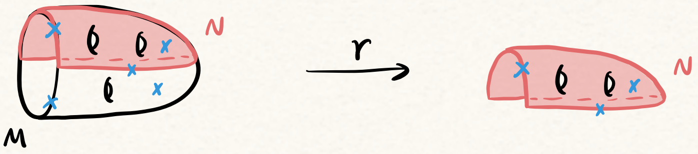
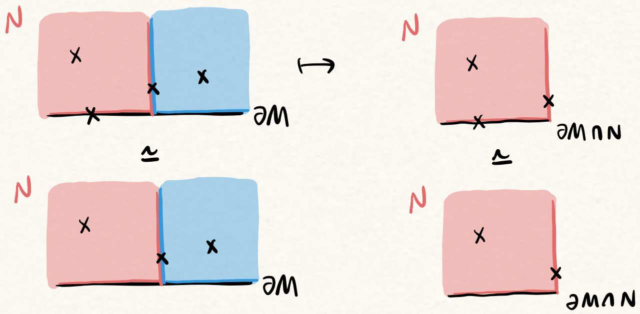
Recall: In talk 7, we assumed that $\tilde{C}(M) \to \tilde{C}(N)$ are quasifibrations for maps in the left diagram.

Goal: justify this assumption.



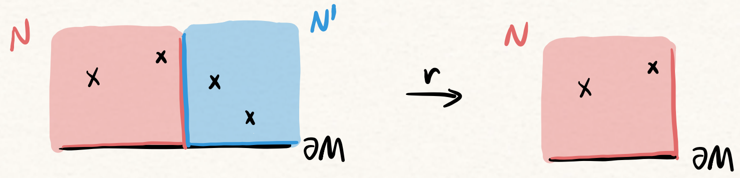
$\tilde{C}(M) \xrightarrow{r} \tilde{C}(N)$ is well-defined if $\underline{\partial M \cap N \subseteq \partial N}$.

automatic if N is a closed submanifold of the same dimension



Observations:

①



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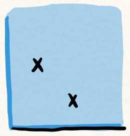


$\tilde{C}(M) \xrightarrow{r} \tilde{C}(N)$ "feels like" a fiber bundle:

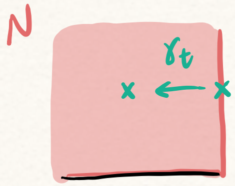
For $[s] \in \tilde{C}(N)$,

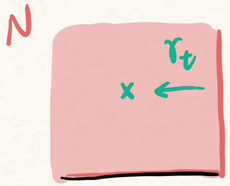
$r^{-1}([s])$ = all relative configurations of M that restrict to $[s]$ on N.

$\cong C(N', \overline{\partial N' - B})$

② $\tilde{C}(M) \to \tilde{C}(N)$ is not a fibration.

Suppose we have a path γ_t in $\tilde{C}(N)$ given by a point sliding in from $B \setminus \partial M$.



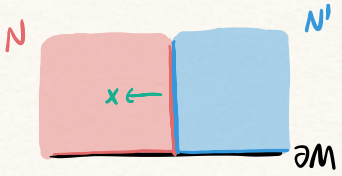


At time 0, we use the empty configuration representative.

this is a valid path in $\tilde{C}(N)$.

Claim: $\nexists\ \tilde{\gamma}$ in $\tilde{C}(M)$ with $\tilde{\gamma}(0) =$ empty configuration in M

proof by picture:

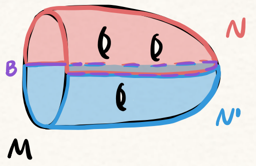


NOT a path in $\tilde{C}(M)$.

points cannot appear / disappear / collide in Int(M).

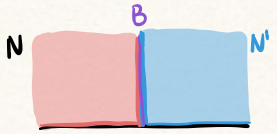
Now, we show that although $\tilde{C}(M) \to \tilde{C}(N)$ is not really a fibration, it is still a quasifibration if $N \subseteq M$ satisfies some conditions.

We impose some conditions on $N \subseteq M$:



$(N, \partial N)$, $(N', \partial N')$ embedded closed submanifolds of same dimension as M.

$M = N \cup N'$, $\quad N \cap N' = \partial N \cap \partial N' =: B$



If $B \subseteq \partial N$, $B \subseteq \partial N'$ are submanifolds,

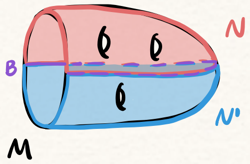
then we say $N \subseteq M$ is nice.

↑ no crazy glueing

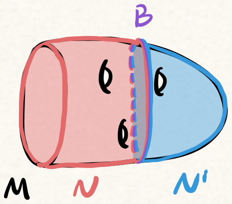
Additional condition:

(*) each connected component of B has nonempty intersection with ∂M.

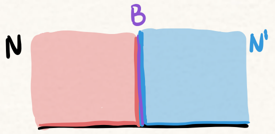
" we have to cut through ∂M "



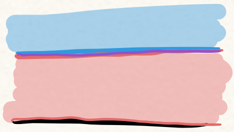
(*) ✓



(*) ✗



(*) ✓



(*) ✗

(*) is naturally satisfied when we do a handle body decomposition of compact manifold with boundary.

Thm : If $N \subseteq M$ is a nice inclusion satisfying $(*)$,

Prop 3.1 in McDuff

then $r : \tilde{C}(M) \to \tilde{C}(N)$ is a quasi-fibration with fiber $F = C(N', \overline{\partial N' - B})$.

Def: A map $r: Y \to X$ is called a quasi-fibration if $r^{-1}(x) \hookrightarrow \mathrm{hoFib}(x)$ induces an iso of homotopy groups for all x.

weak homotopy equivalence

homotopy fiber

The main strategy of proving Prop 3.1 is to use the following Lemma:

Lem 3.3: $X = \bigcup X_k$, X_k closed and $X_1 \subset X_2 \subset \cdots$

Let $r: Y \to X$ be a map.

Suppose that for each k

(i) $r: r^{-1}(X_k - X_{k-1}) \to X_k - X_{k-1}$

is a fibration with fiber F

(ii) $\exists$ open subset U_k of X_k which contains X_{k-1}

and there are homotopies $h_t : U_k \to U_k$

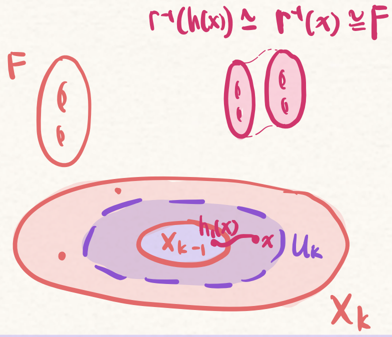
and $H_t : r^{-1}(U_k) \to r^{-1}(U_k)$ such that

(a) $h_0 = \mathrm{id}$, $h_t(X_{k-1}) \subseteq X_{k-1}$, $h_1(U_k) \subseteq X_{k-1}$.

(b) $H_0 = \mathrm{id}$, $r \circ H_t = h_t \circ r$

(c) $H_1 : r^{-1}(x) \to r^{-1}(h_1(x))$ is a homotopy equivalence $\forall x \in U_k$.

Then $r: Y \to X$ is a quasifibration with fiber F.



Intuition: For each level $X_k - X_{k-1}$, we have a fiberation.

We stitch together these fibrations in a coherent way.

using homotopy h_t and H_t

Then we get a quasifibration.

Now, we use Lemma 3.3 to prove Prop 3.1 :)

proof of Prop 3.1

$\tilde{C}^k(N) = \{$ conf of $\leq k$ points in $N \}$

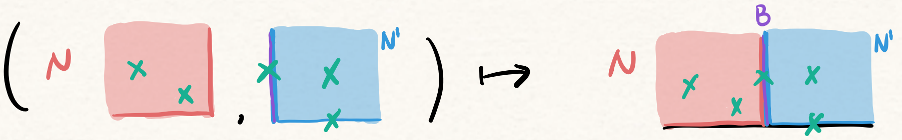
$\tilde{C}^k(N) - \tilde{C}^{k-1}(N) = \{$ conf of k pts in $N - \partial N \}$

(i) $r^{-1}(\overset{= V_k}{\tilde{C}^k(N) - \tilde{C}^{k-1}(N)}) \to \overset{= V_1}{\tilde{C}^k(N) - \tilde{C}^{k-1}(N)}$ is a fibration.

Configurations on M s.t. when restricted to $N - \partial N$, we have exactly k points.

we have homeo

$$V_k \times F \to r^{-1}(V_k);$$
$$([s], [t]) \mapsto [s \cup t]$$

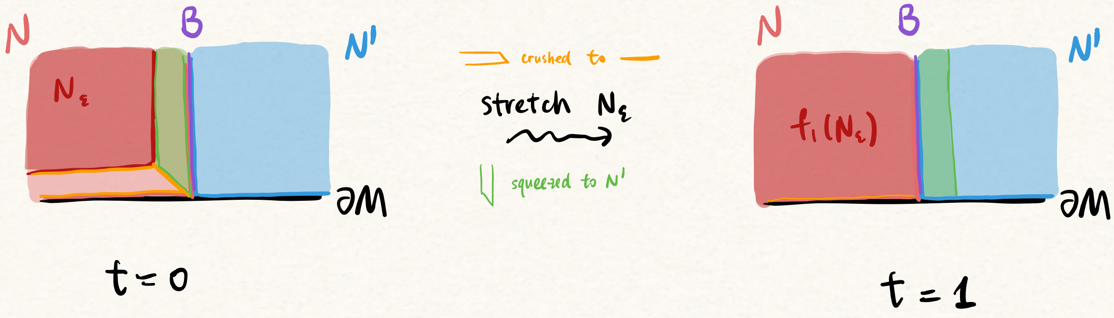


(ii) Now we build the homotopies!

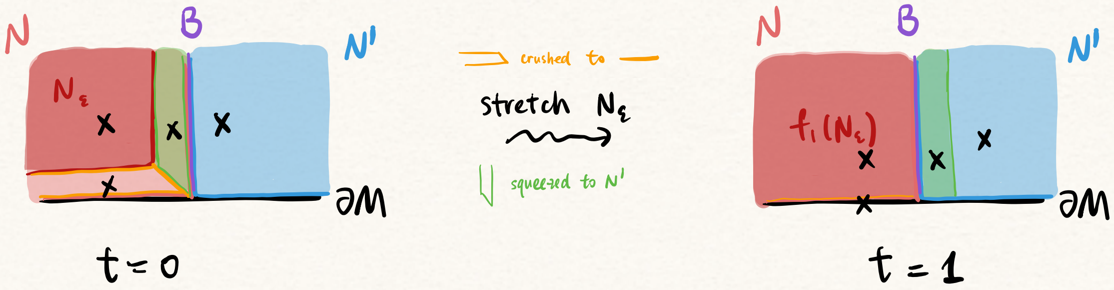
Pick ε sufficiently small.

$N_\varepsilon = N - \mathrm{Nbhd}_\varepsilon(\partial N)$

⇝ There is a homotopy $f_t : (M, \partial M) \to (M, \partial M)$ s.t. $f_0 = \mathrm{id}$, $f_1(N_\varepsilon) = N$, and $f_t|_{f_t^{-1}(M - \partial M)}$ is always injective.



This space level homotopy induces homotopies on configuration spaces.



Let $U_k := \{ [s] \in \tilde{C}^k(N) : |s \cap N_\varepsilon| \leq k-1 \} \supseteq \tilde{C}^{k-1}(N)$

⇝ f_t induces homotopy $h_t : U_k \to U_k$ s.t.

$h_0 = \mathrm{id}$, $h_t(\tilde{C}^{k-1}) \subseteq \tilde{C}^{k-1}$, $h_1(U_k) \subseteq X_{k-1}$ (a)

and homotopy $H_t : r^{-1}(U_k) \to r^{-1}(U_k)$ such that

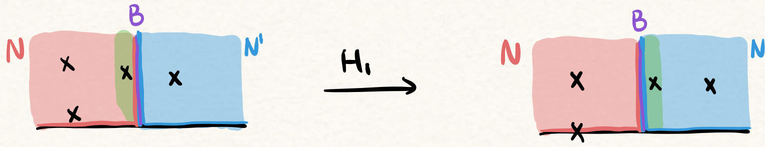
$H_0 = \mathrm{id}$, $r \circ H_t = h_t \circ r$ (b)

Now, we just need to show

$$H_1 : r^{-1}([s]) \xrightarrow[\simeq]{} r^{-1}(h_1[s]) \qquad \forall [s] \in U_k. \qquad (c)$$

Lem 3.4 : If $(*)$ is satisfied, then

$$H_1 : r^{-1}([s]) \xrightarrow[\simeq]{} r^{-1}(h_1[s]) \qquad \forall [s] \in U_k.$$



proof of Lem 3.4 :

$r^{-1}([s])$ and $r^{-1}(h_1[s])$ are both canonically identified with $C(N', \overline{\partial N' - B})$

$$H_1 : C(N', \overline{\partial N' - B}) \longrightarrow C(N', \overline{\partial N' - B})$$
$$[t] \longmapsto [f_1(t)] \cup [w]$$

$[w] = [f_1(s) \cap N']$

"particles squeezed to N'"

$(*) \Rightarrow$ $[w]$ can be connected to the empty configuration through configurations in W.

" we can push points in W to the boundary ∂M ".

$\Rightarrow$ H_1 is homotopic to $[t] \mapsto [f_1(t)]$

$\Rightarrow$ H_1 is homotopic to id.

$\Rightarrow$ H_1 is an homotopy equivalence.

Note: If (*) is not satisfied, then these points in W are trapped.

Lemma 3.4 wouldn't hold.