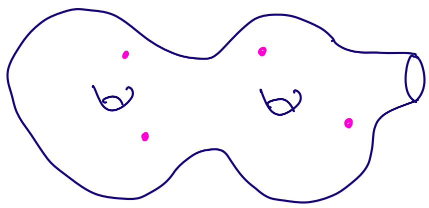
M : oriented, open manifold.

$q = \dim M \geq 2$.

(Unordered) Configuration spaces :

$$C_n(M) := \{(x_1,\dots,x_n) \mid x_i \in M,\ x_i \neq x_j\} / S_n$$



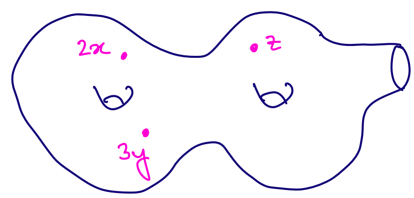
$C_n(M)$: manifold of dim nq

Symmetric Products :

$$A_n(M) := \{(x_1,\dots,x_n) : x_i \in M\} / S_n$$ (So the x_i need not be distinct!)

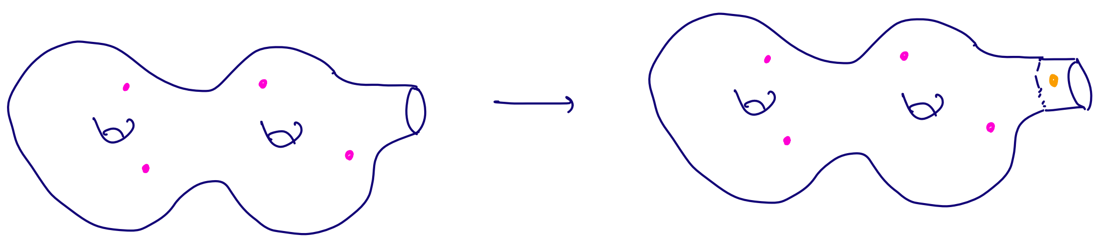
$A_n(M)$: • not necessarily a manifold

• $H^i_c(A_n(M)) = 0$ for $i > nq$



$\in A_6$

Stabilization Map : $C_n(M) \longrightarrow C_{n+1}(M)$



Goal : $C_n(M) \to C_{n+1}(M)$ is a H_*-equiv for $* \leq \lfloor \frac{n}{2} \rfloor$

Equivalently : $\mathbb{R}^q \times C_n(M) \xhookrightarrow{\text{open}} C_{n+1}(M)$ is a H_*-equiv for $* \leq \lfloor \frac{n}{2} \rfloor$

(†) **Dual Goal** : $\mathbb{R}^q \times C_n(M) \xhookrightarrow{\text{open}} C_{n+1}(M)$ is a H^*_c-equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$

will let us use LES for $U \underset{\text{open}}{\hookrightarrow} X$ in H^*_c

This talk : Prove (+) assuming the analogous statement for symmetric products $A_n(M)$.

(will state this explicitly in a moment)

Rmk : • $C_n(M)$ is not a homotopy invariant, but $A_n(M) = M^n/S_n$ is

(eg : $C_2(\mathbb{R}^d) \simeq S^{d-1}$)

• $A_n(\mathbb{C}) \cong \mathbb{C}^n$ sym. products more accessible than $C_n(M)$

(Warning : $H^*_c(A_n)$ is NOT htpy-invariant!)

Notation : Write $\{x, x, y, y, y, z\} \in A_6(M)$ as $2x + 3y + z$

We can write any $\xi \in A_n(M)$ (non-uniquely) as $\xi = 2T + U$ $\quad T \in A_{n-2k} \quad U \in A_k$

Eg: $\{x, x, y, y, y, z\} = \underset{T}{2(x)} + \underset{U}{(3y+z)}$

$= \underset{T}{2(x+y)} + \underset{U}{(y+z)}$

Define $A_{n,k} := \{ \xi = 2T + U \in A_n \mid |T| \geq k \}$

Eg : $\{x, x, y, y, y, z\} = \underset{T}{2(x)} + \underset{U}{(3y+z)} \in A_{n,1}$

$= \underset{T}{2(x+y)} + \underset{U}{(y+z)} \in A_{n,2} \subset A_{n,1}$

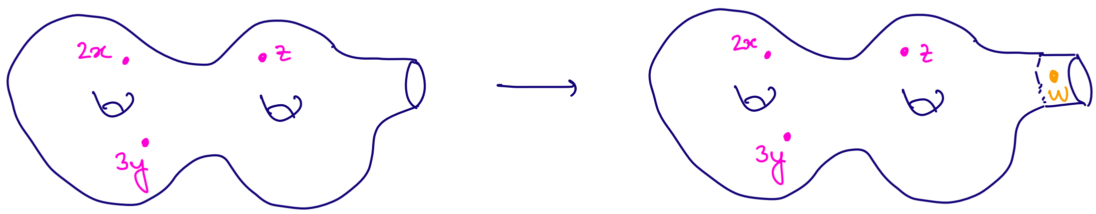
Note : $A_n = A_{n,0} \supset A_{n,1} \supset A_{n,2} \supset \cdots \supset A_{n,\lfloor \frac{n}{2} \rfloor} \supset A_{n,\lfloor \frac{n}{2} \rfloor + 1} = \phi$

"Arnol'd - Segal filtration"

$A_{n,k} - A_{n,k+1} = A_k \times C_{n-2k}$] basis for relating stability statement for C_n to that of A_n

Stabilization map for A_n :

$$\mathbb{R}^q \times A_n \xrightarrow{\text{open}} A_{n+1}$$



This is filtered : $\mathbb{R}^q \times A_{n,k} \longmapsto A_{n+1,k}$

LES for H_c^* :

$$A_{n,k} - A_{n,k+1} = A_k \times C_{n-2k}$$

$$A_k \times C_{n-2k} \overset{\text{open}}{\subset} A_{n,k}$$

$$\mathbb{R}^q \times A_{n,k} - \mathbb{R}^q \times A_{n,k+1} = \mathbb{R}^q \times A_k \times C_{n-2k}$$

Get :

$$\begin{tikzcd}
\cdots & H^i_c(\mathbb{R}^q \times A_{n,k+1}) \arrow[l] \arrow[d] & H^i_c(\mathbb{R}^q \times A_{n,k}) \arrow[l] \arrow[d] & H^i_c(A_k \times (\mathbb{R}^q \times C_{n-2k})) \arrow[l] \arrow[d] & H^{i-1}_c(\mathbb{R}^q \times A_{n,k+1}) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & H^i_c(A_{n+1,k+1}) \arrow[l] & H^i_c(A_{n+1,k}) \arrow[l] & H^i_c(A_k \times C_{n-2k+1}) \arrow[l] & H^{i-1}_c(A_{n+1,k+1}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Recall our goal :

Assuming that

$\mathbb{R}^q \times A_n \longrightarrow A_{n+1}$ is a H_c^*-equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$ $\forall n \geq 1$ (1)

Prove that

$\mathbb{R}^q \times C_n \longrightarrow C_{n+1}$ is a H_c^*-equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$ $\forall n \geq 1$ (2)

<u>Note</u>: (2) holds for $n = 1$

$$\mathbb{R}^q \times C_1(M) \cong \mathbb{R}^q \times M \quad , \text{dim } 2q$$

$$(n+1)q - \lfloor \tfrac{n}{2} \rfloor = 2q - 0 = 2q$$

So we'll induct on n.

<u>Lemma</u>: For $k > 0$, $\mathbb{R}^q \times A_{n,k} \longrightarrow A_{n+1,k}$ is a H_c^* - equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$

<u>Pf</u>: True for $k > \lfloor \frac{n+1}{2} \rfloor$, since in that case $A_{n,k} = \phi = A_{n+1,k}$.

So will downwards induct on k.

$$\begin{tikzcd}
\cdots & H_c^i(\mathbb{R}^q \times A_{n,k+1}) \arrow[l] \arrow[d, "(**)"] & H_c^i(\mathbb{R}^q \times A_{n,k}) \arrow[l] \arrow[d, "\substack{\text{Want} \\ \cong}"] & H_c^i(A_k \times (\mathbb{R}^q \times C_{n-2k})) \arrow[l] \arrow[d, "(*)"] & H_c^{i-1}(\mathbb{R}^q \times A_{n,k+1}) \arrow[l] \arrow[d, "(***)"] & \cdots \arrow[l] \\
\cdots & H_c^i(A_{n+1,k+1}) \arrow[l] & H_c^i(A_{n+1,k}) \arrow[l] & H_c^i(A_k \times C_{n-2k+1}) \arrow[l] & H_c^{i-1}(A_{n+1,k+1}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

(*) : $\cong$ by induction on n, since $n - 2k < n$

(**), (***) : $\cong$ by (downwards) induction on k

$\Longrightarrow$ $\cong$ in desired range

Caveat : We actually used the fact that

$$A_k \times \mathbb{R}^q \times C_{n-2k} \longrightarrow A_k \times C_{n-2k+1}$$

is H_c^* - iso for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$

instead of

$$\mathbb{R}^q \times C_{n-2k} \longrightarrow C_{n-2k+1}$$

This can be easily shown using the Künneth formula — will postpone to end.

For now, we have :

Lemma : For $k > 0$, $\mathbb{R}^q \times A_{n,k} \longrightarrow A_{n+1,k}$ is a H_c^* - equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$

Set $k=0$ in LES (and recall $A_n = A_{n,0}$) :

$$\begin{tikzcd}[column sep=small]
\cdots \leftarrow H_c^i(\mathbb{R}^q \times A_{n,1}) \arrow[d, "*"] & H_c^i(\mathbb{R}^q \times A_n) \arrow[l] \arrow[d, "(2)"] & H_c^i(A_0 \times (\mathbb{R}^q \times C_n)) \arrow[l] \arrow[d, "(1)"] & H_c^{i-1}(\mathbb{R}^q \times A_{n,1}) \leftarrow \cdots \arrow[l] \arrow[d, "**"] \\
\cdots \leftarrow H_c^i(A_{n+1,1}) & H_c^i(A_{n+1}) \arrow[l] & H_c^i(A_0 \times C_{n+1}) \arrow[l] & H_c^{i-1}(A_{n+1,1}) \leftarrow \cdots \arrow[l]
\end{tikzcd}$$

$*, **$: H_c^* - equiv in desired range by Lemma

(2) : H_c^* - equiv in desired range by assumption

$\Longrightarrow (1)$: H_c^* - equiv in desired range

Künneth Claim :

Suppose $Y \to Z$ is a H^*_c-equiv for $* \geq m$ and $H^*_c(X) = 0$ for $* > d$.

Then $X \times Y \to X \times Z$ is a H^*_c-equiv for $* \geq d+m$.

Pf: Künneth : $H^i_c(X \times Y) \cong \bigoplus_{a+b=i} H^a_c(X) \otimes H^b_c(Y)$

Can assume $a \leq d$ since $H^a_c(X) = 0$ o/w.

If $i \geq d+m$, this forces $b = i-a \geq m$.

Apply to : $X = A_k$

$Y = \mathbb{R}^q \times C_{n-2k}$, $Z = C_{n-2k+1}$

$\rightsquigarrow$ $A_k \times \mathbb{R}^q \times C_{n-2k} \longrightarrow A_k \times C_{n-2k+1}$

is H^*_c-equiv for $* \geq (kq) + \left((n-2k+1)q - \lfloor \frac{n-2k}{2} \rfloor\right)$

$$= \underbrace{(n+1)q - kq - \lfloor \tfrac{n}{2} \rfloor + k}_{< (n+1)q - \lfloor \frac{n}{2} \rfloor}$$

$\Longrightarrow$ H^*_c-equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$