Notes for Talk 5 (Scanning 1: Definitions and Theorems)

May 28, 2026

1 Introduction

The goal of this talk is to understand the scanning map and the following theorem.

Theorem 1.1 (McDuff)**.** *Let M be an open, paracompact manifold. Then there exist maps $C_k(M) \to \Gamma^c_k(M)$ which induce an isomorphism*

$$colim_{k\to\infty} H_*(C_k(M)) \to colim_{k\to\infty} H_*(\Gamma^c_k(M))$$

1.1 Motivating the scanning map

We have the following theorem

Theorem 1.2 ((Homological Stability)Segal)**.** *If M is a compact manifold with non-empty boundary, then $H_i(C_k(M)) \xrightarrow{\cong} H_i(C_{k+1}(M))$ for $i \leq \frac{k}{2}$*

Example 1.3.

$$H_1(C_{100}(\mathbb{R}^2)) \cong H_1(C_2(\mathbb{R}^2)) = \mathbb{Z}$$

But what about $H_2(C_{100}(\mathbb{R}^2)) \cong H_2(C_4(\mathbb{R}^2)) = ??$

Scanning gives us a way to figure out the homology of a configuration space by using the space of section on a manifold. This leads us to the following theorem

Theorem 1.4 (McDuff, Segal)**.** *Let M be an open, connected, paracompact manifold. Then $H_i(C_k(M)) \cong H_i(\Gamma^c_k(M))$ for $i \leq \frac{k}{2}$*

The reason that we can go from $colim_{k\to\infty}\ H_i(C_k(M))$ to $H_i(C_k(M))$ is by stability, and from $colim_{k\to\infty}\ H_*(\Gamma^c_k(M))$ to $\Gamma^c_k(M)$ because the space of sections does not depend on k upto homotopy for open connected manifolds.

Theorem 1.2 + Theorem 1.4 gives us a way to answer this question.

$$H_2(C_{100}(\mathbb{R}^2)) \underset{\textbf{Theorem 1.4}}{\cong} H_2(\Gamma^c_4(\mathbb{R}^2)) = \mathbb{Z}/2$$

2 Scanning map

Definition 2.1. We define the following things which would be referenced throughout this and the following talks.

- Let M be a manifold and let E_M be the fiber bundle over M with fiber S^n. and whose fibers over $x \in M$ are $T_xM \cup \{\infty\}$.
- Define $\Gamma^c(M)$ to be the space of cross sections of E_M with compact support.
- Let $\Gamma^c_k(M)$ be the cross sections of degree k.

Recall:

- *Sections of a tangent bundle are vector fields, but sections of E_M are vector fields that can also be infinity.*
- *Degree of a section is intersection number of the section with the section at zero.*

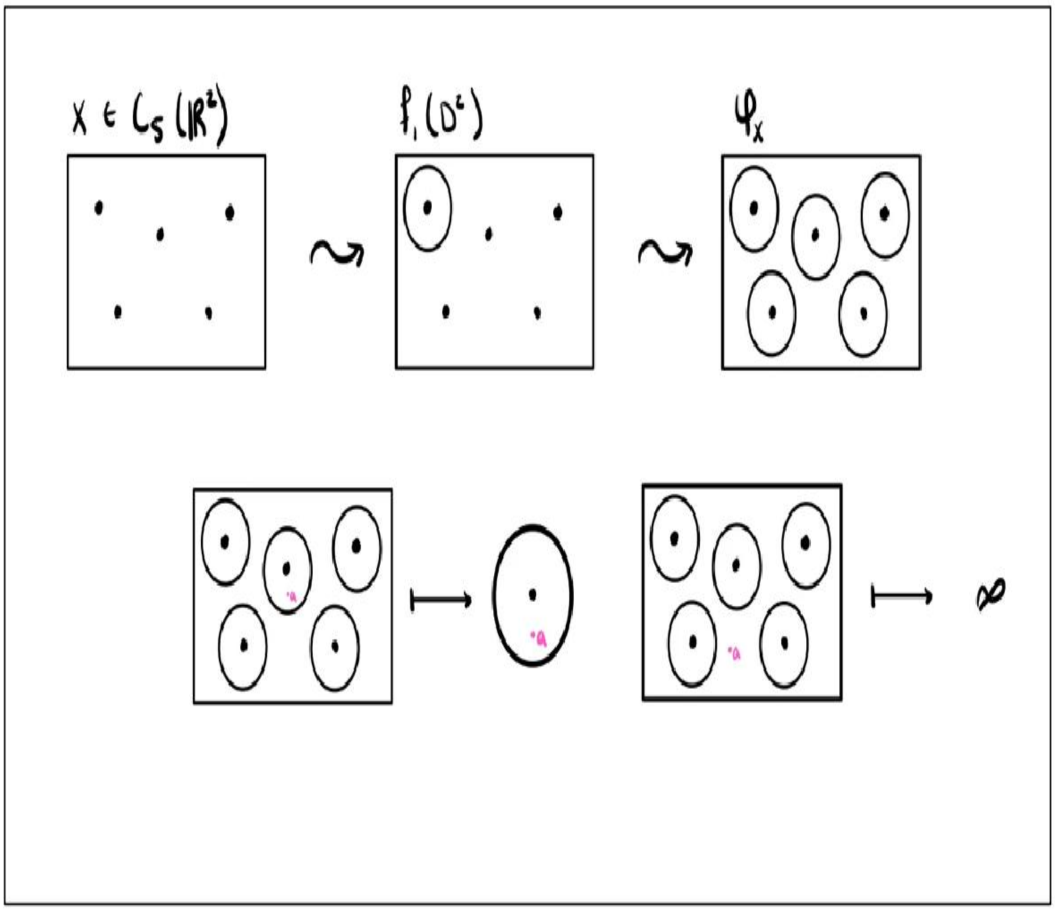
2.1 Visualizing the scanning map (M=R^2)

Let the scanning map on $\mathbb{R}^2$ *be* $s : C_k(\mathbb{R}^2) \to \Gamma_k^c(\mathbb{R}^2)$*. As* $T(\mathbb{R}^2)$ *is trivial,* $\Gamma_k^c(\mathbb{R}^2) \simeq Maps_k^c(\mathbb{R}^2, S^2) \simeq Maps_k^*(S^2, S^2)$*. So on* $\mathbb{R}^2$*,* $s : C_k(\mathbb{R}^2) \longrightarrow Maps_k^*(S^2, S^2)$*.*

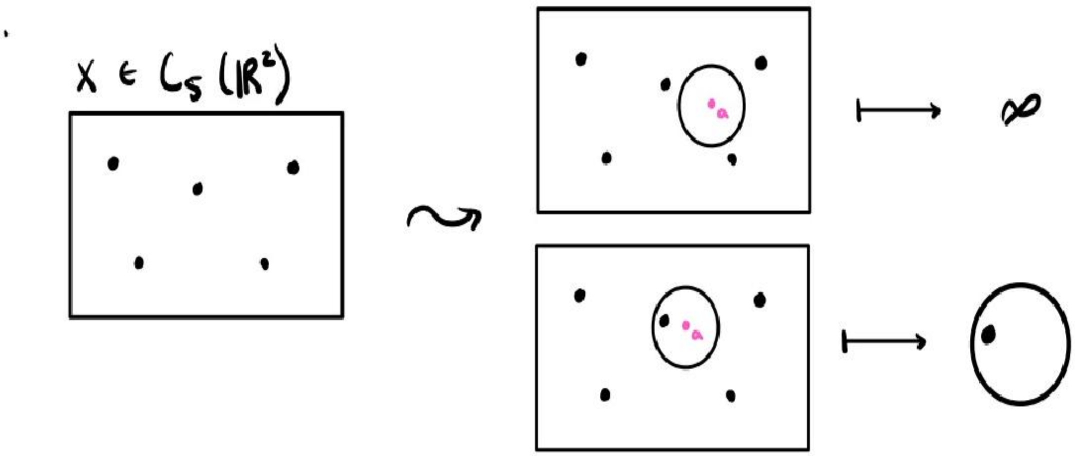
For every $X \in C_k(\mathbb{R}^2)$*, we define a map* $\phi_X : S^2 \to S^2$*, such that* $deg(\phi_X) = k, \phi_X(\infty) = \infty$*. First we define* $f_i : D^2 \to \mathbb{R}^2,\ a \mapsto ra + x_i$ *with* $r = \frac{1}{3}min\{d(x_i, x_j)|\ i \neq j\},\ f_i(0) = x_i,\ f_i(D^2) \cap f_j(D^2) = \phi$*.*

$$\phi_X(a) = \begin{cases} f_i^{-1}(a), & a \in f_i(D^2) \\ \infty, & a \in f_i(D^2),\ \forall i \end{cases}$$

Visualization of the map:



Alternately we can define the scanning map by moving a single magnifying glass over the plane where the field of vision is just a disk of a fixed radius r. instead of k magnifying glasses at the same time.



2.2 *Relative Configuration Spaces*

Definition 2.2. • (Unordered Configuration space) For any manifold M,

$$C_k(M) = \Big(\{(x_1, x_2, \cdots, x_k) \in M^k | \; x_i \in M \;\wedge\; x_i \neq x_j, \; \forall (i \neq j)\}\Big)/S_k,$$

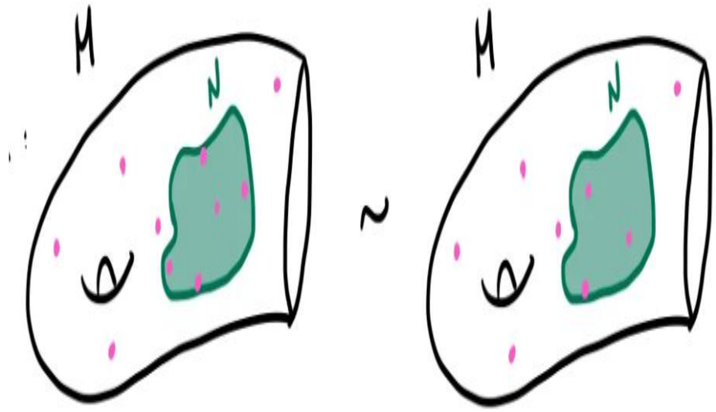
is the unordered configuration space of k points of a manifold M.

- $C(M) = \sqcup_{k \geq 0} C_k(M)$ is the space of finite subsets of M.

We define a more generalized configuration space for the purpose of future talks where points can vanish or appear from a subspace of the manifold.

Definition 2.3. Ley $N \subseteq M$ and let t, s be finite subsets of M (i.e. s$\in$C$_n(M)$, $t \in C_m(M)$, and m and n need not be equal

$$C(M, N) = C(M)/\sim, \text{ where } \Big(s \sim t \iff s \cap (M - N) = t \cap (M - N)\Big).$$



2.3 Scanning map for Relative Configuration Spaces

Let $\bar{M}$ be a compact manifold with boundary and let $M = int(\bar{M})$. The scanning map for relative configuration spaces $s_{M,N}$ is defined as follows

$$\begin{array}{ccc} C(M) & \xrightarrow{s_M} & \Gamma_c(M) \\ \downarrow\!\!\downarrow & & \downarrow \\ C(M,N) & \xrightarrow{s_{M,N}} & \Gamma_c(M,N) \end{array}$$

Here $\Gamma(M,N)$ is the space of sections which are compactly supported over $M - N$. and map to the point $*_x$ $(\partial D_x/\partial D_x) \in \dot{T}M_x$ at N. For simplicity if we assume $T\bar{M}$ (and TM) is trivial which gives us the following equivalence

$$\Gamma_k^c(M) \cong Map_k^c(M, S^n),\ n = \dim(M).$$

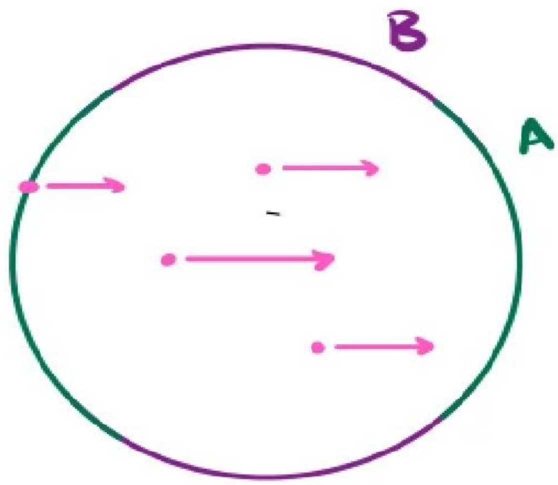
Theorem 2.4 (McDuff)**.** *Let $\partial\bar{M} = A \cup B$, then $C(\bar{M}, A) \xrightarrow{\simeq} (Map(\bar{M}, B), (S^n, \infty))$.*

Note: *There are some point set assumptions on A, B which we are not worried about at this moment.*

Let us look at some examples.

- $\bar{M} = \mathbb{R}^2,\ \partial\bar{M} = S^1$. *Here A is the disjoint union of two arcs.*

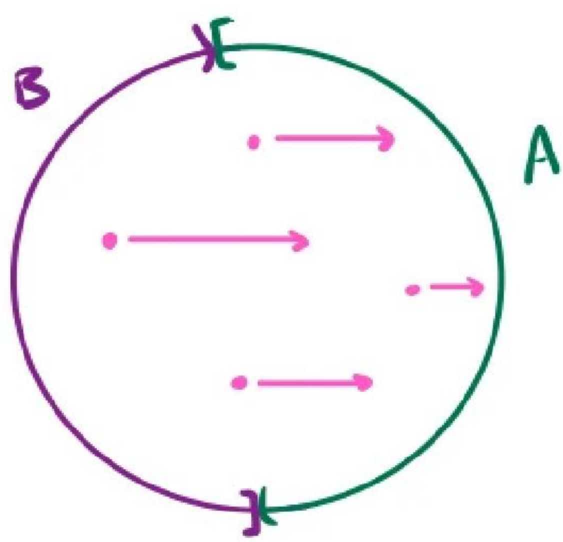
$$H_i(C(D^2, A)) \cong H_i\Big(Map\Big((D^2, B), (S^2, \infty)\Big)\Big) \cong H_i(\Omega S^2) = \mathbb{Z},\ \forall i.\ \because\ D^2/B \simeq S^1$$



The configuration in the above diagram actually depicts a class in $H_4(C(D^2, A))$.

- *Let* $M = D^2$ *and* $A{=}[0, \frac{1}{2})$, $B{=}[\frac{1}{2}, 1)$. *We have*

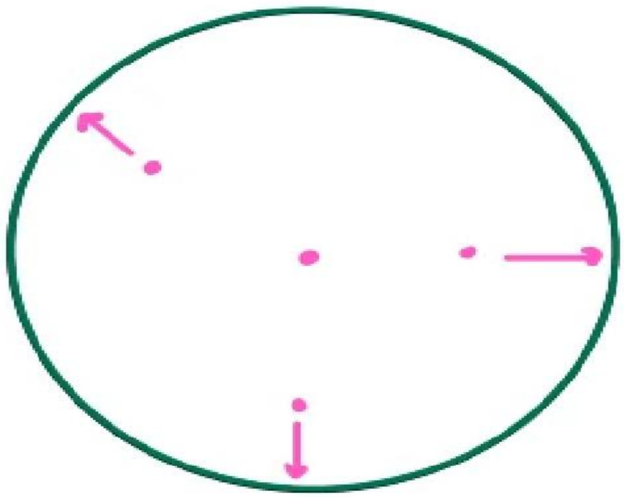
  $C(D^2, [0, \frac{1}{2})) \xrightarrow{\simeq} Map\Big((D^2, [\frac{1}{2}, 1)),\ (S^2, \infty)\Big),\ D^2/[\frac{1}{2}, 1) \simeq *.$



The diagram shows that any configuration can be continuously deformed to arrive at the empty configuration by pushing the particles to the right.

- $A = \partial M$, $B = \phi$. *Here we have*

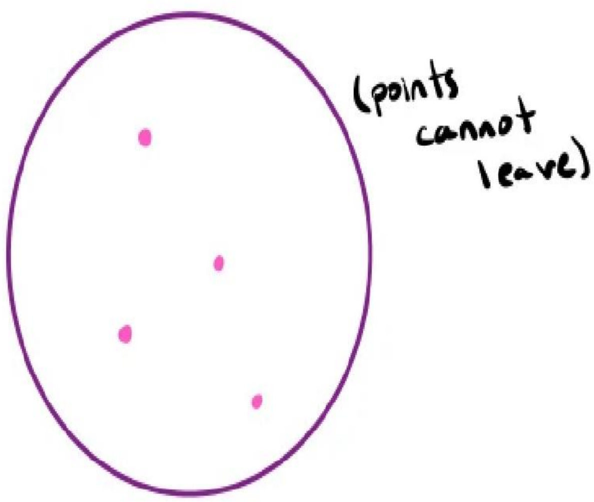
$$C(D^2, S^1) \xrightarrow{\simeq} Maps((D^2, \phi), (S^2, \infty)) = Map(D^2, S^2) \simeq S^2.$$



We can push all the particles radially outward to the boundary except if the particle is "close" to the origin.

- $A = \phi$, $B = \partial M$

  $C(D^2, \phi) \xrightarrow{\not\simeq} Maps((D^2, S^1), (S^2, \infty))$



The scanning map is not a homotopy euivalence in this case but it is an isomorphism in a stable range. For each k, $H_i(C_k(D^2)) \cong H_i(\Omega_k^2 S^2)$, *when* $i \leq \frac{k}{2}$. *Here* Ω *is the based loop space on* S^2. *and* Ω_k *is the degree* k *component. Here* $\Omega_k^2(X) := \Omega_k(\Omega_k(X))$.

3 Computing $\pi_{n+1}(S^n)$

We make some observations:

(*) Suppose $F_k(\mathbb{R}^n)$ is the space of ordered configuration space of k points in $\mathbb{R}^n$. We have the Gauss map $\gamma : F_2(\mathbb{R}^n) \to S^{n-1} \times \mathbb{R}_{>0} \times \mathbb{R}^n,\ \gamma(x_1, x_2) = \left(\frac{x_2 - x_1}{|x_2 - x_1|}, |x_2 - x_1|, \frac{x_1 + x_2}{2}\right)$. This map is an homeomorphism and this induces a homotopy $\bar{\gamma} : C_2(\mathbb{R}^n) \xrightarrow{\simeq} \mathbb{RP}^{n-1}$.

(**) If TM is trivial, then $\Gamma_k^c(M) = Map_k^c(M, S^n)$. In our case we have $M = \mathbb{R}^n$ is contractible and hence TM is trivial, and $Map_k^c(\mathbb{R}^n, S^n) \cong Map_k(S^n, S^n) = \Omega_k^n S^n$.

(***) H_1 is the abelianization of π_1. We have for any connected space $X, n \geq 1,\ \Omega_k^n X \simeq \Omega_0^n X$ and $\pi_i(X)$ is abelian for $i > 1$.

$$\begin{aligned} H_1(C_2(\mathbb{R}^n)) &\underset{\textbf{Theorem 1.2}}{\cong} H_1(C_k(\mathbb{R}^n)) \underset{\textbf{Theorem 1.4}}{\cong} H_1(\Gamma_k^c(\mathbb{R}^n)) \\ &\underset{(**)}{\cong} H_1(\Omega_k^n S^n) \underset{(***)}{\cong} (\pi_1(\Omega_0^n S^n))^{ab} \cong (\pi_{n+1}(S^n))^{ab} = \pi_{n+1}(S^n) \end{aligned}$$

Hence we have

$$\pi_{n+1}(S^n) \cong H_1(C_2(\mathbb{R}^n)) \underset{(*)}{\cong} H_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2, & n > 2 \\ \mathbb{Z}, & n = 2 \\ 0, & n \leq 1 \end{cases}$$