CONFIGURATION SPACES SUMMER SCHOOL
University of Michigan

A brief introduction to representation stability

Rita Jiménez Rolland

INSTITUTO DE MATEMÁTICAS, UNAM - OAXACA

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Our question today

Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ of topological spaces or discrete groups

How does the homology and cohomology groups of X_n change as the parameter n increases?

- Braid groups

  'full' $\{B_n\}_n$ and *pure* $\{P_n\}_n$

- Configuration spaces

  unordered $\{\mathrm{C}_n(M)\}_n$ and *ordered* $\{\mathrm{F}_n(M)\}_n$

- Mapping class groups

  'full' $\{\mathrm{Mod}^n(\Sigma)\}_n$ and *pure* $\{\mathrm{PMod}^n(\Sigma)\}_n$

Sequences of vector spaces

$\mathbb{Q}$**-coefficients:** X_n topological space or discrete group $\rightsquigarrow$ $V_n := H^k(X_n; \mathbb{Q})$ vector space over $\mathbb{Q}$

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How do the vector spaces V_n change as n grows?

If the vector spaces V_n are defined over $\mathbb{Q}$ and have finite dimension

First answer: Understand how $\dim_{\mathbb{Q}} V_n$ changes when n grows

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If the sequences of spaces $\{V_n\}_n$ of interest have "symmetries"

Action of S_n in V_n: Group homomorphism $S_n \to \mathrm{GL}(V_n)$

Second answer: Take into account the symmetries.

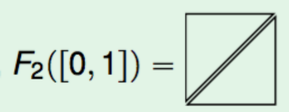
Ordered configuration spaces

M a Hausdorff topological space (eg. $[0, 1]$, $\mathbb{C}$, manifolds)

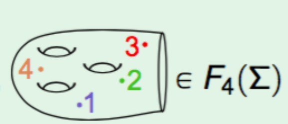
$\mathrm{F}_n(M)$– **configuration space** of n **ordered** points in M

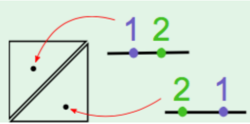
$$F_n(M) = \left\{(z_1, \ldots, z_n) : z_i \in M, \quad z_i \neq z_j \text{ for all } i \neq j\right\}$$

$\mathrm{F}_n(M) = M^n \setminus$ "fat diagonal"

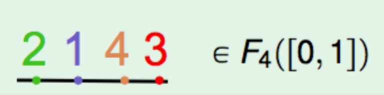


$$\mathrm{F}_n(M) = \Big\{\{1, 2, \ldots, n\} \hookrightarrow M\Big\}$$





$$\Big\{\text{connected comp. of } \mathrm{F}_n([0, 1])\Big\} \longleftrightarrow S_n$$

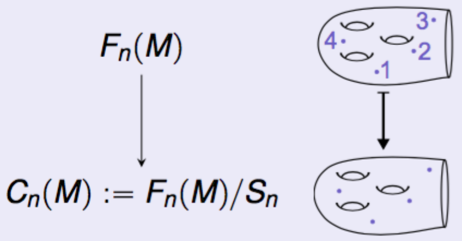


Unordered configuration spaces

$\mathrm{F}_n(M)$ has symmetries: S_n acts permuting the points

$$\sigma \cdot (z_1, \ldots, z_n) = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})$$

The action is free!



The **configuration space** of n **unordered** points in M:

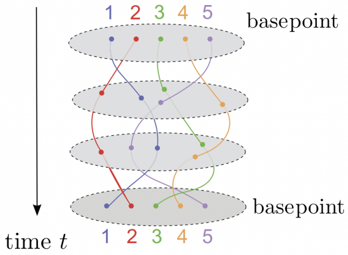
$$\mathrm{C}_n(M) = \{\{z_1, \ldots, z_n\} : z_i \in M, \quad z_i \neq z_j\}$$

Configurations on the plane and braid groups

$$F_n(\mathbb{C}) = \{(z_1, \ldots, z_n) : z_i \in \mathbb{C}, \quad z_i \neq z_j \text{ for all } i \neq j\}$$

$$C_n(\mathbb{C}) = F_n(\mathbb{C})/S_n = \{ \text{ sets of } n \text{ points in the plane } \mathbb{C}\}$$

are Eilenberg-MacLane spaces of type $K(\pi, 1)$



$$H^*(F_n(\mathbb{C}); \mathbb{Q}) \cong H^*(P_n; \mathbb{Q})$$

$$H^*(C_n(\mathbb{C}); \mathbb{Q}) \cong H^*(B_n; \mathbb{Q})$$

Loop of configurations in $F_n(\mathbb{C})$:
pure braid of n strands

$$\pi_1(F_n(\mathbb{C})) = \mathbf{P_n}$$

pure braid group

Loop of configurations in $C_n(\mathbb{C})$:
braid of n strands

$$\pi_1(C_n(\mathbb{C})) = \mathbf{B_n}$$

Artin braid group

$$1 \to P_n \to B_n \to S_n \to 1$$

Cohomology of Artin braid groups

$$H^*(\mathrm{C}_n(\mathbb{C}); \mathbb{Q}) \cong H^*(B_n; \mathbb{Q})$$

$$H^1(B_n; \mathbb{Q}) \cong \mathrm{Hom}(B_n, \mathbb{Q}) \cong \mathbb{Q} = \langle f \rangle$$

$f\big(\text{braid}\big) :=$ total index of the braid

 $\longmapsto$?

Cohomology of Artin braid groups

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$\longmapsto$?

How do the vector spaces $H^1(B_n;\mathbb{Q})$ change when n grows?

$$\dim_{\mathbb{Q}} H^1(B_n;\mathbb{Q}) = 1 \quad \text{si } n \geq 2$$

It does NOT depend of n!

Cohomology of Artin braid groups

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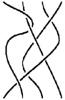

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It does NOT depend of n!

A sequence $\{X_n\}_{n\in\mathbb{N}}$ of discrete groups or spaces satisfies **homological stability** (over a ring R) if for every $k \geq 0$

$$H_k(X_n;R) \cong H_k(X_{n+1};R) \qquad \text{when} \quad n \gg k.$$

With rational coefficients,

$$\dim_{\mathbb{Q}} H_k(X_n,\mathbb{Q}) = \text{constant} \qquad \text{when} \quad n \gg k.$$

(Arnol'd 68, F. Cohen 72) The sequence of braid groups $\{B_n\}_n$ satisfies **(co)homological stability** over $\mathbb{Z}$

First cohomology of pure braid groups

$$H^*(\mathrm{F}_n(\mathbb{C});\mathbb{Q}) \cong H^*(P_n;\mathbb{Q})$$

$$H^1(P_1;\mathbb{Q}) = 0 \qquad H^1(P_2;\mathbb{Q}) = \mathbb{Q} \qquad H^1(P_3;\mathbb{Q}) = \mathbb{Q}^3$$

$$H^1(P_n;\mathbb{Q}) \cong \mathrm{Hom}(P_n;\mathbb{Q})$$

$\omega_{\mathrm{i},\mathrm{j}}$: (pure braid) $\longmapsto$ (# times the string i wraps around the string j)

Theorem (Arnol'd 1968) If $n \geq 2$, the homomorphisms $\omega_{i,j}$ give a basis for $H^1(P_n;\mathbb{Q})$ i.e.

$$H^1(P_n;\mathbb{Q}) = \mathbb{Q}^{\binom{n}{2}}$$

First cohomology of pure braid groups

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$$H^1(P_n;\mathbb{Q}) = \mathbb{Q}^{\binom{n}{2}}$$

$\dim_{\mathbb{Q}} H^1(P_n;\mathbb{Q}) = \frac{n(n-1)}{2} \to \infty$ when $n \to \infty$

There is NOT (co)homological stability!!

Instability: why?

There are symmetries!

$$S_n \curvearrowright \mathrm{F}_n(\mathbb{C}) \quad \rightsquigarrow \quad S_n \curvearrowright H^k(\mathrm{F}_n(\mathbb{C});\mathbb{Q}) = H^k(P_n;\mathbb{Q})$$

$S_n \curvearrowright H^1(P_n;\mathbb{Q}) = \mathrm{Span}_{\mathbb{Q}}\langle \omega_{i,j} : 1 \leq i < j \leq n\rangle$ by $\sigma \cdot \omega_{i,j} = \omega_{\sigma(i).\sigma(j)}$

an we have,

$$\left\{H^k(P_n;\mathbb{Q})\right\}_n \textbf{ is a sequence of } S_n\textbf{-representations}$$

Fact: If V is a $\mathbb{Q}$-vector space with a non-trivial (and non-sign) action of S_n ,

then $\dim_{\mathbb{Q}}(V) \geq n-1$

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$$\text{then } \dim_{\mathbb{Q}}(V) \geq n-1$$

Non-trivial symmetries $\Longrightarrow$ there is NO (co)homological stability

S_n-representations

Groups as people are known by their actions

(Guillermo Moreno)

- *representation:* $\mathbb{Q}$-vector space with a lineal action of S_n

$$\text{Vector space } V \text{ over } \mathbb{Q}$$
$$+$$
$$\text{group homomorphism } \phi : S_n \to GL(V)$$

- *subrepresentation:*

$W \subseteq V$ subspace invariant under the action of S_n

"Obvious" subrepresentations: $0 \subseteq V$ and $V \subseteq V$

- *irreducible representation*:

V **does not** have subrepresentations different from 0 and V

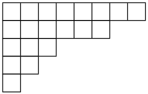
S_n-representations

Maschke Theorem. Any S_n-representation of finite dimension decomposes over $\mathbb{Q}$ in direct sum of irreducibles.

Young. The irreducible S_n-representations are indexed by partitions of n.

Each *partition of n* can be represented by a Young diagram.

For example,



$$\lambda = (8, 6, 3, 2, 1)$$

corresponds to a partition

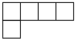
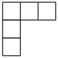
$$n = 8 + 6 + 3 + 2 + 1 = 20$$

Comparing representations

How to compare irreducibles for different values of n?

Two irreducibles are "the same"

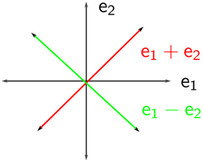
if they only differ in the first row of their Young diagrams.

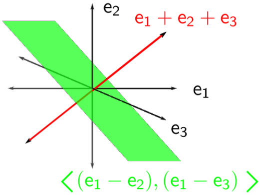
irreducible S_n-representations	**Partitions of n**	**Uniform notation**
Trivial representation $\mathbb{Q}$	$n = n + 0$	V  $\ldots$
Standard rep $\mathbb{Q}^n/\mathbb{Q}$	$n = (n-1) + 1$	V  $\ldots$
$\bigwedge^2$(Standar rep)	$n = (n-2) + 1 + 1$	V  $\ldots$

Example: permutation representation

$V_n = \mathbb{Q}^n = \langle e_1, e_2, \ldots, e_n \rangle$ with the action $\sigma(e_i) = e_{\sigma(i)}$ for $\sigma \in S_n$ decomposes in **two** invariant subspaces:

$$\mathbb{Q}^n = \{a(e_1 + e_2 + \ldots + e_n)\} \oplus \{a_1 e_1 + a_2 e_2 + \ldots a_n e_n : \sum a_i = 0\}$$





$V_1 = \mathbb{Q}^1 = V_{\square}$

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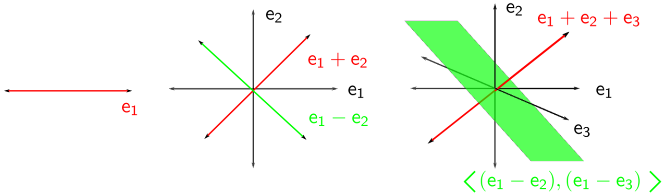

$V_1 = \mathbb{Q}^1 = V_{\square}$
$V_2 = \mathbb{Q}^2 = V_{\square\square} \oplus V_{\substack{\square\\\square}}$

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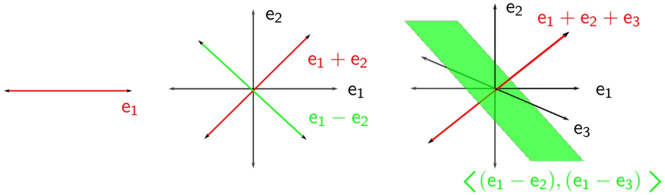

$V_1 = \mathbb{Q}^1 = V_{\ydiagram{1}}$

$V_2 = \mathbb{Q}^2 = V_{\ydiagram{2}} \oplus V_{\ydiagram{1,1}}$

$V_3 = \mathbb{Q}^3 = V_{\ydiagram{3}} \oplus V_{\ydiagram{2,1}}$

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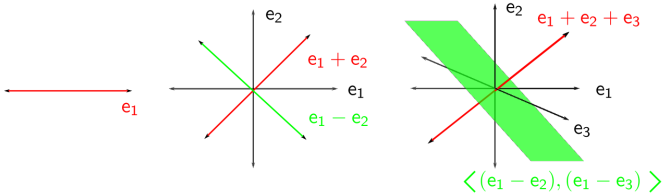

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$$V_4 = \mathbb{Q}^4 = V_{\yng(4)} \oplus V_{\yng(3,1)}$$

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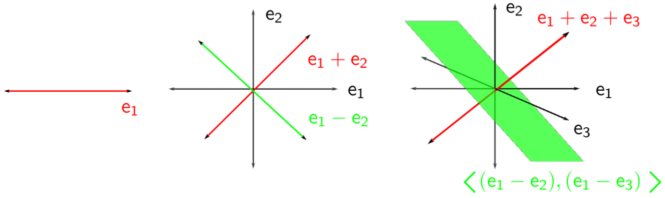

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$$V_n = \mathbb{Q}^n = V_{\square\square\square\square} \ldots \oplus V_{\begin{smallmatrix}\square\square\square\\ \square\end{smallmatrix}} \ldots$$

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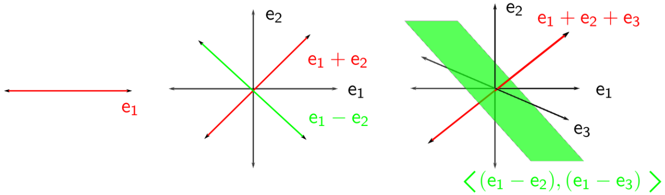

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$V_4 = \mathbb{Q}^4 = V_{\ydiagram{4}} \oplus V_{\ydiagram{3,1}}$

$V_n = \mathbb{Q}^n = V_{\ydiagram{4}\ldots} \oplus V_{\ydiagram{3,1}\ldots}$

For $n \geq 2$ we can find the decomposition of V_{n+1} from the decomposition of V_n *by "adding a box to the top row" of each Young diagram.*

First cohomology of pure braid groups as S_n-representations

$S_n \curvearrowright H^1(P_n;\mathbb{Q}) = \mathrm{Span}_{\mathbb{Q}}\langle \omega_{i,j} : 1 \le i < j \le n\rangle$ by $\sigma \cdot \omega_{i,j} = \omega_{\sigma(i).\sigma(j)}$

$$
\begin{array}{lcllll}
H^1(P_2;\mathbb{Q}) & \cong & V_{\ydiagram{2}} & & & \\
H^1(P_3;\mathbb{Q}) & \cong & V_{\ydiagram{3}} & \oplus & V_{\ydiagram{2,1}} & \\
H^1(P_4;\mathbb{Q}) & \cong & V_{\ydiagram{4}} & \oplus & V_{\ydiagram{3,1}} & \oplus \; V_{\ydiagram{2,2}} \\
H^1(P_5;\mathbb{Q}) & \cong & V_{\ydiagram{5}} & \oplus & V_{\ydiagram{4,1}} & \oplus \; V_{\ydiagram{3,2}} \\
H^1(P_6;\mathbb{Q}) & \cong & V_{\ydiagram{6}} & \oplus & V_{\ydiagram{5,1}} & \oplus \; V_{\ydiagram{4,2}} \\
H^1(P_7;\mathbb{Q}) & \cong & V_{\ydiagram{7}} & \oplus & V_{\ydiagram{6,1}} & \oplus \; V_{\ydiagram{5,2}}
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\end{array}$$

How do the vector spaces $H^1(P_n;\mathbb{Q})$ change when n grows?

For all $n \geq 4$ this pattern continues: *can obtain one row of the decomposition from the previous by "adding a box to the top row" of each Young diagram*

$$H^1(P_n;\mathbb{Q}) = V_{\yng(4)} \dots \oplus V_{\yng(3,1)} \dots \oplus V_{\yng(2,2)} \dots$$

Multiplicity stability

Definition (2010): Grosso modo...

A sequence $\{V_n\}_n$ of S_n-rational representations satisfies
multiplicity stability
if the decomposition of V_n into irreducible representations is eventually independent of n.

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How do the vector spaces $H^k(P_n; \mathbb{Q})$ change when n grows?

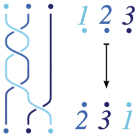
Theorem (Church–Farb, 2013) For each $k \geq 0$,
the sequence of S_n-representations

$\{H^k(P_n; \mathbb{Q})\}_n$ satisfies *multiplicity stability*

stabilizing for $n \geq 4k$.

- In general, decomposition into irreps is not known explicitly
- Church-Farb's first proof was combinatorial.

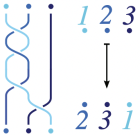
Multiplicity stability: a consequence for braid groups



$$1 \to P_n \to B_n \to S_n \to 1 \rightsquigarrow S_n \curvearrowright H^q(P_n;\mathbb{Q})$$

$$\begin{aligned} \dim_{\mathbb{Q}} H^k(B_n;\mathbb{Q}) &\cong \dim_{\mathbb{Q}} H^k(P_n;\mathbb{Q})^{S_n} \\ &= \text{multiplicity of the trivial rep} \\ &\quad \text{in the decomposition} \end{aligned}$$

Multiplicity stability: a consequence for braid groups



$$1 \to P_n \to B_n \to S_n \to 1 \rightsquigarrow S_n \curvearrowright H^q(P_n;\mathbb{Q})$$

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$$H^1(P_n;\mathbb{Q}) = V_{\ydiagram{4}}\dots \oplus V_{\ydiagram{3,1}}\dots \oplus V_{\ydiagram{2,2}}\dots \qquad \text{for } n \geq 4$$

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$$\vdots$$

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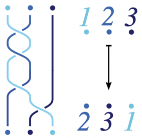

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$$\vdots$$

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Multiplicity stability for $H^k(P_n;\mathbb{Q})$ $\Rightarrow$ **Homological stability over $\mathbb{Q}$ for $H^k(B_n;\mathbb{Q})$**

Multiplicity stability: further examples

There are **multiplicity stability** patterns in the (co)homology of

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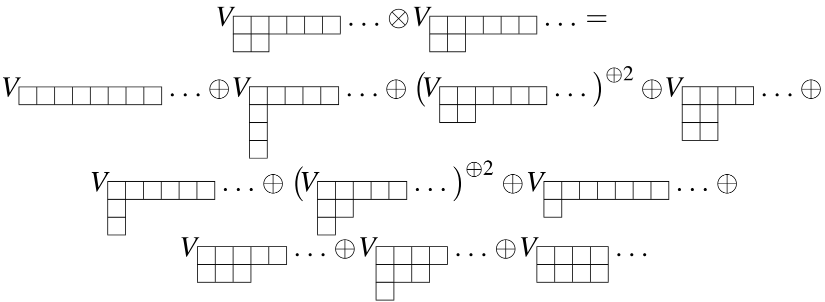
imply **rational homological stability** for

- Artin braid groups B_n (Arnold 198, Cohen 1972)
- unordered configuration spaces $\mathrm{C}_n(M)$ of manifolds of dim≥ 2:
  - (McDuff 1975, Segal 1979) for connected open manifolds
  - (Church 2012, Randal-Williams 2013) for closed manifolds
- 'full' mapping class groups $\mathrm{Mod}^n(\Sigma)$ "by punctures" (Handbury 2009; Hatcher–Wahl 2010)

Multiplicity stability: further examples

When $n \geq 8$



Theorem (Murnaghan 1938) In the decomposition

$$V(\lambda[n]) \otimes V(\mu[n]) = \bigoplus_{\nu[n]} g_{\lambda[n],\mu[n]}^{\nu[n]} V(\nu[n])$$

the **Kronecker coefficients** $g_{\lambda[n],\mu[n]}^{\nu[n]}$ are eventually constant.

$\left\{V(\lambda[n]) \otimes V(\mu[n])\right\}_n$ satisfies multiplicity stability

Underlying structure: FI-modules

What underlying structure is driving the multiplicity stability patterns?

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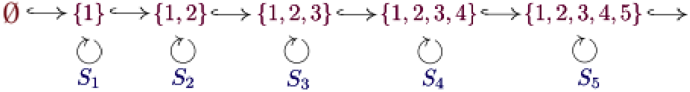
CHURCH–ELLENBERG–FARB: these sequences of S_n-representations are *finitely generated* FI-*modules*

Category FI: objects are finite sets and morphisms are injective maps.

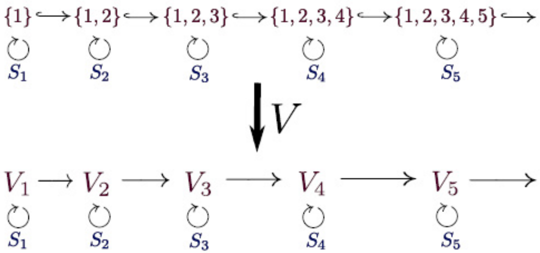


R a commutative unital ring.

An **FI-module (over** R**)** is a functor $V : \mathrm{FI} \to R\text{-Mod}$ from FI to the category of R-modules

The category of FI-modules

The data of an FI-module is a sequence of S_n-representations with equivariant maps.



Category of FI-modules over *R*: objects are FI-modules over R and morphisms are natural transformations between FI-modules.

It is an *abelian category*:
there are well-defined notions of submodules, quotients, kernels, cokernels, direct sums, tensor products.

Examples of FI-modules

Examples:
The following sequences of S_n-representations, along with the obvious inclusions, assemble to form FI-modules over $\mathbb{Q}$

- $V_n = \mathbb{Q}$ (trivial S_n-representations), with isomorphisms
- $V_n = \mathbb{Q}^n$ (canonical permutation representations)
- $V_n = \mathbb{Q}[x_1, \ldots, x_n]$ (diagonal action of S_n on monomials by permuting indices)
- $V_n = H^k(P_n; \mathbb{Q})$, "forget the (i)th strand" induced morphisms
- $V_n = H^k(\mathrm{F}_n(M); \mathbb{Q})$, "forget the (i)th point" induced morphisms
- $V_n = H^k(\mathrm{PMod}^n(\Sigma); \mathbb{Q})$, with "forget the (i)th puncture" induced morphisms

Non-examples:

- $V_n = \mathbb{Q}$ (alternating S_n-representation), with isomorphisms
- $V_n = \mathbb{Q}[S_n]$ (regular S_n-representations), with maps induced by inclusions $S_n \hookrightarrow S_{n+1}$ of groups

Generators of FI-modules

- An FI-module $V = \{V_n\}$ is **generated by a set** $S \subset \bigsqcup_{n \geq 0} V_n$ if V is the smallest FI-submodule containing S.
- An FI-module $V = \{V_n\}$ is generated in degree $\leq d$ if V is generated by the set $\bigsqcup_{0 \leq n \leq d} V_n$. If $d < \infty$, we say that V **is generated in finite degree**.
- If V is generated by a finite set S we say that V **is finitely generated**.

Finite generation is preserved when taking finite sums, tensor products and extensions.

Theorem (Noetherian property). Let R be a commutative Noetherian ring. Then any submodule of a finitely generated FI-module over R is itself finitely generated.

Cohomology of pure braid groups: again!

For $k \geq 0$, the sequence $\{H^k(P_n;\mathbb{Q})\}_n$ assemble to form
an FI-module $H^k(P_\bullet;\mathbb{Q})$ over $\mathbb{Q}$

Theorem (Arnold 1969). The cohomology algebra $H^*(P_n;\mathbb{Q})$ can be described as a certain quotient of the exterior algebra on the symbols $\omega_{i,j} \in H^1(P_n;\mathbb{Q})$

$$H^*(P_n;\mathbb{Q}) \cong \bigwedge \langle \omega_{i,j} \rangle / \sim$$

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$$\omega_{i,j} = \omega_{j,i},\ i \neq j,\ i,j \in [n].$$

- The vector spaces $H^1(P_n;\mathbb{Q})$ are spanned by monomials $\omega_{i,j}$.
  $\Rightarrow$ the FI-module $H^1(P_\bullet;\mathbb{Q})$ is generated by $\omega_{1,2} \in H^1(P_2;\mathbb{Q})$
- The FI-module $H^2(P_\bullet;\mathbb{Q})$ is generated by
  $\omega_{1,2} \wedge \omega_{2,3} \in H^2(P_3;\mathbb{Q})$ and $\omega_{1,2} \wedge \omega_{3,4} \in H^2(P_4,;\mathbb{Q})$

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- For $q \geq 0$,
  $H^k(P_\bullet;\mathbb{Q})$ is a finitely generated FI-module in degree $\leq 2k$.

Consequences of finite generation

Theorem (Church–Ellenberg–Farb 2014). Let V be an FI-module over $\mathbb{Q}$ that is finitely generated in degree $\leq d$. Then the following hold:

- **(Multiplicity stability)** The decomposition of V_n into irreducible representations is independent of n for all n sufficiently large.
- **(Polynomial dimension growth)** $\exists$ a polynomial p of degree $\leq d$ s.t.

$$\dim_{\mathbb{Q}}(V_n) = p(n) \text{ for } n \text{ sufficiently large}$$

- **(Polynomial characters)** $\exists$ a character polynomial P s.t.

$$\chi_{V_n}(\sigma) = P(\sigma) \text{ for all } \sigma \in S_n \text{ and all } n \text{ sufficiently large.}$$

- **(Stable inner products)** If Q is any character polynomial, then $\langle \chi_{V_n}, Q \rangle_{S_n}$ is independent of n for all n sufficiently large.
- **(Finite presentability)** V is finitely presentable as an FI-module.

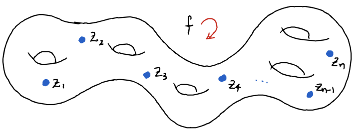
Representation stability and FI-modules

sequence $\{V_n\}_n$ of S_n-representations	Functor $V : \mathrm{FI} \longrightarrow \mathbb{Q} - \mathit{Vect}$
multiplicity stability	Finite generation Noetherian property
Murnaghan Stability	V, W f.g. $\Rightarrow V \otimes W$ f.g.
inductive description	presented in finite degrees

Mapping class groups

Σ compact surface and $z_1, z_2, \ldots, z_n$ a set of n distinct points in $\mathring{\Sigma}$
the punctures or *marked points*

Mapping class group
$\mathrm{Mod}^n(\Sigma) :=$ Group of isotopy classes of (orientation-preserving) diffeomorphisms $f : \Sigma \to \Sigma$, with $f|_{\partial\Sigma} = \mathrm{id}_\Sigma$, that fix the set $\{z_i\}$



$\mathrm{Mod}^n(\Sigma) \curvearrowright \{z_1, z_2, \ldots, z_n\}$

$$1 \to \mathrm{PMod}^n(\Sigma) \to \mathrm{Mod}^n(\Sigma) \to S_n \to 1$$

with kernel the
pure mapping class group $\mathrm{PMod}^n(\Sigma)$

Examples: $\mathrm{PMod}^1(\mathbb{T}^2) \cong \mathrm{SL}_2(\mathbb{Z})$
$\mathrm{Mod}^n(\mathbb{D}^2) \cong B_n$ *Artin braid group*
$\mathrm{PMod}^n(\mathbb{D}^2) \cong P_n$ *pure braid group*

Cohomology of pure mapping class groups

Let Σ be a surface such that:

- $\Sigma = \mathbb{S}^2, \mathbb{T}^2$ or $\Sigma = \Sigma_{g,r}$ with $2g + r > 2$, if Σ is orientable, or
- $\Sigma = \mathbb{R}P^2, \mathbb{K}$ or $\Sigma = N_{g,r}$ with $g \geq 3$ and $r \geq 0$ if Σ is non-orientable,

Theorem (J.R. 2019) Let A be any abelian group. For any $k \geq 0$ the sequence of cohomology groups

$$\{H^k\big(\mathrm{PMod}^n(\Sigma); A\big)\}_n$$

assembles into a a finitely generated FI-module $H^k\big(\mathrm{PMod}^\bullet(\Sigma); A\big)$.

$\Rightarrow$ The sequence $\{H^k\big(\mathrm{PMod}^n(\Sigma); \mathbb{Q}\big)\}_n$ satisfies **multiplicity stability** for n sufficiently large

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In particular, for $g \geq 2$, the sequence

$\{H^k(\mathcal{M}_{g,n}; \mathbb{Q}) \cong H^k(\mathrm{PMod}^n(\Sigma_g); \mathbb{Q})\}_n$ is multiplicity stable for $n \gg k$

Cohomology of pure mapping class groups

$$H^2(\mathrm{PMod}^n(\Sigma_g);\mathbb{Q}) = \mathbb{Q}^{n+1} = \langle \kappa_1, \psi_1, \psi_2 \, \ldots \psi_n \rangle \quad \text{for } g \geq 5, n \geq 1$$

The group S_n acts trivially on κ_1 and permutes the ψ_i-classes.

The FI-module $H^2(\mathrm{PMod}^\bullet(\Sigma);\mathbb{Q})$ is finitely generated by κ_1 and ψ_1 in degree 1

- **Multiplicity stability:**

$$\begin{array}{rlll}
H^2(\mathrm{PMod}^1(\Sigma_g);\mathbb{Q}) = \mathbb{Q}^2 = & V_{\ydiagram{1}} & \oplus\ V_{\ydiagram{1}} & \\
H^2(\mathrm{PMod}^2(\Sigma_g);\mathbb{Q}) = \mathbb{Q}^3 = & V_{\ydiagram{2}} & \oplus\ V_{\ydiagram{2}} & \oplus\ V_{\ydiagram{2}} \\
H^2(\mathrm{PMod}^3(\Sigma_g);\mathbb{Q}) = \mathbb{Q}^4 = & V_{\ydiagram{3}} & \oplus\ V_{\ydiagram{3}} & \oplus\ V_{\ydiagram{2,1}} \\
H^2(\mathrm{PMod}^n(\Sigma_g);\mathbb{Q}) = \mathbb{Q}^{n+1} = & V_{\ydiagram{4}\ldots} & \oplus\ V_{\ydiagram{4}\ldots} & \oplus\ V_{\ydiagram{3,1}\ldots}
\end{array}$$

- **Polynomial Betti numbers:** $\dim_{\mathbb{Q}} \left(H^2(\mathrm{PMod}^n(\Sigma_g);\mathbb{Q}) \right) = n + 1$
- **Polynomial characters:** $\chi_{H^2(\mathrm{PMod}^n(\Sigma_g);\mathbb{Q})} = Z_1 + 1$
- **Rational homological stability for $\{\mathrm{Mod}^n(\Sigma_g)\}_n$:**
  $\dim_{\mathbb{Q}} H^2(\mathrm{Mod}^n(\Sigma_g);\mathbb{Q}) = \text{multiplicity of } V_{\ydiagram{3}\ldots} = 2 \text{ for } n \geq 3$

In general, we do not have such a explicit description of $H^k(\mathrm{PMod}^n(\Sigma);\mathbb{Q})$ as a $\mathbb{Q}$-vector space nor as an S_n-representation

Cohomology of the classifying space of $\mathrm{PDiff}^n(M)$

Let M be a connected real manifold with (possibly empty) boundary of dimension $d \geq 2$ and of finite type and $z_1, z_2, \ldots, z_n$ a set of n distinct points in M

$\mathrm{PDiff}^n(M) =$ Group of diffeomorphisms $f : M \to M$ s.t. $f(z_i) = z_i$

Theorem (J. R. 2019) Let A be any abelian group. For $k \geq 0$, the cohomology groups

$$\{H^k(\mathrm{BPDiff}^n(M); A)\}_n$$

assemble to form an FI-module $H^k(\mathrm{BPDiff}^\bullet(M); A)$ which is presented in finite degrees

\+ extra finiteness conditions on M and $\mathrm{BDiff}(M) \Rightarrow$ finite generation

Remark: For $k \geq 0$ and Σ a surface as before the FI-module

$$H^k\big(\mathrm{BPDiff}^\bullet(\Sigma); A\big) = H^k\big(\mathrm{PMod}^\bullet(\Sigma); A\big)$$

Main ingredients of the proof

The inclusion $\mathrm{PDiff}^n(M) \hookrightarrow \mathrm{Diff}(M)$ induces

$$\mathrm{BPDiff}^n(M) \longrightarrow \mathrm{BDiff}(M)$$

a **fiber bundle** with fiber $\mathrm{Diff}(M)/\,\mathrm{PDiff}^n(M) \approx \mathrm{F}_n(\mathring{M})$

This gives us a functor from

$$\mathrm{FI}^{op} \to \mathrm{Fib}\left(\mathrm{BDiff}(M)\right)$$

and a cohomologically graded first quadrant **Serre spectral sequence** of FI-modules with E_2-term

$$E_2^{p,q} = H^p\left(\mathrm{BDiff}(M); H^q(\mathrm{F}_\bullet(\mathring{M}); A)\right)$$

converging to $H^{p+q}(\mathrm{BPDiff}^\bullet(M); A)$.

For $\Sigma = \Sigma_g^r$ with $2g + r > 2$ or $\Sigma = N_g^r$ with $g \geq 3$ we obtain the **Birman exact sequence**

$$1 \to \pi_1(\mathrm{F}_n(\mathring{\Sigma})) \to \mathrm{PMod}^n(\Sigma) \to \mathrm{Mod}(\Sigma) \to 1$$

Main ingredients of the proof

Spectral sequence of FI-modules

$$E_2^{p,q} = H^p\big(\mathrm{BDiff}(M); H^q(\mathrm{F}_\bullet(\mathring{M}); A)\big) \Rightarrow H^{p+q}(\mathrm{BPDiff}^\bullet(M); A)$$

- **(Church–Ellenberg–Farb, CEF–Nagpal 2014):**

  - Finite generation of FI-modules is preserved under extensions and subquotients

  - Over $\mathbb{Q}$ can keep track of weight and stability degree under spectral sequences

- **(CEF 2015; Miller–Wilson 2019):** For connected manifolds M of finite type, the FI-modules $H^q(\mathrm{F}_\bullet(M); \mathbb{Z})$ are presented in finite degs

- **(Church–Miller–Nagpal–Reinhold 2018)**:

  - Prove general results concerning spectral sequences of FI-modules that can be used to improve stable ranges

  - Obtain linear bounds for the presentation degree of $H^q(\mathrm{F}_\bullet(M); \mathbb{Z})$

Representation stability with the FI-module perspective

- This re-framing gives a conceptual explanation and easy-to-check criterion for multiplicity stability.
- The definition of a finitely generated FI-module makes sense for representations over the integers and other coefficients.
- The notion of a f.g. FI-module can be generalized to other groups or to maps with additional data and it makes sense even in situations where multiplicity stability is not well-defined.
- In this framework we can draw on the tools of commutative or homological algebra to study our sequences of representations.

Other categories

Encode a sequence of representations as a
representation of a category $\mathcal{C}$:
a functor from $\mathcal{C}$ to the category of vector spaces.

Name	Definition	Application
$\mathbf{FI}$	finite sets / injections	Cohomology of configuration spaces [1]
$\mathbf{FI}_d$	finite sets / injections with a d-coloring on the complement of the image	Configuration spaces of graphs [6] Syzygies of Segre embeddings [10]
$\mathbf{FIM}$	finite sets / injections with a perfect matching on the complement of the image	Secondary stability [4] Representations of $\mathbf{O}_\infty$ [7]
$\mathbf{OI}$	finite totally ordered sets / order preserving injections	Homology of unipotent groups [5]
$\mathbf{FS}^{\mathrm{op}}$	finite sets / surjections (opposite category)	Homology of $\overline{\mathcal{M}}_{g,n}$ [11]
$\mathbf{VA}(\mathbf{F})$	finite dimensional vector spaces over $\mathbf{F}$ / linear maps	Steenrod algebra [3]

More stability patterns in configuration and moduli spaces

- **(Harer 1986, Wahl 2008)** The sequences

$$\{\mathrm{PMod}^n(\Sigma_{\mathbf{g},r})\}_{\mathbf{g}},\ \{\mathcal{M}_{\mathbf{g},n}\}_{\mathbf{g}} \text{ and } \{\mathrm{PMod}^n(N_{\mathbf{g},r})\}_{\mathbf{g}}$$

  satisfy homological stability

- **(J.R.–Maya Duque 2015)**: The cohomology of the *pure cactus group* $\left\{H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{R});\mathbb{Q})\right\}_n$ is a f.g. FI-module.
- **(Tosteson 2018)**: The cohomology of the *Deligne–Mumford compactification* $\left\{H^k(\overline{\mathcal{M}}_{g,n};\mathbb{Q})\right\}_n$ is a FS^{op}-module f.g.

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- **(Patzt 2018)** Representation stability for filtrations of *Torelli groups*
- **(Miller–Wilson 2019, Wawrykow 2022)** *Secondary* representation stability in the homology of configuration spaces
- **(Bibby–Gadish 2019)** Representation stability in the homology of *orbit configuration spaces*
- **(Himes 2024)** *Secondary* homological stability for $\{\mathrm{C}_n(M)\}_n$
- **(Baron–Pal–Wang–Wilson–Yang 2024)** Representation stability in the (co)homology of *vertical configuration spaces*

Thanks

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The figures used in this presentation were made by Jenny Wilson

rita@im.unam.mx