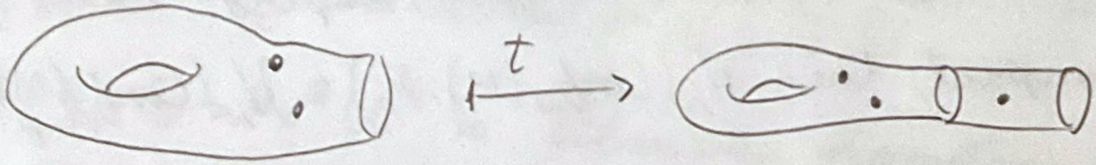
Periodicity for configuration spaces of compact manifolds

M - connected mfld

$\mathrm{Conf}_N(M) :=$ unordered configuration space of N pts on M

Thm [McDuff-Segal]: If M is non-compact, then $\exists$ a map $t: H_i(\mathrm{Conf}_N(M)) \to H_i(\mathrm{Conf}_{N+1}(M))$ which is an iso for $N \geq 2i$

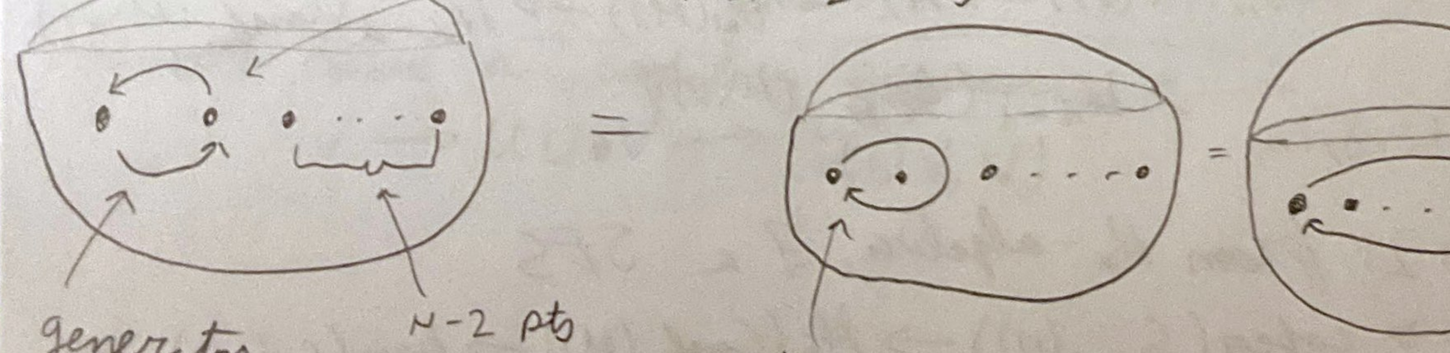


today: What happens if M is compact?

Thm [Fadell-Van Buskirk]

For $N \geq 2$, $H_1(\mathrm{Conf}_N(S^2); \mathbb{Z}) \cong \mathbb{Z}/(2N-2)$

do this $2N-2$ times



$N-2$ pts

generators of H_1 is given by 2 pts doing a half-twist

do this $N-1$ times

$= 0$

The universal coefficient thm $\Rightarrow$
if p is an odd prime (for $N \geq 2$)

$$H_1(\mathrm{Conf}_N(S^2); \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & N \equiv 1 \pmod{p} \\ 0 & \text{else} \end{cases}$$

$\Rightarrow H_1(\mathrm{Conf}_N(S^2); \mathbb{F}_p) \cong H_1(\mathrm{Conf}_{N+p}(S^2); \mathbb{F}_p)$ for $N \geq 2$

Ex:

N	2	3	4	5	6	7
$H_1(\mathrm{Conf}_N(S^2); \mathbb{F}_3)$	0	0	$\mathbb{F}_3$	0	0	$\mathbb{F}_3$

~~dfn #~~ Thm [Cantero-Palmer, Nagpal, Kupers-Miller]:
If M is compact, then $H_i(\mathrm{Conf}_N(M); \mathbb{F}_p) \cong H_i(\mathrm{Conf}_{N+p}(M); \mathbb{F}_p)$ for $N \gg i$.

Pf strategy of Kupers-Miller (using ideas of ~~Randal-Williams~~ Randal-Williams)
Let $d = \dim(M)$

Step 1: use an open cover of $\mathrm{Conf}_N(M)$ to construct a LES

$$\to H_i(\mathrm{Conf}_N(M \setminus *)) \to H_i(\mathrm{Conf}_N(M)) \to H_{i-d}(\mathrm{Conf}_{N-1}(M \setminus *)) \xrightarrow{\delta_i} H_{i-1}(\mathrm{Conf}_N(M \setminus *))$$

$\delta_i := \delta_i(N)$

Step 2: from H_*-algebra, $\exists$ a SES

$$0 \to \mathrm{coker}(\delta_{i+1}(N)) \to H_i(\mathrm{Conf}_N(M)) \to \ker(\delta_i(N)) \to 0$$

so it suffices to show that $\ker(\delta_i(N)) \cong \ker(\delta_i(N+p))$ & $\mathrm{coker}(\delta_i(N)) \cong \mathrm{coker}(\delta_i(N+p))$ for $N \gg i$

Step 3: use that $H_i(\mathrm{Conf}_N(M\setminus *);\mathbb{F}_p) \cong H_i(\mathrm{Conf}_{N+p}(M);\mathbb{F}_p)$ for $N >> i$ to show that $\ker(\delta_i(N))$ & $\mathrm{coker}(\delta_i(N))$ periodically stabilize

Step 1:

Fix a disk $D \subset M$ of dim d & let $* := 0 \in D$

Let $U := \left\{ \xi \in \mathrm{Conf}_N(M) : \xi \text{ has a unique closest pt in } D \text{ to } * \right\}$

Lemma: $f : U \to D$, $\xi \mapsto$ closest pt of ξ to D is a fibration with fiber $\cong \mathrm{Conf}_{N-1}(M\setminus *)$

Since $D \simeq *$, U is fiberwise homotopy equivalent to $D \times \mathrm{Conf}_{N-1}(M\setminus *) \simeq \mathrm{Conf}_{N-1}(M\setminus *)$

Let $V := \mathrm{Conf}_N(M\setminus *)$

Have $U \cap V \simeq (D\setminus *) \times \mathrm{Conf}_{N-1}(M\setminus *)$

$\{U, V\}$ is an open cover of $\mathrm{Conf}_N(M)$

Upshot: have a cofiber sequence

$$V \to U \cup V \to (U \cup V, V)$$

$\Rightarrow$ have a LES

$$\to H_i(V) \to H_i(U \cup V) \to H_i(U \cup V, V) \xrightarrow{\delta_i} H_{i-1}(V)$$

$$H_i(V) = H_i(\mathrm{Conf}_N(M \setminus *)) \qquad H_i(U\cup V) = H_i(\mathrm{Conf}_N(M))$$

Lemma: $H_i(U \cup V, V) = H_{i-d}(\mathrm{Conf}_{N-1}(M \setminus *))$

Pf: excision $\Rightarrow H_i(U \cup V, V) = H_i(U, U \cap V)$

sketch

$$(U, U \cap V) = (D, D \setminus *) \wedge \mathrm{Conf}_{N-1}(M \setminus *)$$
$$= \Sigma^d \, \mathrm{Conf}_{N-1}(M \setminus *)$$

Step 3:

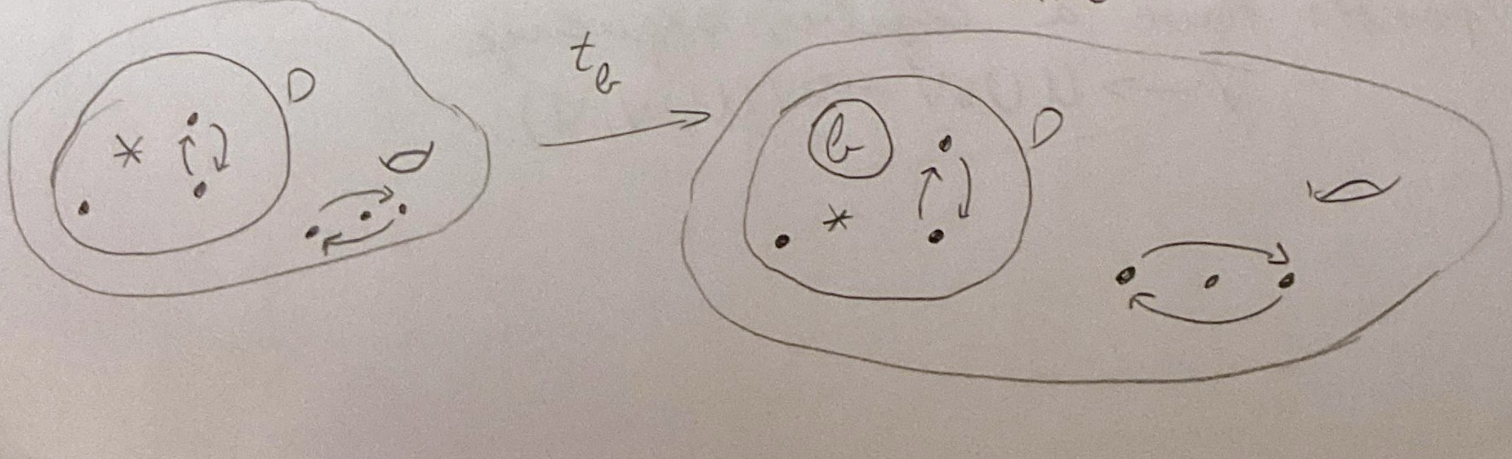
M is non-compact $\Rightarrow \exists$ an embedding ~~$(M\setminus *) \sqcup \mathbb{R}^d \to$~~

$M \setminus * \sqcup \mathbb{R}^d \to M \setminus *$

$\rightsquigarrow$ get a map $\mathrm{Conf}_k(M \setminus *) \times \mathrm{Conf}_\ell(\mathbb{R}^d) \to \mathrm{Conf}_{k+\ell}(M \setminus *)$

~~for any group~~ ~~get the~~ for any $b \in H_j(\mathrm{Conf}_\ell(\mathbb{R}^d))$, there is a map

$t_b : H_*(\mathrm{Conf}_k(M \setminus *)) \to H_{*+j}(\mathrm{Conf}_{k+\ell}(M \setminus *))$



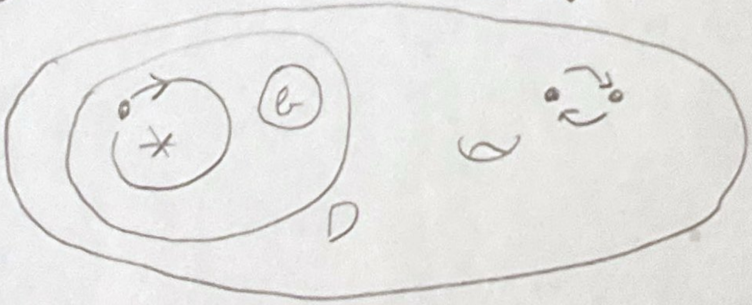
$$\begin{tikzcd} H_{i-d}(\mathrm{Conf}_{N-1}(M\setminus *)) \arrow[r, "\delta_i(N)"] \arrow[d, "t_b"'] & H_{i-1}(\mathrm{Conf}_N(M\setminus *)) \arrow[d, "t_b"] \\ H_{i-d+j}(\mathrm{Conf}_{N-1+l}(M\setminus *)) \arrow[r, "\delta_{i+j}(N+l)"'] & H_{i-1+j}(\mathrm{Conf}_{N+l}(M\setminus *)) \end{tikzcd} \qquad (**)$$

t_b induces maps $\ker(\delta_i(N)) \to \ker(\delta_{i+j}(N+l))$

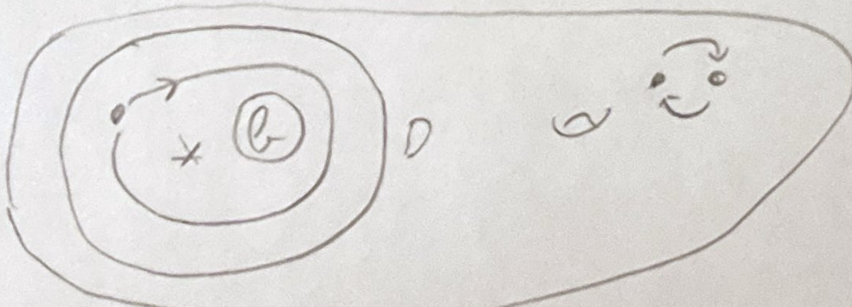
$\mathrm{coker}(\delta_i(N)) \to \mathrm{coker}(\delta_{i+j}(N+l))$

precisely if $(**)$ commutes

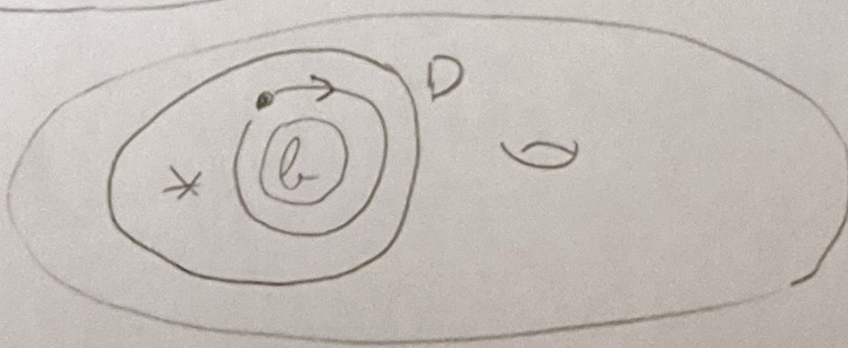
$\delta t_b =$



$t_b \delta =$



$\delta t_b - t_b \delta =$



Let $e \in H_0(\mathrm{Conf}_1(\mathbb{R}^d))$ be the class of a pt

Claim: the failure of S & t_b commuting is measured by a homology class ~~$[b,e]$~~ $[b,e]$ in $\mathrm{Conf}(D)$ called the Browder bracket of b & e

Lemma: If $b = e^p \in H_0(\mathrm{Conf}_p(\mathbb{R}^d); \mathbb{F}_p)$, then $[b, e] = 0$

$\Rightarrow t_{e^p}$ & b commute

Pf of thm:

if $b = e^p$, then $(**)$ commutes

since $t_{e^p} = (t_e)^p$ is an iso for $N >> i$, the induced maps on ker & coker are also isos for $N >> i$.