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Michigan 2026

<u>Def</u>

Let X be a space. Let

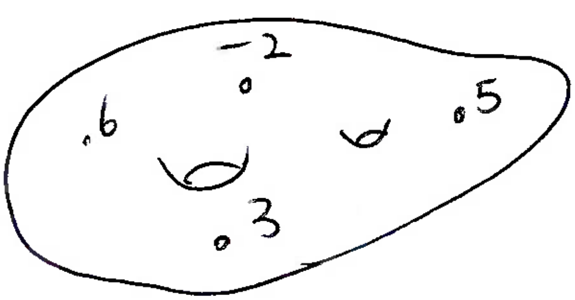
$$\mathbb{Z}X = \left(\coprod_n X^n \times \mathbb{Z}^n\right)/\sim$$

$$(x_1, x_2, \dots; n_1, \dots) \sim (x_{\sigma(1)}, x_{\sigma(2)} \dots; n_{\sigma(1)} \dots)$$

$\sigma \in S_n$

$$(x_1, x_2, x_3 \dots; n_1, n_2, \dots) \sim (x_1, x_3, \dots; n_1+n_2, n_3 \dots)$$

if $x_1 = x_2$.



$\in \mathbb{Z}M$ $\quad M =$ 

converges to

Thm (Dold-Thom / Dold-Kan)

$\pi_i(\mathbb{Z}X) = H_i(X)$.

Eilenberg-Steenrod axioms

F_i: Pairs of spaces $\to$ Ab $\qquad F_i(X,Y) \cong H_i(X,Y)$ if

$\partial: F_i(X,Y) \to F_{i-1}(Y)$

1) F_i are homotopy invariant
2) F_i satisfy excision
3) $F_i(pt) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{else} \end{cases}$
4) $F_i(\sqcup X_\alpha) = \bigoplus F_i(X_\alpha)$
5) LES:

$$F_i(Y) \to F_i(X) \to F_i(X,Y) \to F_{i-1}(Y) \to \dots$$

Let $F_i(X,Y) = \pi_i(\mathbb{Z}X/\mathbb{Z}Y)$

for $Y \subseteq X$.

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5) follows from showing

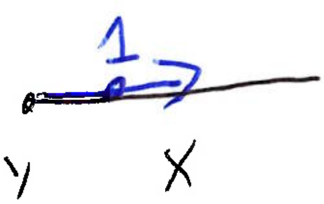
$\mathbb{Z}X \to \mathbb{Z}X / \mathbb{Z}Y$ is a fibration.

Ex

$X = [0,1]$

$Y = \{0\}$

~~AA~~ $h : [0,1] \to \mathbb{Z}X / \mathbb{Z}Y$



$\tilde{h}_0 =$

$\tilde{h} =$

Cor

$$\mathbb{Z} D^n / \mathbb{Z} S^{n-1} \simeq K(\mathbb{Z}, n)$$

Proof

$$\pi_i\, \mathbb{Z} D^n / \mathbb{Z} S^{n-1} = H_i(D^n, S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Thm

$$\pi_0\, \mathrm{Map}(X, K(A, n)) = H^n(X; A)$$

Cor

$$\pi_i\, \mathrm{Map}(X, K(A, n)) = H^{n-i}(X; A)$$

Proof

$$\pi_i\, \mathrm{Map}(X, K(A,n)) \cong$$

$$\pi_0\, \Omega^i\, \mathrm{Map}(X, K(A,n)) \cong$$

$$\pi_0\, \mathrm{Map}(X, \Omega^i K(A,n)) \cong$$

$$\pi_0\, \mathrm{Map}(X, K(A, n-i)) \cong$$

$$H^{n-i}(X; A)$$

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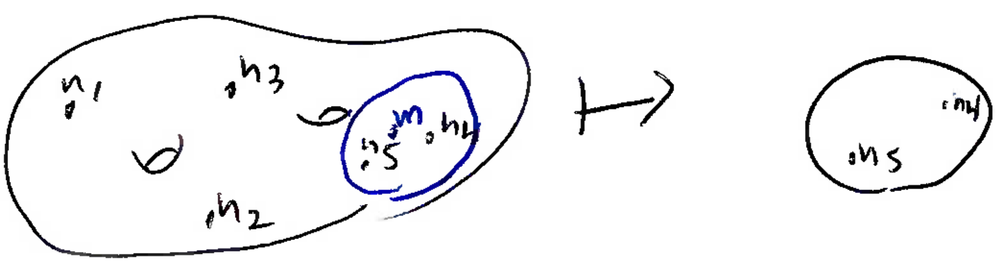
Observation

If M is a compact oriented n-manifolds

$$\pi_i \mathbb{Z}M = H_i(M) \cong H^{n-i}(M) = \pi_i(\mathrm{Map}(M, K(\mathbb{Z}, n)))$$
$$= \pi_i(\mathrm{Map}(M, \mathbb{Z}D^n/\mathbb{Z}S^{n-1}))$$

$$S : \mathbb{Z}M \to \mathrm{Map}(M, \mathbb{Z}D^n/\mathbb{Z}S^{n-1})$$

$$\underset{\sum n_i x_i}{\mathbb{Z}M} \times \underset{m}{M} \to \mathbb{Z}D^n/\mathbb{Z}S^{n-1}$$



Can prove $S : \mathbb{Z}M \xrightarrow{\simeq} \mathrm{Map}(M, \mathbb{Z}D^n/\mathbb{Z}S^{n-1})$ same way as for configuration spaces.

~~Def~~ ~~Remark~~

~~For A a commutative topological ab group, can define $H_i(X; A)$ =~~

(6)

Remark

For A a topological ab ~~group~~ monoid, can define AX. For A discrete group

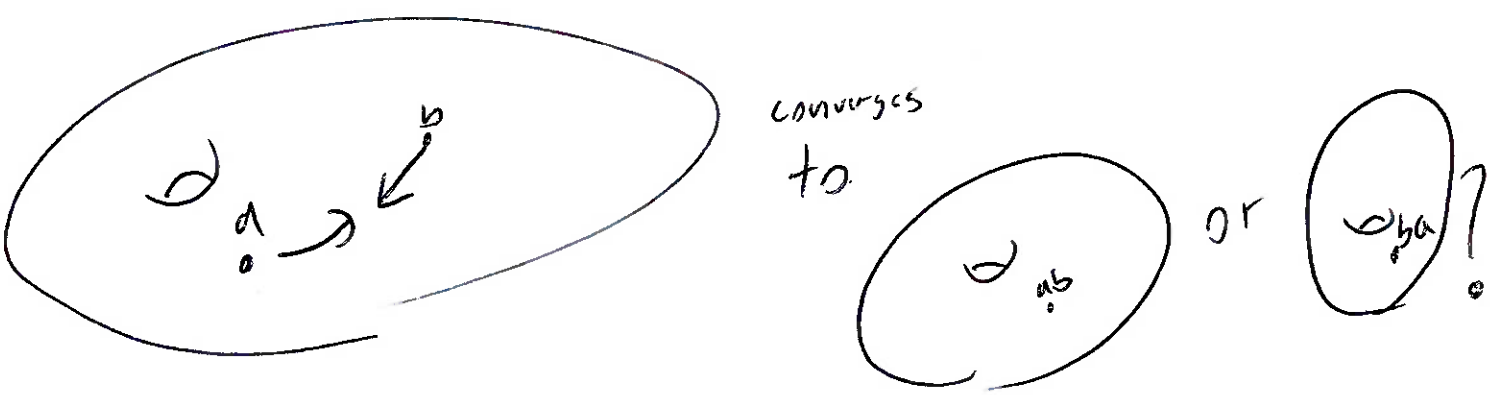
$$\pi_i(AX) \cong H_i(X; A).$$

Ex

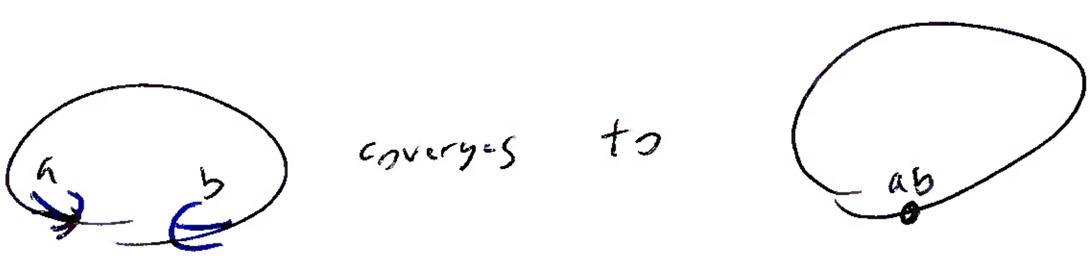
$$H_i(X; S^1) = H_{i-1}(X; \mathbb{Z}).$$

Question

What about A non commutative?



Dim $M = 1$



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For A any monoids $\dim M = 1$, can define AM. For $M = S^1$, related to Hochschild homology

Interpolating between Associative and Commutative

__Def__

Emb^n = category of n topological smooth manifolds and smooth embeddings.

(sym monoidal with disjoint union)

$\mathrm{Disk}^n \subseteq \mathrm{Emb}^n$ subcategory on objects $\bigsqcup_n \mathbb{D}^n$ open n disk.

A Disk^n-algebra A is a strong monoidal functor

$$A : \mathrm{Disk}^n \to \mathrm{Top}$$

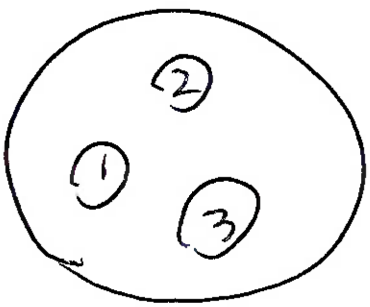
(8) conflate $A(\mathbb{D}^n)$ and A

$$A\Big(\coprod_k \mathbb{D}^n\Big) = A(\mathbb{D}^n)^k = A^k$$

$$\mathrm{Emb}\Big(\coprod_k \mathbb{D}^n, \mathbb{D}^n\Big) \times A^k \to A$$

Ex

$A = \mathrm{Conf}(\mathbb{D})$

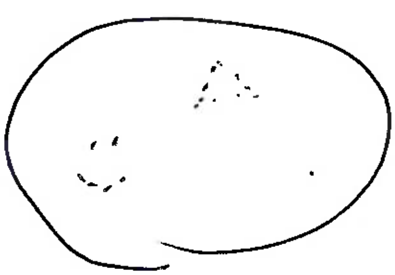


$\in \mathrm{Emb}\big(\coprod_3 \mathbb{D}^2, \mathbb{D}^2\big)$

$\times$

$\in \mathrm{Conf}(\mathbb{D})$

$\downarrow$



Ex

$A = \mathbb{Z}$

$\times (7, -10) \mapsto -3$

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Def

For A an n-disk alg

let $AM = \tilde{A}M$

Homotopy left Kan extension $\tilde{A}$

$$\begin{tikzcd} \mathrm{Emb}^n \arrow[r, "\tilde{A}"] & \mathrm{Top} \\ \mathrm{Disk}^n \arrow[u] \arrow[ur, "A"'] & \end{tikzcd}$$

Ex $A = \mathrm{Conf}(\mathbb{D})$

$AM = \mathrm{Conf}(M)$

Ex $A = \mathbb{Z}$

$\mathbb{Z}M = \mathbb{Z}M$

Thm (Thurston, Salvatore, Lurie)

$$AM \simeq \Gamma_c\left(\begin{matrix} E^A \\ \downarrow \\ M \end{matrix}\right) \text{ if } \pi_0 A \text{ is a group.}$$

E^A a bundle with fibers AD^n / AS^{n-1}.

Im/ Sander and I proved scanning and stability if $\pi_0 A$ is not a group.