F_\nu^\omega(t)$ is always an integer multiple of $t^{\langle\nu,\omega\rangle}$, but the computational benefit of this is minimal.

2. Since W_ω permutes Ψ^+, it follows that the coefficients $F_\nu^\omega(t)$ are W_ω-invariant, and thus the expansion (5.1) may be compactly encoded by recording the (nonzero) coefficients $F_\nu^\omega(t)$ only for those $\nu\in\mathcal{Q}$ that are dominant with respect to W_ω. The remaining terms may be generated on-demand by an efficient depth-first traversal of each W_ω-orbit. (See [**St3**] for a discussion of the traversal problem.) In the case of $\mathcal{E}_7$, this reduces the storage requirements to 486 terms, one for each $W(\mathcal{E}_6)$-orbit (see Table 5.1).

3. Let G_ω be a semisimple Lie group with root system $\Phi_\omega=\{\alpha\in\Phi:\langle\omega,\alpha\rangle=0\}$. Aside from the change of variable $t\to -t$, the left side of (5.1) may be recognized as the graded character of the exterior algebra of a G_ω-representation. In these terms, $F_\nu^\omega(-t)$ is the graded multiplicity of the weight ν in this exterior algebra. Furthermore, using a trick based on the Weyl denominator formula (see Theorem 2.1 of [**St4**]), it is possible to explicitly describe the decomposition of this graded character into irreducible G_ω-characters; viz.,

$$\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})=\sum_{x\in Y_0^\omega}(-t)^{\ell(x)}\chi^\omega(x^{-1}\rho-\rho), \tag{5.3}$$

where $\chi^\omega(\nu)=J_\omega(e^{\nu+\rho})/J_\omega(e^\rho)$ denotes a Weyl character associated to Φ_ω and (as in Proposition 2.5) Y_0^ω is the set of minimum-length coset-representatives for W/W_ω. There are standard algorithms based on Freudenthal's formula for computing dominant weight multiplicity in irreducible representations (see the discussion in Section 6 of [**St3**]), so by computing these multiplicities for each of the irreducible characters appearing in (5.3), we may obtain the coefficients $F_\nu^\omega(t)$ for W_ω-dominant ν without having to compute the non-dominant coefficients as well. In the case of $\mathcal{E}_7$, there are only 56 terms in (5.3) (the index of $W(\mathcal{E}_6)$ in $W(\mathcal{E}_7)$) and this weight multiplicity computation is negligible compared to the overall cost of solving the equations that determine the multiplicities $\bar{C}_\lambda(q,t)$.

Turning now to the quasi-minuscule case, Theorem 3.4 provides a triangular system of equations for the graded multiplicities that is easily expressible in terms of the transition

Φ	$\lvert W\rvert$	m	$\mathrm{NT}(\omega)$	$\mathrm{NT}(\omega)/{\sim}$
$\mathcal{C}_n$	$2^n n!$	$2n$	$(n+8)3^{n-2}$	$n(n+5)/2$
$\mathcal{E}_8$	696729600	58	1136387035	13024
$\mathcal{F}_4$	1152	16	3124	178
$\mathcal{G}_2$	12	6	33	19

TABLE 5.2. The number of terms and W_ω-orbits of terms in (5.5).

matrix $A_{\lambda\mu}(q,t)$ $(\lambda,\mu\in\mathcal{Q}^+)$, or equivalently, the Weyl character decomposition of

$$P_\omega(\lambda)-qtP_\omega(\lambda-\widetilde{\alpha})=J\Big(e^{\lambda+\rho}(1-qte^{-\widetilde{\alpha}})\cdot\textstyle\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})\Big)\Big/J(e^\rho),$$

where $\Psi^+=\{\alpha\in\Phi:\langle\omega,\alpha\rangle>0\}$ and $\omega=\widetilde{\alpha}^\vee$ is the quasi-minuscule co-weight. In this case, we have

$$P_\omega(\lambda)-qtP_\omega(\lambda-\widetilde{\alpha})=\sum_{\mu\leqslant\lambda}A_{\lambda\mu}(q,t)\chi(\mu)=\sum_{\nu\in\mathcal{Q}}F^\omega_\nu(q,t)\chi(\lambda+\nu) \tag{5.4}$$

for all nonzero $\lambda\in\mathcal{Q}^+$, where the polynomials $F^\omega_\nu(q,t)$ are defined by the expansion

$$(1-qte^{-\widetilde{\alpha}})\cdot\prod_{\alpha\in\Psi^+}(1-te^{-\alpha})=\sum_{\nu\in\mathcal{Q}}F^\omega_\nu(q,t)e^\nu. \tag{5.5}$$

Thus, one may generate the linear equation that determines $\bar{C}_\lambda(q,t)$ by an algorithm completely analogous to the minuscule case.

Again, the main difficulty is that of computing the coefficients $F^\omega_\nu(q,t)$. As in the minuscule case, the number of weights ν such that $F^\omega_\nu(q,t)\neq0$ is generally much smaller than the obvious bound of 2^m, where $m=1+|\Psi^+|=\langle2\rho,\widetilde{\alpha}^\vee\rangle$. Also, it is clear that (5.5) is W_ω-invariant, so it suffices to determine $F^\omega_\nu(q,t)$ only for those ν that are dominant with respect to W_ω. In Table 5.2, we list the number of weights ν such that $F^\omega_\nu(q,t)\neq0$, along with the number of W_ω-orbits of such weights, for various root systems.

As in the minuscule case, one may recognize (5.5) as the graded character of the exterior algebra of a G_ω-representation. In particular, note that as a weight for G_ω, $\widetilde{\alpha}$ is trivial. Moreover, there is a quasi-minuscule analogue of (5.3) (see Theorem 3.2 of [**St4**]); viz.,

$$\prod_{\alpha\in\Psi_1}(1-te^{-\alpha})=\sum_{x\in Y^\omega_+}(-t)^{\ell(x)}\frac{1-(t^2e^{-\widetilde{\alpha}})^{m/2-\ell(x)}}{1-t^2e^{-\widetilde{\alpha}}}\chi^\omega(x^{-1}\rho-\rho), \tag{5.6}$$

where $\Psi_1=\Psi^+-\{\widetilde{\alpha}\}=\{\alpha\in\Phi:\langle\omega,\alpha\rangle=1\}$ and Y^ω_+ denotes the set of shortest coset representatives for W/W_ω of length $<m/2$, the maximum possible being $m-1=\ell(s_{\widetilde{\alpha}})$. Note that the cosets of W/W_ω correspond to short co-roots, so the number of summands is the number of short positive co-roots. By computing the dominant weight multiplicities

for each Weyl character appearing in (5.6) via the Freudenthal formula, and multiplying the results by the W_ω-invariant factor $(1-qte^{-\widetilde{\alpha}})(1-te^{-\widetilde{\alpha}})$, we may thereby obtain the coefficients $F^\omega_\nu(q,t)$ for those ν that are W_ω-dominant without having to compute the non-dominant coefficients as well.

We remark that for most root systems, the minuscule algorithm is faster than the quasi-minuscule version (assuming that a minuscule co-weight is available); the only exceptions are the root systems $\mathcal{C}_n$. In these cases, there are $(n+8)3^{n-2}$ nonzero terms in (5.5), whereas the analogous minuscule number is exponentially larger than $2^n n! = |W|$.

A general feature of both the minuscule and quasi-minuscule versions of the algorithm is that once the one-time expense of determining the expansion in (5.1) or (5.5) is finished, the number of orbit-sorts required to generate the equation that determines $\bar{C}_\lambda(q,t)$ is independent of λ. Furthermore, as λ moves deeper into the dominant chamber, these orbit-sorts will run faster, since the terms $\lambda+\nu$ in (5.2) and (5.4) will be closer to the dominant chamber. Once λ is sufficiently deep in the dominant chamber, all such terms will be dominant, and the entire calculation is trivial. In these cases, we have $A^\omega_{\lambda\mu}(t) = F^\omega_{\mu-\lambda}(t)$ and (in the quasi-minuscule case) $A_{\lambda\mu}(q,t) = F^\omega_{\mu-\lambda}(q,t)$ for all $\mu \in \mathcal{Q}^+$.

B. The symmetrization strategy.

The main flaw of the preceding algorithm is that it is simply not practical for $\mathcal{E}_8$. Indeed, as can be seen from Table 5.2, it takes about 1.1 billion orbit-sorts to generate each linear equation. However, we will now describe a second algorithm for computing graded multiplicities in the Macdonald kernel that is practical for $\mathcal{E}_8$, and faster for $\mathcal{E}_7$. Beyond the improvements in these two cases, it is also useful to have this second, independent algorithm as a sanity check.

For the minuscule and quasi-minuscule versions of the second algorithm, the starting points are not the systems of linear equations provided by Theorems 3.3 and 3.4, but an alternative pair of systems that may be derived from the (heretofore unused) functional equations provided in Lemmas 3.1(b) and 3.2(b).

To describe these new systems, it will be important to note that the definition of $C_\lambda(q,t)$ (see (1.4)) is sensible even for non-dominant λ. Following this convention, we have

$$\lambda+\rho = w(\mu+\rho) \quad \Rightarrow \quad C_\lambda(q,t) = \operatorname{sgn}(w) C_\mu(q,t)$$

for all $w \in W$, and $C_\lambda(q,t) = 0$ if $\lambda+\rho$ has a nontrivial stabilizer. Thus $C_\lambda(q,t)$ may be reduced to standard (dominant) form by a ρ-shifted orbit-sort in exactly the same way that a non-dominant Weyl character $\chi(\lambda)$ is reduced.

Proposition 5.1. *If $\omega \in \mathcal{P}^\vee$ is dominant and minuscule, then for all $\lambda \in \mathcal{Q}$, we have*

$$\sum_{x\in W} \Big(t^{-\langle \rho, x\omega\rangle} - q^{\langle\lambda,\omega\rangle} t^{\langle\rho,x\omega\rangle}\Big) C_{x\lambda}(q,t) = 0.$$

Proof. It is evident that

$$\begin{aligned}
&[e^\lambda]\varphi_{\omega,q}\Big(J(e^\rho)\Delta(q,t)\cdot\varphi_{\omega,t^{-1}}(J(e^{-\rho}))\Big)\\
&\qquad= q^{\langle\lambda,\omega\rangle}[e^\lambda]J(e^\rho)\Delta(q,t)\cdot\sum_{x\in W}\operatorname{sgn}(x)t^{\langle x\rho,\omega\rangle}e^{-x\rho}\\
&\qquad= q^{\langle\lambda,\omega\rangle}\sum_{x\in W}\operatorname{sgn}(x)t^{\langle x\rho,\omega\rangle}[e^{\lambda+x\rho}]J(e^\rho)\Delta(q,t)\\
&\qquad= q^{\langle\lambda,\omega\rangle}\sum_{x\in W}t^{\langle\rho,x^{-1}\omega\rangle}[e^{x^{-1}\lambda+\rho}]J(e^\rho)\Delta(q,t)\\
&\qquad= q^{\langle\lambda,\omega\rangle}\sum_{x\in W}t^{\langle\rho,x\omega\rangle}C_{x\lambda}(q,t),
\end{aligned}$$

using the anti-symmetry of $J(e^\rho)\Delta(q,t)$ for the third equality. One similarly obtains

$$[e^\lambda]J(e^\rho)\Delta(q,t)\cdot\varphi_{\omega,t}(J(e^{-\rho})) = \sum_{x\in W}t^{-\langle\rho,x\omega\rangle}C_{x\lambda}(q,t),$$

and this coincides with the previous expression by Lemma 3.1(b). □

Note that $(x\lambda+\rho)^+ \leqslant \lambda^+ + \rho$, so if $C_{x\lambda}(q,t)$ reduces to $\pm C_\mu(q,t)$ in standard form, then $\mu \leqslant \lambda^+$. Furthermore, $\mu = \lambda^+$ occurs only if there is some $w\in W$ such that $wx\lambda$ and $w\rho$ are simultaneously at the highest points of their orbits (namely, λ^+ and ρ). However, the orbit of ρ is regular, so this occurs only if $w=1$ and $x\lambda=\lambda^+$. From this it is easy to deduce that if the relation in Proposition 5.1 is reduced to standard form, then the condition $\langle\lambda,\omega\rangle\neq 0$ is sufficient to force the coefficient of $C_{\lambda^+}(q,t)$ to be nonzero.

Thus for nonzero $\lambda\in\mathcal{Q}^+$, the above relations provide a triangular system of equations that uniquely determine the normalized multiplicities $\bar{C}_\lambda(q,t)$. To recast these equations in a more efficient computational form, note that the term indexed by x depends only on the pair $(x\lambda, x\omega)$, so it may be rewritten as a sum over the W-orbit of (λ,ω). Equivalently, if $w_1,\dots,w_l$ are coset representatives for W/W_λ and $\{\theta_1,\dots,\theta_k\}$ is the W_λ-orbit of ω, then Proposition 5.1 is equivalent to the relation

$$\sum_{i=1}^{l}C_{w_i\lambda}(q,t)\sum_{j=1}^{k}\Big(t^{-\langle\rho,w_i\theta_j\rangle} - q^{\langle\lambda,\omega\rangle}t^{\langle\rho,w_i\theta_j\rangle}\Big) = 0.$$

Thus for each $\lambda\in\mathcal{Q}^+$, one may generate an equation for $\bar{C}_\lambda(q,t)$ by running a depth-first traversal of the quotient W/W_λ and computing a total of $l=|W/W_\lambda|$ orbit-sorts.

Note that for a generic λ, the number of orbit-sorts will be $|W|$, and one can see from Table 5.1 that this compares unfavorably with our previous algorithm. Another negative factor is that the cost of computing the coefficient that accompanies each orbit-sort is usually higher in this new algorithm. However, it should be noted that the parameter k (i.e., the size of the W_λ-orbit of ω) is generally very small; indeed, $k > 1$ only if λ is on the wall of the dominant chamber normal to ω. In the case of $\mathcal{E}_7$, the maximum possible value for k is 12, and this occurs only if λ is a multiple of the highest root.

A third negative factor is that as λ moves deeper into the dominant chamber, the set of points that are orbit-sorted; i.e., $\{x\lambda + \rho : x \in W\}$ does not move toward the dominant chamber, and thus a simple comparison of the number of orbit-sorts again tends to understate the advantage of the first algorithm.

To summarize, this new algorithm is an improvement only when λ has a sufficiently large stabilizer. The effect is most dramatic for the exceptional root systems $\mathcal{E}_6$ and $\mathcal{E}_7$. In these cases, the chambers are so narrow that all dominant weights with small stabilizers are very high in the partial order. For example, in our implementation of the two algorithms, we measured the amount of time taken by both algorithms to generate the linear equations for all in $\lambda \in \mathcal{Q}^+$ up to some pre-determined height (recall that $\mathrm{ht}(\lambda) = \langle \lambda, \rho^\vee \rangle$). We found that in the case of $\mathcal{E}_6$, the point where the first algorithm became faster (on average) did not occur until after processing the 79 dominant weights of height $\leqslant 54$. In the case of $\mathcal{E}_7$, we processed the 113 dominant weights of height $\leqslant 100$, and found only one of these for which the first algorithm was significantly faster, and four others in which the performance was roughly comparable.

In the quasi-minuscule case, we have the following.

Proposition 5.2. *If $\widetilde{\alpha}$ is the highest root and $\omega = \widetilde{\alpha}^\vee$, then for all $\lambda \in \mathcal{Q}$, we have*

$$\sum_{\mu,\beta} \sum_{i \geqslant 0} \left(f_i^\beta(q,t) - q^{\langle \lambda, \omega \rangle} f_i^\beta(q^{-1}, t^{-1}) \right) C_{\mu - i\beta}(q,t) = 0, \tag{5.7}$$

where (μ, β) ranges over $\{(x\lambda, x\widetilde{\alpha}) : x \in W, x\widetilde{\alpha} > 0\}$ and $f_i^\beta(q,t) \in \mathbf{Z}[q, t^{\pm 1}]$ is defined by

$$(1 - qtz)(1 - tz) \frac{1 - t^{\langle 2\rho, \beta^\vee \rangle} z^{\langle \rho, \beta^\vee \rangle}}{1 - t^2 z} = \sum_{i \geqslant 0} t^{\langle \rho, \beta^\vee \rangle} f_i^\beta(q,t) z^i.$$

In particular, $f_i^\beta(q,t) = 0$ for $i \geqslant \langle \rho, \beta^\vee \rangle + 2$. Furthermore, if λ is dominant and nonzero, then the coefficient of $C_\lambda(q,t)$ in the standard reduction of (5.7) is nonzero, and every term is indexed by a dominant weight $\leqslant \lambda$.

Proof. Given $x \in W$, either $x\widetilde{\alpha} > 0$ or $x\widetilde{\alpha} < 0$, and the involution $x \mapsto xs_{\widetilde{\alpha}}$ interchanges these two cases. Hence,

$$\begin{aligned}\varphi_{\omega,t}(J(e^{-\rho})) &= \sum_{x \in W} \operatorname{sgn}(x) t^{-\langle \rho, x\widetilde{\alpha}^\vee \rangle} e^{-x^{-1}\rho} \\ &= \sum_{x\colon x\widetilde{\alpha}>0} \operatorname{sgn}(x) \big(t^{-\langle \rho, x\widetilde{\alpha}^\vee \rangle} e^{-x^{-1}\rho} - t^{\langle \rho, x\widetilde{\alpha}^\vee \rangle} e^{-s_{\widetilde{\alpha}} x^{-1}\rho} \big) \\ &= \sum_{x\colon x\widetilde{\alpha}>0} \operatorname{sgn}(x) t^{-\langle \rho, x\widetilde{\alpha}^\vee \rangle} e^{-x^{-1}\rho} \big(1 - t^{\langle 2\rho, x\widetilde{\alpha}^\vee \rangle} e^{\langle \rho, x\widetilde{\alpha}^\vee \rangle \widetilde{\alpha}}\big).\end{aligned}$$

Recalling the definition of $\Omega_2(q,t)$ in (3.1), it follows that

$$\Omega_2(q,t) = \sum_{x\colon x\widetilde{\alpha}>0} \operatorname{sgn}(x) e^{-x^{-1}\rho} \sum_{i \geqslant 0} f_i^{x\widetilde{\alpha}}(q,t) e^{i\widetilde{\alpha}},$$

and thus for all $\lambda \in \mathcal{Q}$, we have

$$\begin{aligned}{}[e^\lambda] J(e^\rho)\Delta(q,t) \cdot \Omega_2(q,t) &= \sum_{x\colon x\widetilde{\alpha}>0} \operatorname{sgn}(x) \sum_{i \geqslant 0} f_i^{x\widetilde{\alpha}}(q,t) [e^{\lambda + x^{-1}\rho - i\widetilde{\alpha}}] J(e^\rho)\Delta(q,t) \\ &= \sum_{x\colon x\widetilde{\alpha}>0} \sum_{i \geqslant 0} f_i^{x\widetilde{\alpha}}(q,t) C_{x\lambda - ix\widetilde{\alpha}}(q,t).\end{aligned}$$

Similarly, we also have

$$[e^\lambda]\varphi_{\omega,q}\big(J(e^\rho)\Delta(q,t) \cdot \Omega_2(q^{-1},t^{-1})\big) = q^{\langle \lambda, \omega \rangle} \sum_{x\colon x\widetilde{\alpha}>0} \sum_{i \geqslant 0} f_i^{x\widetilde{\alpha}}(q^{-1},t^{-1}) C_{x\lambda - ix\widetilde{\alpha}}(q,t).$$

These two expressions are equal by Lemma 3.2(b), so

$$\sum_{x\colon x\widetilde{\alpha}>0} \sum_{i \geqslant 0} \Big(f_i^{x\widetilde{\alpha}}(q,t) - q^{\langle \lambda, \omega \rangle} f_i^{x\widetilde{\alpha}}(q^{-1},t^{-1}) \Big) C_{x\lambda - ix\widetilde{\alpha}}(q,t) = 0.$$

Each term of this sum depends only on the pair $(x\lambda, x\widetilde{\alpha})$, so it may be rewritten in an equivalent form as a sum over (half of) the orbit of $(\lambda, \widetilde{\alpha})$, yielding (5.7).

Now suppose that $\lambda \in \mathcal{Q}^+$ is nonzero, and consider the result of reducing $C_{\mu - i\beta}(q,t)$ to standard form, where $(\mu, \beta) = (x\lambda, x\widetilde{\alpha})$ for some $x \in W$, $\beta > 0$, and $0 \leqslant i \leqslant \langle \rho, \beta^\vee \rangle + 1$. In the case $i = 0$, it follows by the same reasoning used following the proof of Proposition 5.1 that the resulting graded multiplicity (if nonzero) is indexed by a dominant weight $\leqslant \lambda$, and equality occurs only if $\mu = \lambda$ and $x \in W_\lambda$. The contribution of these terms to the coefficient of $C_\lambda(q,t)$ is therefore

$$\sum \Big(f_0^\beta(q,t) - q^{\langle \lambda, \omega \rangle} f_0^\beta(q^{-1},t^{-1}) \Big) = \sum \big(t^{-\langle \rho, \beta^\vee \rangle} - q^{\langle \lambda, \omega \rangle} t^{\langle \rho, \beta^\vee \rangle} \big), \tag{5.8}$$

where the sum ranges over all positive roots β in the W_λ-orbit of $\widetilde{\alpha}$.

The remaining possibility is that $0 \leqslant i-1 \leqslant \langle \rho, \beta^\vee \rangle$. In these cases, $\rho - (i-1)\beta$ is on a weight string between ρ and $s_{\widetilde{\alpha}}(\rho)$, and $\mu - \beta$ is on a weight string between μ and $s_\beta(\mu)$, since $\lambda \neq 0$ implies $\langle \mu, \beta^\vee \rangle = \langle \lambda, \widetilde{\alpha}^\vee \rangle \geqslant 1$. It follows via (1.1) that

$$(\mu - i\beta + \rho)^+ \leqslant (\mu - \beta)^+ + (\rho - (i-1)\beta)^+ \leqslant \lambda + \rho,$$

and thus when $C_{\mu - i\beta}(q,t)$ is reduced, the result is (if nonzero) indexed by a dominant weight $\leqslant \lambda$. Furthermore, equality occurs only if there is an element $w \in W$ that simultaneously maps $\mu - \beta$ to λ and $\rho - (i-1)\beta$ to ρ. The latter requires $i = 1$ or $i = \langle \rho, \beta^\vee \rangle + 1$, and since the orbit of ρ is regular, this forces $w = 1$ or $w = s_\beta$. However, $\mu - \beta < \mu \leqslant \lambda$, so the former is impossible, and the latter requires λ to be the top of the β-string through μ; hence $\lambda = \mu$ and $\langle \lambda, \widetilde{\alpha}^\vee \rangle = \langle \mu, \beta^\vee \rangle = 1$. In such cases, we have $C_{\mu - i\beta}(q,t) = -C_\lambda(q,t)$ and $f_i^\beta(q,t) = qt^{\langle \rho, \beta^\vee \rangle}$, and it is not hard to see that the net contribution to the coefficient of $C_\lambda(q,t)$ exactly matches (5.8). Thus in general, the coefficient of $C_\lambda(q,t)$ is either given by (5.8), or if $\langle \lambda, \widetilde{\alpha}^\vee \rangle = 1$, by twice that amount. $\square$

As in the minuscule case, the above relations for $\lambda \in \mathcal{Q}^+$ provide a triangular system of equations for computing the normalized multiplicities $\bar{C}_\lambda(q,t)$. However, the number of orbit-sorts required to reduce (5.7) to standard form is considerably larger. First, the number of terms in the outer sum is half the size of the W-orbit of $(\lambda, \widetilde{\alpha})$, instead of being controlled directly by the (potentially smaller) orbit of λ. Second, the summand indexed by (μ, β) requires $\langle \rho, \beta^\vee \rangle + 2$ orbit-sorts. Since every root in the orbit of $\widetilde{\alpha}$ (i.e., every long root) occurs the same number of times, it follows that the total number of orbit-sorts required to reduce (5.7) is $(a+2)/2$ times the size of the orbit of $(\lambda, \widetilde{\alpha})$, where a denotes the average height of a (positive) short co-root.

In the cases $\mathcal{C}_n$, $\mathcal{E}_8$, $\mathcal{F}_4$ and $\mathcal{G}_2$, we have $a = (n+1)/2$, $31/3$, $23/6$, and 2, and in the worst (i.e., generic) cases, the total numbers of orbit-sorts required are $2^{n-2}(n+5)n!$, 4.3 billion, 3360, and 24 (respectively). Comparing this with the data in Table 5.2, it is not surprising that the first algorithm is much faster on average for $\mathcal{C}_n$, whereas for $\mathcal{F}_4$ and $\mathcal{G}_2$, the numbers and performance are roughly competitive.

In the case of $\mathcal{E}_8$, the sharpness of the dominant chamber means that every dominant weight λ for which it is remotely feasible to *solve* the equation that determines $\bar{C}_\lambda(q,t)$ has a very large stabilizer, and requires only a tiny fraction of the potential billions of orbit-sorts that would be needed in the worst case. For example, we had no difficulty generating and solving the equations for the 25 dominant weights of height $\leqslant 126$; this involved a total of 32 million orbit-sorts. For the 110 dominant weights of height $\leqslant 200$, the total would be about 880 million, which is still less than the number that would be used by the first algorithm to reduce a single equation.

6. Denominator systems

Recall that by Corollary 3.5, we know that each of the normalized multiplicities $\bar{C}_\lambda(q,t)$ is a rational function with a (not necessarily least) denominator consisting of a product of factors of the form $1-q^a t^b$, where $a \geqslant 0$ and $b \geqslant 1$. The goal of this section is to describe these denominators in an explicit form.

Again for simplicity, we assume throughout this section that Φ is irreducible.

THEOREM 6.1. *For all $\lambda \in \mathcal{Q}^+$, there exist integers $a_1, \dots, a_n \geqslant 0$ such that*

$$\prod_{1 \leqslant j \leqslant n} (1-qt^{e_j})(1-q^2t^{e_j}) \cdots (1-q^{a_j}t^{e_j}) \tag{6.1}$$

is a denominator for $\bar{C}_\lambda(q,t)$, where $e_1, \dots, e_n$ are the exponents of Φ.

Before proving this result, we first need to review some basic properties of the cyclotomic polynomials $Z_m(t)$. To be precise, we let $Z_m(t) \in \mathbf{Z}[t]$ denote the minimal polynomial of a primitive mth root of unity, normalized so that $Z_m(0) = 1$. One knows that

$$1 - t^m = \prod_{d|m} Z_d(t). \tag{6.2}$$

LEMMA 6.2. *If p is a prime such that $p \nmid m$, then $Z_m(t^p) = Z_m(t) Z_{mp}(t)$.*

Proof. We have

$$\prod_{d|m} Z_d(t^p) = 1 - t^{mp} = \prod_{d|mp} Z_d(t) = \prod_{d|m} Z_d(t) Z_{dp}(t),$$

so the result follows by induction with respect to m. $\square$

LEMMA 6.3. *If $\gcd(a,b) = 1$, then $Z_m(q^a t^b)$ is irreducible in $\mathbf{Z}[q,t]$.*

Proof. By Dirichlet's Theorem, there are infinitely many positive integers k such that $a + kb = p$ is prime (or we are in the trivial case $a = 1$, $b = 0$). If k is chosen large enough so that $p > m$, Lemma 6.2 implies that $Z_m(q^{a+kb}) = Z_m(q) Z_{m(a+kb)}(q)$ is a product of exactly two irreducible polynomials in $\mathbf{Z}[q]$. On the other hand, a nonconstant polynomial in $\mathbf{Z}[q,t]$ can have only finitely many constant specializations of the form $t = q^k$, so if $Z_m(q^a t^b)$ failed to be irreducible, it would have to have a divisor, say $f(q,t)$, with the property that $f(q, q^k) = Z_m(q)$ for infinitely many k. Hence, $f(q,t)$ must be independent of t and $f(q,t) = Z_m(q)$ divides $Z_m(q^a t^b)$. However, the latter does not vanish when q is a root of unity, a contradiction. $\square$

For integers $a, b \geqslant 0$, it will be convenient to introduce the notation

$$Z_{a,b}(q,t) := Z_m(q^{a/m} t^{b/m}) \qquad (m = \gcd(a,b),\ (a,b) \neq (0,0)).$$

The previous lemma shows that these polynomials are irreducible; they may be viewed as bivariate cyclotomic polynomials in the sense that

$$1 - q^a t^b \;=\; \prod_{d \,|\, \gcd(a,b)} Z_{a/d,b/d}(q,t). \tag{6.3}$$

LEMMA 6.4. *If $b \geqslant 1$, then $Z_{a,b}(q,t)$ divides a given polynomial $N(q,t)$ in $\mathbf{Z}[q,t]$ if and only if $Z_{a+kb}(q)$ divides $N(q,q^k)$ for all k sufficiently large.*

Proof. Let $m = \gcd(a,b)$. Since $Z_{a,b}(q,t)$ is irreducible (Lemma 6.3) and cannot divide any polynomial depending only on q, we may replace $N(q,t)$ by $Z_m(q)N(q,t)$ if necessary, and suppose that $Z_m(q)Z_{a+kb}(q)$ divides $N(q,q^k)$ for all sufficiently large k. As noted in the proof of Lemma 6.3, we have $Z_{a,b}(q,q^k) = Z_m(q)Z_{a+kb}(q)$ for infinitely many k, so there are also infinitely many k such that $Z_{a,b}(q,q^k)$ divides $N(q,q^k)$.

Replacing $N(q,t)$ by some $\mathbf{Z}[q]$-multiple if necessary, we may assume that the division of $N(q,t)$ by $Z_{a,b}(q,t)$ over the ground field $\mathbf{Q}(q)$ yields a quotient and remainder in $\mathbf{Z}[q,t]$. In particular, the remainder $R(q,t)$ must have t-degree smaller than $Z_{a,b}(q,t)$, so the degree of $R(q,q^k)$ is smaller than the degree of $Z_{a,b}(q,q^k)$ for k sufficiently large. It follows that the remainder of $N(q,q^k)$ divided by $Z_{a,b}(q,q^k)$ is $R(q,q^k)$ for all such k. However, the conclusion of the previous paragraph shows that there are infinitely many such choices for which the remainder is zero; i.e., $R(q,t)$ has infinitely many roots of the form $t = q^k$. Thus $R(q,t) = 0$ and $Z_{a,b}(q,t)$ divides $N(q,t)$.

For the converse, note that $q \mapsto q^{(ak+b)/m}$ maps roots of unity of order $ak+b$ to roots of unity of order m, so $Z_{a+kb}(q)$ is a divisor of $Z_{a,b}(q,q^k) = Z_m(q^{(a+kb)/m})$ for all $k > 0$. Thus the stated condition is clearly necessary. $\square$

For each positive integer b, let us define $r_b = r_b(\Phi)$ to be the number of exponents of Φ (counted according to multiplicity) that are multiples of b.

LEMMA 6.5. *If $b \geqslant 1$ and $Z_{a,b}(q,t)$ is a factor of the least denominator for $\bar{C}_\lambda(q,t)$, then $a \neq 0$ and b divides some exponent of Φ. Furthermore, the multiplicity of $Z_{a,b}(q,t)$ as an irreducible factor of this denominator is at most $r_b(\Phi)$.*

Proof. Write $\bar{C}_\lambda(q,t) = N(q,t)/D(q,t)$ where N and D are polynomials in $\mathbf{Z}[q,t]$ with no common factor, and suppose that $Z_{a,b}(q,t)$ occurs $r \geqslant 1$ times as a factor of $D(q,t)$. We must have $Z_{a,b}(q,t) \nmid N(q,t)$, so by Lemma 6.4, there are infinitely many choices for k such that $Z_{a+kb}(q)$ does not divide $N(q,q^k)$. On the other hand, the Macdonald kernel $\Delta(q,t)$ has polynomial coefficients at $t = q^k$, so $C_\lambda(q,q^k) = C_0(q,q^k)N(q,q^k)/D(q,q^k)$ must be a polynomial for all $k \geqslant 1$. Thus there are infinitely many choices for k such that $Z_{a+kb}(q)^r$ divides $C_0(q,q^k)$. However, Cherednik's evaluation of $C_0(q,t)$ (see (1.5)) implies

$$C_0(q,q^k) = \prod_{1 \leqslant j \leqslant n} (1 - q^{ke_j+1})(1 - q^{ke_j+2}) \cdots (1 - q^{ke_j+k-1}),$$

and $Z_d(q)$ occurs as an irreducible factor of $1 - q^m$ (necessarily of multiplicity one) if and only if $d \mid m$ (see (6.2)), so there must be infinitely many k such that there are at least r multiples of $a + kb$ among the integers $ke_j + i$ ($1 \leqslant j \leqslant n$, $1 \leqslant i < k$). This immediately eliminates the possibility $a = 0$, since none of these integers are multiples of k.

For each multiple of $a + kb$ among the integers $ke_j + i$ for fixed j, there must be an integer c in the range

$$\frac{ke_j + 1}{a + kb} \leqslant c \leqslant \frac{ke_j + k - 1}{a + kb} < \frac{e_j + 1}{b}.$$

The above lower bound approaches e_j/b as $k \to \infty$, so for k sufficiently large, the only integers c trapped in this range satisfy $e_j/b \leqslant c < (e_j + 1)/b$. However, there is at most one integer with this property; namely, $c = e_j/b$, and it occurs only if $b \mid e_j$. Thus we conclude that at least r of the exponents e_j must be multiples of b. □

Proof of Theorem 6.1. Write $\bar{C}_\lambda(q, t) = N(q, t)/D(q, t)$ where N and D are polynomials in $\mathbf{Z}[q, t]$ with no common factor. By Corollary 3.5, we know that $D(q, t)$ divides a product of cyclotomic polynomials $Z_m(t)$ and polynomials of the form $1 - q^a t^b$ with $a, b \geqslant 1$. However, Lemma 6.5 in the special case $(a, b) = (0, m)$ shows that $Z_m(t)$ cannot occur as a factor of $D(q, t)$, so we conclude that the irreducible factors of $D(q, t)$ are polynomials of the form $Z_{a,b}(q, t)$ for some $a, b \geqslant 1$.

Let I denote the finite set of pairs (a, b) such that $\gcd(a, b) = 1$ and $Z_{ma,mb}(q, t)$ occurs as a factor of $D(q, t)$ for some $m \geqslant 1$. By Lemma 6.5, the maximum multiplicity of the factor $Z_{ma,mb}(q, t)$ is r_{mb}, so $D(q, t)$ is a divisor of

$$\prod_{(a,b)\in I} \prod_{m\geqslant 1} Z_{ma,mb}(q, t)^{r_{mb}} = \prod_{(a,b)\in I} \prod_{j:b\mid e_j} \prod_{m\mid e_j/b} Z_{ma,mb}(q, t) = \prod_{(a,b)\in I} \prod_{j:b\mid e_j} 1 - q^{ae_j/b} t^{e_j},$$

the second equality being an instance of (6.3). The factors in the above product involving a fixed index j must be distinct. Otherwise, we would have $a/b = a'/b'$ for some pair $(a, b), (a', b') \in I$. However, we also have $\gcd(a, b) = \gcd(a', b') = 1$, so a/b and a'/b' represent the same rational in lowest terms, and hence $(a, b) = (a', b')$. Thus the factors involving the index j must divide $(1 - qt^{e_j})(1 - q^2 t^{e_j}) \cdots (1 - q^a t^{e_j})$ for some $a \geqslant 0$. □

Having established that each normalized multiplicity $\bar{C}_\lambda(q, t)$ has a denominator of the form described in (6.1), it is natural to try to determine the parameters $a_1, \ldots, a_n$ for these denominators explicitly as a function of λ. However, it should be emphasized that this is not a well-defined problem. Indeed, it is not always the case that the *least* denominator for $\bar{C}_\lambda(q, t)$ has this form. Furthermore, it can (and does) happen that for some λ, the set of all denominators for $\bar{C}_\lambda(q, t)$ having the form of (6.1) does not have a unique minimum with respect to divisibility. Nevertheless, we shall see that it is possible to describe explicit parameters $a_1, \ldots, a_n$ that are (empirically) close to being minimal.

For each $J \subseteq [n]$, let Φ_J denote the parabolic root subsystem of Φ generated by the simple roots indexed by J, and define

$$\lambda_J := \sigma(\Phi^+ - \Phi_J) = 2\rho - 2\rho_J \in \mathcal{Q},$$

where ρ_J denotes half the sum of the positive roots of Φ_J. We remark that these weights are related to the η-functional that we introduced in Theorem 3.3: if μ is dominant and $J = \{i : \langle \mu, \alpha_i \rangle = 0\}$, then $\eta(\mu) = \lambda_J$.

Note that the weights $\mathcal{B}(\Phi) := \{\lambda_J : J \subseteq [n]\}$ form a Boolean algebra in $(\mathcal{Q}, \leqslant)$; i.e.,

$$\lambda_J \leqslant \lambda_K \;\Leftrightarrow\; J \supseteq K.$$

We remark that s_i permutes $\Phi^+ - \Phi_J$ for $i \in J$, whence $\langle \lambda_J, \alpha_i^\vee \rangle = 0$ in such cases. On the other hand, for $i \notin J$, we have $\langle \beta, \alpha_i^\vee \rangle \leqslant 0$ for $\beta \in \Phi_J$, and hence $\langle \lambda_J, \alpha_i^\vee \rangle \geqslant 2\langle \rho, \alpha_i^\vee \rangle = 2$. Thus λ_J is dominant and belongs to the face of the dominant chamber whose stabilizer is generated by $\{s_i : i \in J\}$. In particular, if $J = [n] - \{j\}$, then λ_J is an integer multiple of the jth fundamental weight.

If $\lambda \in \mathcal{Q}$ is in the sublattice $b\mathcal{Q}$ for some integer $b \geqslant 1$, we say that λ is *b-divisible.*

Theorem 6.6. *For all $\lambda \in \mathcal{Q}^+$, we have*

$$\bar{C}_\lambda(q,t) \cdot \prod_{a,b \geqslant 1} Z_{a,b}(q,t)^{k_{a,b}(\lambda)} \in \mathbf{Z}[q,t],$$

where $k_{a,b}(\lambda) = \min(r_b, l_{a,b}(\lambda))$ and $l_{a,b}(\lambda) = l$ denotes the length of the longest chain

$$\lambda \geqslant (a/b)\lambda_{J_1} > (a/b)\lambda_{J_2} > \cdots > (a/b)\lambda_{J_l} > 0,$$

in which $\lambda_{J_1}, \ldots, \lambda_{J_l}$ are each b-divisible.

Remark 6.7. We have used the methods of Section 5 to explicitly compute several thousand graded multiplicities $\bar{C}_\lambda(q,t)$, including at least 100 for every irreducible root system of rank $\leqslant 6$, as well as $\mathcal{D}_7$ and $\mathcal{E}_7$. (And 25 in the case of $\mathcal{E}_8$.) We found that in the vast majority of these cases, the denominator described in the above result is either the least denominator, or has only one extraneous irreducible factor.

In the following, it will be convenient to compute greatest common divisors in the Laurent polynomial ring $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. Note that this ring is still a unique factorization domain, but q and t are now units. In this context, the definition of $Z_{a,b}(q,t)$ remains sensible, and Lemma 6.3 remains valid, if we allow a or b to be negative.

As noted previously, for each nonzero $\lambda \in \mathcal{Q}^+$, we have $\eta(\lambda) = \lambda_J \neq 0$ for some J. Letting $b_J = b$ denote the largest integer such that λ_J is b-divisible (i.e., the greatest common divisor of the simple root coordinates for λ_J), there must exist some $\theta \in \mathcal{P}^\vee$ such that $\langle \lambda_J, \theta \rangle = b_J$. One knows that the minuscule co-weights are coset representatives for $\mathcal{P}^\vee/\mathcal{Q}^\vee$ (e.g., see Exercise VI.2.5 of [**B**]), so there is a unique nonzero minuscule or quasi-minuscule dominant co-weight ω such that θ is in the coset $\omega + \mathcal{Q}^\vee$.

Lemma 6.8. *Given λ, ω, and J as above, we have*

$$\gcd\{1 - q^{\langle \nu,\omega\rangle} t^{\langle \eta(\nu),\omega\rangle} : \nu \in W\lambda\} = \begin{cases} 1-q^c t^{b_J} & \text{if } \lambda = (c/b_J)\lambda_J \text{ for some } c \geqslant 1, \\ 1 & \text{otherwise.} \end{cases}$$

If ω is quasi-minuscule and we add the constraint $\langle \nu, \omega\rangle \geqslant 1$, the same result holds.

Proof. Let $G_\lambda(q,t) = \gcd\{1 - q^{\langle \nu,\omega\rangle} t^{\langle \eta(\nu),\omega\rangle} : \nu \in W\lambda\}$. Since η intertwines the action of W (i.e., $\eta(w\nu) = w\eta(\nu)$ for $w \in W$), it follows that the same common divisor would be obtained by fixing $\nu = \lambda$ and letting ω vary over its W-orbit; i.e.,

$$G_\lambda(q,t) = \gcd\{1 - q^{\langle \lambda,\theta\rangle} t^{\langle \eta(\lambda),\theta\rangle} : \theta \in W\omega\}.$$

One sees from (6.3) and Lemma 6.3 that $1 - q^a t^b$ and $1 - q^c t^d$ have a common divisor if and only if (a,b) and (c,d) are proportional, or equivalently, if $L = \mathbf{Z}(a,b) + \mathbf{Z}(c,d)$ is a lattice of rank one. In that case, the greatest common divisor is $1 - q^u t^v$, where (u,v) is any generator of L. Thus we deduce that $G_\lambda(q,t) = 1$ unless

$$\{(\langle \lambda, \theta\rangle, \langle \eta(\lambda), \theta\rangle) : \theta \in W\omega\} \subset \mathbf{Z}^2 \tag{6.4}$$

generates a lattice of rank one. In the quasi-minuscule case (i.e., $\omega = \widetilde{\alpha}^\vee$, where $\widetilde{\alpha}$ denotes the highest root), adding the constraint $\langle \nu, \omega\rangle \geqslant 1$ is equivalent to restricting θ to those co-roots $\beta^\vee$ in the orbit of $\widetilde{\alpha}^\vee$ such that $\langle \lambda, \beta^\vee\rangle \geqslant 1$. However, any orbit of roots is centrally symmetric, and $\langle \lambda, \beta^\vee\rangle = 0$ implies $s_\beta(\lambda) = \lambda$, $s_\beta(\eta(\lambda)) = \eta(\lambda)$, and hence $\langle \eta(\lambda), \beta^\vee\rangle = 0$, so this extra constraint does not change the lattice.

It is well-known and easy to show that the lattice generated by any nonzero orbit in $\mathcal{P}^\vee$ includes a basis of $\operatorname{Span}\Phi$, so the lattice generated by (6.4) cannot be of rank one unless λ and $\eta(\lambda) = \lambda_J$ are proportional. In that case, we must have $\lambda = (c/b_J)\lambda_J$ for some $c \geqslant 1$, and each generator in (6.4) is a multiple of (c, b_J). However, we chose ω precisely so that some member of the lattice has the form (a, b_J) for some a, so we conclude that (c, b_J) generates the lattice and $G_\lambda(q,t) = 1 - q^c t^{b_J}$. $\square$

Proof of Theorem 6.6. Given a nonzero $\lambda \in \mathcal{Q}^+$ with $\eta(\lambda) = \lambda_J$, let ω be a nonzero dominant minuscule or quasi-minuscule co-weight such that $\langle \lambda_J, \theta\rangle = b_J$ for some θ in the coset $\omega + \mathcal{Q}^\vee$. If ω is minuscule, Theorem 3.3 implies that for every ν in the W-orbit of λ, there is a linear relation of the form

$$\sum_{\mu \leqslant \lambda} B^\omega_{\nu\mu}(q,t)\bar{C}_\mu(q,t) = 0,$$

where $B^\omega_{\nu\lambda}(q,t) = (1 - q^{\langle \nu,\omega\rangle} t^{\langle \eta(\nu),\omega\rangle}) A^\omega_{\nu\lambda}(t)$. It follows that for all such ν,

$$\left(1 - q^{\langle \nu,\omega\rangle} t^{\langle \eta(\nu),\omega\rangle}\right) A^\omega_{\nu\lambda}(t) D_\lambda(q,t)\bar{C}_\lambda(q,t) \in \mathbf{Z}[q^{\pm 1}, t^{\pm 1}], \tag{6.5}$$

where $D_\lambda(q,t)$ denotes the least common denominator for $\{\bar{C}_\mu(q,t) : \mu < \lambda\}$. Furthermore, Theorem 6.1 shows that the least denominator for $\bar{C}_\lambda(q,t)$ has no irreducible factor that depends only on t, so $A^\omega_{\nu\lambda}(t)$ may be omitted from (6.5) whenever it is nonzero. However, the forms $\langle\,\cdot\,,\omega\rangle$ and $\langle\eta(\cdot),\omega\rangle$ are both constant on W_ω-orbits, and Lemma 2.4 shows that every W_ω-orbit in $W\lambda$ includes at least one member ν such that $A^\omega_{\nu\lambda}(t) \neq 0$, so there is no harm in omitting $A^\omega_{\nu\lambda}(t)$ from (6.5) even when it is zero, and we conclude that the greatest common divisor of the terms $(1 - q^{\langle\nu,\omega\rangle}t^{\langle\eta(\nu),\omega\rangle})D_\lambda(q,t)$ is a denominator for $\bar{C}_\lambda(q,t)$.

If ω is quasi-minuscule, Theorem 3.4 may be used to reach the same conclusion, the only modification being the need for the additional constraint $\langle\nu,\omega\rangle \geqslant 1$. (Note also that the extraneous factors of 2 that occur when $\langle\nu,\omega\rangle = 1$ may be omitted, by Theorem 3.4(c).)

In both cases, Lemma 6.8 shows that the greatest common divisor in question is either $D_\lambda(q,t)$ or $(1 - q^c t^{b_J})D_\lambda(q,t)$, the latter occurring if and only if $\lambda = (c/b_J)\lambda_J$ for some integer $c \geqslant 1$. In this case, note that the irreducible factors of $1 - q^c t^{b_J}$ are $Z_{a,b}(q,t)$, where a and b range over all integers $\geqslant 1$ such that $a/b = c/b_J$ and λ_J is b-divisible (i.e., $b \mid b_J$). By induction, it follows that $l_{a,b}(\lambda)$ (as defined in the statement of the theorem) is an upper bound for the multiplicity k of $Z_{a,b}(q,t)$ in the least denominator for $\bar{C}_\lambda(q,t)$. On the other hand, we know that every irreducible factor of the least denominator is necessarily of the form $Z_{a,b}(q,t)$ for some $a, b \geqslant 1$, and Lemma 6.5 shows that $r_b = r_b(\Phi)$ is also an upper bound for k; i.e., $k \leqslant \min(r_b, l_{a,b}(\lambda))$. $\square$

Now let the exponents of Φ be sorted into some canonical order, say $e_1 \leqslant \cdots \leqslant e_n$, and let $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_n)$ be a list of subsets of subsets of $[n]$. For each $\lambda \in \mathcal{Q}^+$, define $a_j = a_j(\lambda)$ to be the largest integer $a \geqslant 0$ such that $e_j\lambda \geqslant a\lambda_J$ for some $J \in \mathcal{M}_j$. We say that $\mathcal{M}$ is a *denominator system* for $\Delta(q,t)$ if for all $\lambda \in \mathcal{Q}^+$, we have

$$\bar{C}_\lambda(q,t) \cdot \prod_{1 \leqslant j \leqslant n} (1 - qt^{e_j})(1 - q^2t^{e_j}) \cdots (1 - q^{a_j(\lambda)}t^{e_j}) \in \mathbf{Z}[q,t].$$

A sufficient condition for denominator systems that reduces to the analysis of chains in the Boolean algebra $\mathcal{B}(\Phi) = \{\lambda_J : J \subseteq [n]\}$ may be deduced from Theorem 6.6.

Corollary 6.9. *The family $\mathcal{M}$ is a denominator system for $\Delta(q,t)$ if for all $b \geqslant 1$ and all $J \subset [n]$ such that λ_J is the top of a $\mathcal{B}(\Phi)$-chain of b-divisible elements of length $k \leqslant r_b$, at least k of the sets $\mathcal{M}_j$ with $b \mid e_j$ include subsets of $[n]$ that contain J.*

Proof. It suffices to show that if $Z_{a,b}(q,t)$ has multiplicity k in the least denominator for $\bar{C}_\lambda(q,t)$, then there are at least k indices j such that $Z_{a,b}(q,t)$ divides $1 - q^c t^{e_j}$ for some $c \leqslant a_j(\lambda)$. Given that this multiplicity is k, Theorem 6.6 implies that $k \leqslant r_b(\Phi)$ and that there must be some $J \subset [n]$ such that $\lambda \geqslant (a/b)\lambda_J$ and λ_J is the top element of a b-divisible chain in $\mathcal{B}(\Phi)$ of length k. By the hypothesis on $\mathcal{M}$, it follows that there exist subsets $K \in \mathcal{M}_j$ containing J for at least k indices j such that $b \mid e_j$. For each such K, we have $e_j\lambda \geqslant e_j(a/b)\lambda_J \geqslant (ae_j/b)\lambda_K$, and thus $a_j(\lambda) \geqslant ae_j/b$. Since $Z_{a,b}(q,t)$ divides $1 - q^c t^{e_j}$ when $c = ae_j/b$, the result follows. $\square$

REMARK 6.10. (a) It is easy to see that denominator systems exist: if λ_J is the top element of a b-divisible chain in $\mathcal{B}(\Phi)$ of length $\leqslant r_b(\Phi)$, it suffices to include J in enough of the sets $\mathcal{M}_j$ to force the condition in Corollary 6.9 to hold. A more difficult problem is that of finding an optimal denominator system, or a near-optimal system that can be described uniformly for all root systems.

(b) Consider $\Phi = \mathcal{A}_4$, with the Dynkin diagram numbered 1–2–3–4. The exponents are $1, 2, 3, 4$, and one can easily check that in simple root coordinates,

$$\lambda_{\{1,2\}} = (2,4,6,4), \quad \lambda_{\{3,4\}} = (4,6,4,2), \quad \lambda_{\{1,4\}} = (3,6,6,3),$$
$$\lambda_{\{2,3\}} = (4,4,4,4), \quad \lambda_{\varnothing} = 2\rho = (4,6,6,4).$$

In all other cases, the coefficients of λ_J are co-prime (i.e., $b_J = 1$). Thus there is only one (nonzero) 4-divisible weight in $\mathcal{B}(\Phi)$, one 3-divisible weight, and there are three 2-divisible chains of length 2. From this it is not hard to show that

$$\mathcal{M}_1 = \{\{1,2,3\},\{2,3,4\}\},\ \mathcal{M}_2 = \{\{1,2,4\},\{1,3,4\}\},\ \mathcal{M}_3 = \{\{1,4\}\},\ \mathcal{M}_4 = \{\{2,3\}\}$$

fits the hypothesis of Corollary 6.9, and thus yields an $\mathcal{A}_4$-denominator system.

(c) If λ_J is e_j-divisible for every $J \in \mathcal{M}_j$, then $a_j(\lambda)$ may be redefined as the largest integer a such that $\lambda \geqslant (a/e_j)\lambda_J$ for some $J \in \mathcal{M}_j$. However, not every root system has a denominator system of this form. Indeed, when $\Phi = \mathcal{A}_4$ and $\lambda = \lambda_{\{1,3\}}$, one sees that there is no 2-divisible element $\leqslant \lambda_{\{1,3\}}$ in $\mathcal{B}(\Phi)$, so we would have $a_2(\lambda) = 0$ in such a system. On the other hand, one can show that $Z_{1,2}(q,t) = 1 - qt^2$ occurs with multiplicity 2 in the least denominator for $\bar{C}_\lambda(q,t)$ (matching the upper bound r_2), forcing $a_2(\lambda) \geqslant 1$.

Let Γ denote the group of permutations of $[n]$ induced by automorphisms of the Dynkin diagram of Φ. We define the *primary weights* of Φ to be those (nonzero) weights λ_J such that the (nonempty) complement $[n] - J$ is a subset of a single Γ-orbit.

In all cases, the set of primary weights includes $\lambda_{\langle 1\rangle}, \dots, \lambda_{\langle n\rangle}$, where $\langle j\rangle = [n] - \{j\}$. In addition, if there is a Γ-orbit containing i and j, then $\lambda_{\langle i,j\rangle}$ is also a primary weight, where $\langle i,j\rangle = [n] - \{i,j\}$. This accounts for all of the primary weights except in $\mathcal{D}_4$, where the Γ-orbits are $\{1,2,3\}$ and $\{4\}$ (say), and $\lambda_{\langle 1,2,3\rangle} = \lambda_{\{4\}}$ is also a primary weight.

Note that Γ permutes the set of primary weights. Since each Γ-orbit $J \subset [n]$ of size k produces exactly k orbits of nonempty subsets of J, it follows that there are exactly n orbits of primary weights. Numbering these orbits arbitrarily from 1 to n, we define the jth *divisibility index* of Φ to be the largest integer b_j such that λ_J is b_j-divisible for some (equivalently, any) primary weight in the jth Γ-orbit. See Table 6.1.

EMPIRICAL FACT 6.11. *For every irreducible root system except $\mathcal{A}_n$ ($n = 4$ or $n \geqslant 6$), the divisibility indices b_j may be ordered so that $e_j \mid b_j$. Moreover, this ordering is unique.*

Sketch of Proof. Consider $\Phi = \mathcal{A}_n$, with the Dynkin diagram numbered $1–2–\cdots–n$. To compute the divisibility index of (the Γ-orbit of) $\lambda_{\langle j\rangle}$, note that there are $n-1$ positive

Φ	$b_1, \ldots, b_n$	$e_1, \ldots, e_n$
$\mathcal{A}_{2n-1}$	$n;\ n+j, \gcd(j, 2n) : 1 \leqslant j < n$	$1, 2, \ldots, 2n-1$
$\mathcal{A}_{2n}$	$n+j, \gcd(j, 2n+1) : 1 \leqslant j \leqslant n$	$1, 2, \ldots, 2n$
$\mathcal{B}_n$	$n, n+1, \ldots, 2n-1$	$1, 3, \ldots, 2n-1$
$\mathcal{C}_{2n}$	$n+j, 2n+2j-1 : 1 \leqslant j \leqslant n$	$1, 3, \ldots, 4n-1$
$\mathcal{C}_{2n+1}$	$2n+1;\ n+j, 2n+2j+1 : 1 \leqslant j \leqslant n$	$1, 3, \ldots, 4n+1$
$\mathcal{D}_{2n}$	$n+j, 2n+2j-1 : 0 \leqslant j < n$	$2n-1;\ 1, 3, \ldots, 4n-3$
$\mathcal{D}_{2n+1}$	$2n;\ n+j, 2n+2j+1 : 0 \leqslant j < n$	$2n;\ 1, 3, \ldots, 4n-1$
$\mathcal{E}_6$	$3, 4, 5, 7, 8, 11$	$1, 4, 5, 7, 8, 11$
$\mathcal{E}_7$	$5, 7, 8, 9, 11, 13, 17$	$1, 5, 7, 9, 11, 13, 17$
$\mathcal{E}_8$	$9, 11, 13, 14, 17, 19, 23, 29$	$1, 7, 11, 13, 17, 19, 23, 29$
$\mathcal{F}_4$	$5, 7, 8, 11$	$1, 5, 7, 11$
$\mathcal{G}_2$	$3, 5$	$1, 5$

TABLE 6.1. Divisibility indices and exponents.

roots $\beta \in \Phi_{\langle j \rangle}$ such that $\langle \beta, \alpha_j^\vee \rangle = -1$, whence $\langle 2\rho_{\langle j \rangle}, \alpha_j^\vee \rangle = -(n-1)$ and

$$\langle \lambda_{\langle j \rangle}, \alpha_j^\vee \rangle = \langle 2\rho - 2\rho_{\langle j \rangle}, \alpha_j^\vee \rangle = n+1.$$

It follows that $\lambda_{\langle j \rangle}/(n+1)$ is the jth fundamental weight.

For $\mathcal{A}_n$, we have $\lambda \in \mathcal{Q}$ if and only if $\lambda \in \mathcal{P}$ (i.e., $\langle \lambda, \alpha_i^\vee \rangle \in \mathbf{Z}$ for all i) and $\langle \lambda, \delta \rangle = 0$ modulo $n+1$, where $\delta = \sum i\alpha_i^\vee$. It follows that $\lambda_{\langle j \rangle}$ is b-divisible if and only if $b \mid n+1$ (i.e., $(1/b)\lambda_{\langle j \rangle} \in \mathcal{P}$) and $b \mid j$ (i.e., $\langle \lambda_{\langle j \rangle}, \delta \rangle = (n+1)j/b \in (n+1)\mathbf{Z}$). Hence, the divisibility index of the Γ-orbit $\{\lambda_{\langle j \rangle}, \lambda_{\langle n+1-j \rangle}\}$ is $\gcd(j, n+1)$.

The remaining primary weights for $\mathcal{A}_n$ have the form $\lambda_{\langle i,j \rangle}$, where $i+j = n+1$ and $i < j$. In these cases, there are $j-2$ positive roots $\beta \in \Phi_{\langle i,j \rangle}$ such that $\langle \beta, \alpha_j^\vee \rangle = -1$, whence $\langle \lambda_{\langle i,j \rangle}, \alpha_i^\vee \rangle = j$ and $\langle \lambda_{\langle i,j \rangle}, \alpha_j^\vee \rangle = j$ by Γ-symmetry. It follows that $\langle \lambda_{\langle i,j \rangle}, \delta \rangle = (n+1)j$, so j is the divisibility index of $\lambda_{\langle i,j \rangle}$.

The calculation of divisibility indices in the remaining root systems is similar (and usually easier). We omit the details, but have recorded the results in Table 6.1.

For $\mathcal{A}_n$, it is easy to see that the divisibility indices coincide with the exponents for $n = 1, 2, 3$ and 5. In all other cases of type $\mathcal{A}$, the index $b_j = 1$ occurs at least twice, and the arrangement $e_j \mid b_j$ is clearly impossible. In the remaining root systems, the existence and uniqueness of the desired ordering of divisibility indices is easiest to see by eliminating from each b_j every prime factor that does not divide one or more exponents. In most cases, the result is identical to the exponent sequence. For example, in the case of $\mathcal{B}_n$, this amounts to the fact that for every odd integer $m \leqslant 2n-1$, there is a unique power of 2 such that $n \leqslant 2^r m \leqslant 2n-1$. This elimination argument fails only when $\Phi = \mathcal{D}_{2n+1}$. However in this case, $2n$ is the largest even divisibility index, so it is the only one that is divisible by the exponent $2n$ (the only even exponent). Furthermore, if we discard factors of 2 from the remaining indices, the result matches the remaining (odd) exponents. $\square$

Now let $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_n)$ denote the particular subset family obtained by requiring $J \in \mathcal{M}_j$ if and only if λ_J is in the jth Γ-orbit of primary weights.

CONJECTURE 6.12. *If the Γ-orbits of primary weights are ordered so that $e_j \mid b_j$ for $j = 1, \ldots, n$, then $\mathcal{M}$ is a denominator system for the Macdonald kernel $\Delta(q,t)$.*

REMARK 6.13. (a) One can use Corollary 6.9 to prove this conjecture in the following cases: $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_5$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_6$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_7$, $\mathcal{D}_4$, $\mathcal{D}_8$, $\mathcal{E}_8$, $\mathcal{F}_4$, and $\mathcal{G}_2$ (and no others). In each case, this is a finite (but sometimes lengthy) calculation; we omit the details.

(b) Among the cases where Corollary 6.9 cannot be applied, we have extensive data supporting the conjecture (see Remark 6.7) for $\mathcal{B}_5$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{D}_5$, $\mathcal{D}_6$, $\mathcal{D}_7$, $\mathcal{E}_6$, and $\mathcal{E}_7$.

(c) Each weight λ_K such that $|K| = n - k$ is the top element of a chain in $\mathcal{B}(\Phi)$ of length k, and it is easy to show that there exist subsets $J \in \mathcal{M}_j$ containing K for exactly k of the sets $\mathcal{M}_j$. Thus the criterion of Corollary 6.9 holds for $\mathcal{M}$ in the case $b = 1$.

(d) The sets $\mathcal{M}_j$ are Γ-stable, so the denominator that the conjecture would predict for $\bar{C}_\lambda(q,t)$ is constant on the Γ-orbit of each $\lambda \in \mathcal{Q}^+$. Since the Macdonald kernel is also Γ-invariant, one knows that the least denominators share this same invariance.

(e) Empirical Fact 6.11 shows that the conjecture is applicable to every irreducible root system except $\mathcal{A}_4$ and $\mathcal{A}_n$ for $n \geqslant 6$. Even in these inapplicable cases, it is plausible that some ordering of $\mathcal{M}$ is a denominator system. In many cases, one may use Corollary 6.9 to prove the validity of the ordering in which $\mathcal{M}_j = \{\langle j \rangle, \langle n+1-j \rangle\}$ for $j \leqslant (n+1)/2$ and $\mathcal{M}_j = \{\langle j, n+1-j \rangle\}$ for $j \geqslant (n+1)/2$ (with the exponents ordered so that $e_j = j$). For example, this is valid for $n = 4$ (see Remark 6.10(b)), 7, and 11, but not $n = 6$ or 9. The smallest n for which we are unable to prove that some ordering of $\mathcal{M}$ produces a valid $\mathcal{A}_n$-denominator system is $n = 12$.

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