# L-functions, braid groups, and Burau representations

## ① Configuration space and Burau representation

$$\mathrm{UConf}_n\,\mathbb{C} = \{ S \subset \mathbb{C} \mid |S| = n \} \cong \{ f \in \mathbb{C}[X] \mid f \text{ monic, square-free, } \deg f = n \}$$

$$\{x_1, \dots, x_n\} \longmapsto (X - x_1) \cdots (X - x_n)$$

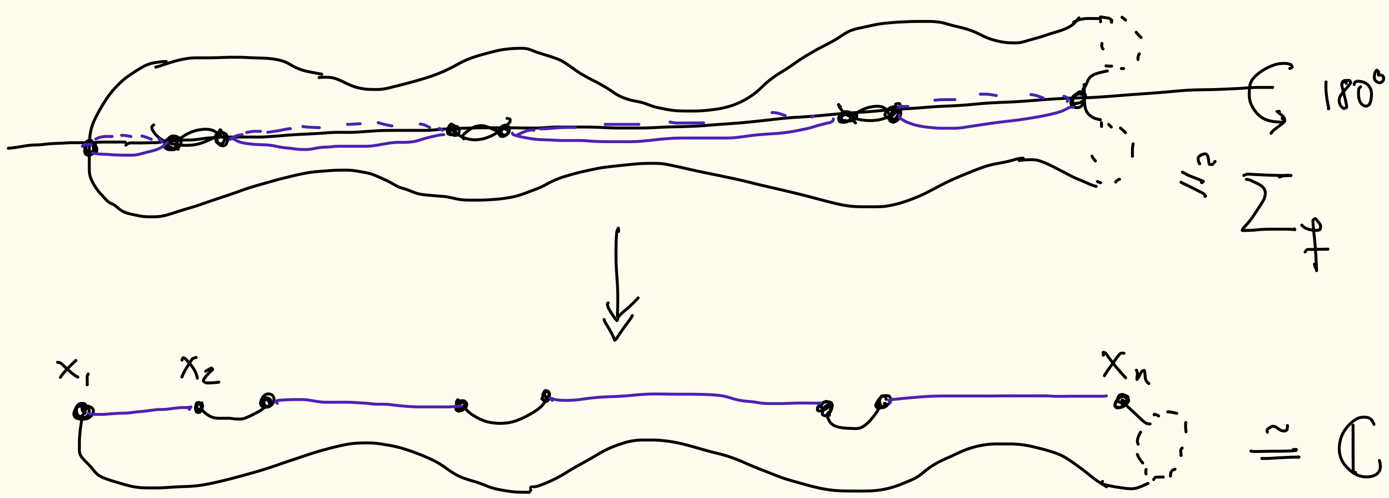
Given $f(X) \in \mathrm{UConf}_n\,\mathbb{C}$

$$\Sigma_f = \{ (x,y) \in \mathbb{C}^2 \mid y^2 = f(x) \} \twoheadrightarrow \mathbb{C}$$
$$(x,y) \longmapsto x$$

$$\dim_{\mathbb{C}} \Sigma_f = 1 \Rightarrow \dim_{\mathbb{R}} \Sigma_f = 2$$

$|\pi^{-1}(x)| = 2$ unless $f(x) = 0$, then $|\pi^{-1}(x)| = 1$

$\leadsto$ branched double cover:



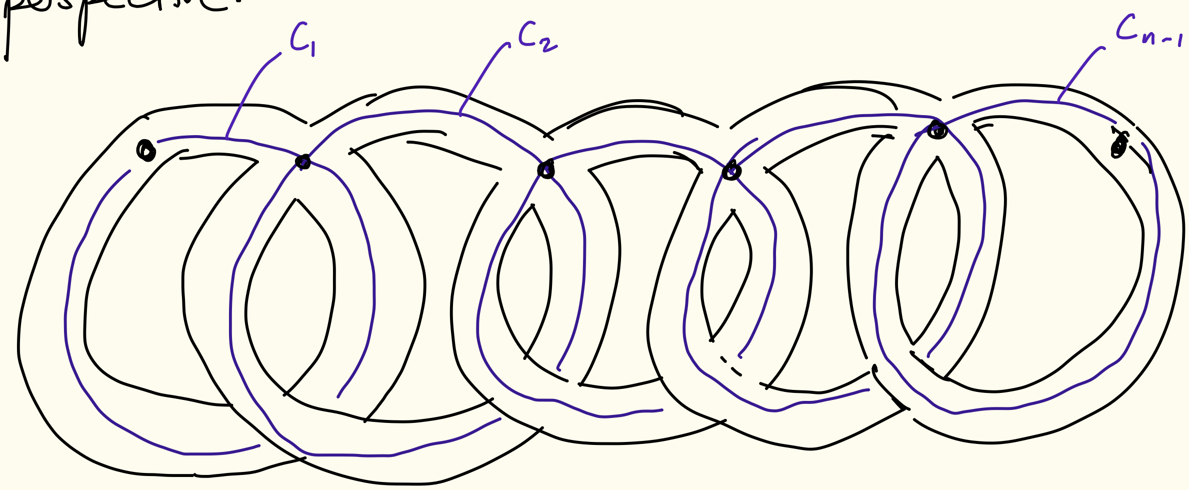
$$\Rightarrow \Sigma_f \cong \begin{cases} \Sigma_{g,1} & n = 2g+1 \\ \Sigma_{g,2} & n = 2g+2 \end{cases}$$

$$\rightsquigarrow Br_n \twoheadrightarrow \mathrm{Mod}(\mathbb{C}, \{x_1, \dots, x_n\}) \longrightarrow \mathrm{Mod}(\Sigma_f)$$

$$\circlearrowright$$

$$H_1(\Sigma_f) \cong \mathbb{Z}^{n-1}$$

another perspective:

$\Sigma_f \cong$ 



$\circlearrowleft$

$Br_n$ w/ $\sigma_i$ acts via Dehn twist about $c_i$.

$\{c_1, \dots, c_{n-1}\}$ basis of $H_1(\Sigma_f) \rightsquigarrow$

$$\sigma_i \longmapsto \begin{pmatrix} I_{i-2} & & \\ & \boxed{\begin{matrix} 1 & & \\ -1 & 1 & 1 \\ & & 1 \end{matrix}} & \\ & & I_{n-i-2} \end{pmatrix}$$

(red)

<u>Burau representation</u>: $Br_n \circlearrowright \mathbb{Z}[t,t^{-1}]^{n-1}$

$$\sigma_i \longmapsto \begin{pmatrix} I_{i-1} & & \\ & \boxed{\begin{matrix} 1 & & \\ t & -t & 1 \\ & & 1 \end{matrix}} & \\ & & I_{n-i-2} \end{pmatrix}$$

So $Br_n \xrightarrow{Bur} GL_{n-1}\mathbb{Z}[t,t^{-1}]^{n-1} \xrightarrow{t=-1} GL_{n-1}\mathbb{Z}$

isomorphic to $Br_n \circlearrowright H_1(\Sigma_f)$.

## ② Moments of L-functions

$$M_r(n) := \sum_{\substack{f \in \mathbb{F}_q[X] \\ \text{monic} \\ \text{square free} \\ \deg f = n}} L\left(\tfrac{1}{2}, \chi_f\right)^r$$

If $f$ is irreducible

$$\chi_f : \mathbb{F}_q[X] \longrightarrow \{0, -1, 1\}$$

$$g \longmapsto \begin{cases} 0 & g \text{ and } f \text{ are not coprime, else:} \\ 1 & g \equiv h^2 \pmod{f} \\ -1 & g \not\equiv h^2 \pmod{f} \end{cases}$$

If $f = f_1 \cdots f_k$ w/ $f_i$ irreducible then

$$\chi_f(g) = \chi_{f_1}(g) \cdots \chi_{f_k}(g)$$

$$L(s, \chi) = \sum_{\substack{g \in \mathbb{F}_q[X] \\ \text{monic}}} \frac{\chi(g)}{q^{s \cdot \deg g}}$$

## ③ Homological stability

Grothendieck-Lefschetz trace formula implies that

$$M_r(n) = q^n \sum_i (-1)^i \, tr\left(\mathrm{Frob}_q \circlearrowright H_i\left(Br_n ; \left(\Lambda^* H_1(\Sigma_f)\right)^{\otimes r}\right)\right)$$

Idea: Homological stability tells us about the asymptotics of $M_r(n)$ (for $n \to \infty$).

$$\left(\Lambda^* H_1(\Sigma_f)\right)^{\otimes r} \cong \bigoplus_\lambda V_\lambda(n)^{\oplus p_\lambda(n)}$$

$V_\lambda(n)$ irred $Br_n$-reps

$p_\lambda(n)$ polynomial for large $n$

$$\rightsquigarrow M_r(n) \approx \sum_{i,\lambda} (-1)^i q^{n-i} \dim H_i(Br_n ; V_\lambda(n)) \cdot p_\lambda(n)$$

Need: • $\dim H_i(Br_\infty ; V_\lambda(\infty))$ Bergström - Diaconu - Petersen - Westerland

• $H_i(Br_n ; V_\lambda(n)) \xrightarrow{\cong} H_i(Br_{n+1} ; V_\lambda(n+1))$

for $n >> i$ (independent of $\lambda$!)

Thm (Randal-Williams – Wahl 2017)

$$H_i(Br_n ; V_\lambda(n)) \xrightarrow{\cong} H_i(Br_{n+1} ; V_\lambda(n+1))$$

for $n \geq 2(i + |\lambda|)$ ← not independent of $\lambda$

Thm (Miller – P. – Petersen – Randal-Williams)
(not exactly)

Given:
- $G_n \twoheadrightarrow Q_n$
- $V_n$ $Q_n$-reps
- Homological stability $H_i(Q_n ; V_{n+k})$ $\forall k$
- Homological stability $H_i(G_n)$

Then: Homological stability $H_i(G_n ; V_{n+k})$ $\forall k$

Application to $H_k(Br_n ; V_\lambda(n))$:

- $Br_n \twoheadrightarrow \mathrm{im}(Bur(-1)) \approx Sp_{n-1}\mathbb{Z}$
- $V_\lambda(n)$ $Sp_{n-1}\mathbb{Z}$-rep
- $H_i(Sp_{n-1}\mathbb{Z} ; V_\lambda(n))$ stable uniform in $\lambda$ (Borel Stability)
- $H_i(Br_n)$ stable

$\Longrightarrow H_i(Br_n; V_\lambda(n))$ stable uniform in $\lambda$

Proof idea:

There is a spectral sequence (roughly like)

$$E^1_{pq} = H_q(G_{n-p}, G_{n-1-p}; V_n, V_{n-1})$$

$$\Longrightarrow H_{p+q}(G_n, G_{n-1}; V_n, V_{n-1}) = 0$$