Stability for symmetric products

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This paper belongs to a series of papers from the Configuration Spaces Summer School 2026 at the University of Michigan, and this paper's goal is to show the homological stability for compactly supported cohomology of symmetric products in order to prove homological stability for the unordered configuration space [4, Proposition A.2, A.3]. This is done since homology satisfies local to global principles (i.e. we can compute homology of simple pieces and then use Mayer–Vietoris or a spectral sequence). The steps that we will follow are:

1. Introduce symmetric products and their relation to configuration spaces and the stabilization map.
2. Reduce the homological stability problem to the local Euclidean structure.
3. Prove homological stability for $q = 2$
4. Outline the $q \geq 2$ proof.

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1 Introduction

Definition 1.1. Given X a topological space, the n-th *symmetric product* of X is the quotient space $\mathrm{SP}_n(X) = X^n/S_n$ where S_n acts on X^n by $\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

In this way its relation with the unordered configuration space $B_k(X)$ is apparent, as one way to describe this space is

$$B_k(X) = \mathrm{Conf}_k(X)/S_n = (X^n - \Delta_X^n)/S_n$$

where $\Delta_X^n = \{(x_i)_{i=1}^n \mid x_i \in M, x_i \neq x_j \text{ for all } i \neq j\}$ is the fat diagonal of X^n. The key difference between the symmetric product and the unordered configuration lies in admitting multiple times the same point in the same unordered collection.

Definition 1.2. If $\eta \in \mathrm{SP}_n(X)$, we say its *degree* is $\deg(\eta) = n$.

Elements in $B_n(X)$ are subsets $\{x_1, \dots, x_n\} \subset X$ with n distinct elements, whereas elements in $\mathrm{SP}_n(X)$ may have repetition. We will denote the image of $(x_1, \dots, x_n)$ in $\mathrm{SP}_n(X)$ by

$$\sum_{i=1}^{n} x_i = x_1 + \cdots + x_n$$

to emphasize the lack of order of the points. This also allows us to write an element in $\mathrm{SP}_n(X)$ as

$$\eta = \sum_{i=1}^{k} n_i x_i,$$

where n_i is the multiplicity of x_i. Note that

$$\deg(\eta) = \sum_{i=1}^{k} n_i.$$

If M is a q-manifold with $q \geq 2$, then so are $\mathrm{Conf}_n(M)$ and $B_n(M)$. But $\mathrm{SP}_n(M)$ does not have to be a manifold as S_n does not act freely on the fat diagonal Δ_M^n.

Now we will move to study the stabilization map. This map is studied more deeply in the first talk of the Summer School but we will give a brief summary of how this map is defined. The reader who knows about these considerations may skip to the next section.

Naively, this is a map

$$s \colon B_n(M) \longrightarrow B_{n+1}(M)$$

that adds an additional point. However, we would need to we assure that the point added to the set is not repeated in it?

The construction that solves this issue is that for open manifolds (i.e. the interior of manifolds with nonempty boundary), there is an embedding $e \colon M \to M$ that is isotopic to $\mathrm{id}_M \colon M \to M$ and its image is a proper submanifold. One can think of the isotopy to pull points near the boundary further away from it. Let us fix a point $x_0 \in M \setminus (e)$. Then, we can define

$$s(\{x_1, \dots, x_n\}) = \{e(x_1), \dots, e(x_n), x_0\}.$$

However, this does not work for closed manifolds. For manifolds with nonempty boundary, its configuration space is homotopy equivalent to the configuration space of its interior. So if homological stability applies to one, it applies to the other. This is done through a deformation retraction: because of the existence of collar neighborhoods near its boundary, a configuration of points in M can be continuously pushed slightly away from the boundary and into the interior, this gives such deformation retraction.

The stabilization map $t\colon B_n(M) \to B_{n+1}(M)$ extends to an open embedding

$$t\colon \mathbb{R}^{\dim M} \times B_n(M) \longrightarrow B_{n+1}(M)$$

by

$$t(y, \{x_1, \ldots, x_n\}) = \{e(x_1), \ldots, e(x_n), \phi(y)\},$$

where ϕ is a homeomorphism from $\mathbb{R}^{\dim M}$ to an open neighborhood of x_0.

2 Reduction to the Euclidean case

Let M be the interior of a compact q-manifold with nonempty boundary. Observe that the stabilization map

$$t\colon \mathbb{R}^q \times B_n(M) \longrightarrow B_n(M)$$

extends to $t\colon \mathbb{R}^q \times \mathrm{SP}_n(M) \to \mathrm{SP}_{n+1}(M)$. This is an open embedding so it induces a map

$$t_*\colon H_c^i(\mathbb{R}^q \times \mathrm{SP}_n(M)) \longrightarrow H_c^i(\mathrm{SP}_{n+1}(M))$$

on compactly supported cohomology via extension by zero. This map is often called the *wrong way map* due to it being covariant, and not contravariant.

Definition 2.1. An open embedding $X \to Y$ is an *H_c^*-equivalence above dimension n* if the induced map $H_c^*(X) \to H_c^*(Y)$ is an isomorphism if $* > n$ and a epimorphism if $* = n$.

Now we state the result we are mainly interested in. This is [4, Proposition A.2].

Proposition 2.2. *The map*

$$t\colon \mathbb{R}^q \times \mathrm{SP}_n(M) \longrightarrow \mathrm{SP}_{n+1}(M)$$

is an H_c^*-equivalence above dimension $(n+1)q - \lceil n/2 \rceil$.

In order to do this we will need to decompose our manifold into simpler parts:

Proposition 2.3. *Let M be the interior of a connected compact q-manifold with nonempty boundary. There exists a closed subspace M_0 of M such that M_0^+ is a finite $(q-1)$-dimensional CW-complex and M^+ can be obtained by attaching a single q-cell $U := M \backslash M_0$ to M_0^+.*

This statement is *asserted* in Segal's paper [4] but proved in Kupers and Miller's paper [2]. They mention in their paper that their proofs depends on deep results in topological manifold theory, some of which were not available at the time. But Segal may have had a different proof in mind. We refer the reader to the Lemmas 2.1 and 2.2 of [2] for a proof.

Definition 2.4. For $\xi = \sum_i n_i x_i \in \mathrm{SP}_n(M)$, we define

$$\xi \cap M_0 = \sum_{i \text{ such that } x_i \in M_0} n_i x_i.$$

Note that $\deg(\xi \cap M_0)$ is the number of points in ξ that are in M_0, counted with multiplicity.

Definition 2.5. Let us define the filtration for $\mathrm{SP}_n(M)$ given by:

$$\mathrm{SP}_n(M) = \mathrm{SP}_n^0(M) \supset \mathrm{SP}_n^1(M) \supset \cdots \supset \mathrm{SP}_n^n(M) \supset \mathrm{SP}_n^{n+1}(M) = \varnothing$$

where $\mathrm{SP}_n^k(M) := \{\xi \in \mathrm{SP}_n(M) \mid \deg(\xi \cap M_0) \geq k\}$

$\mathrm{SP}_n^k(M)$ is the subset of $\mathrm{SP}_n(M)$ with elements whose total charge of points in M_0 is at least k. With this in mind, we can easily observe:

$$\begin{aligned} \mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M) &= \{\xi \in \mathrm{SP}_n(M) \mid k+1 > \deg(\xi \cap M_0) \geq k\} \\ &= \{\xi \in \mathrm{SP}_n(M) \mid \deg(\xi \cap M_0) = k\} \end{aligned}$$

As $M_0 \cap U = \emptyset$, we see that

$$\mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M) = \mathrm{SP}_k(M_0) \times \mathrm{SP}_{n-k}(U)$$

and write $\xi = (\xi_0, \xi_1) \in \mathrm{SP}_k(M_0) \times \mathrm{SP}_{n-k}(U)$ for $\xi \in \mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M)$.

We may assume that the open neighborhood of x_0 that we used to define t is contained in U. Then, we can restrict the stabilization map to

$$t \colon \mathbb{R}^q \times (\mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M)) \longrightarrow \mathrm{SP}_{n+1}^k(M) \backslash \mathrm{SP}_{n+1}^{k+1}(M).$$

Lemma 2.6. *If*

$$\mathbb{R}^q \times \mathrm{SP}_{n-k}(U) \longrightarrow \mathrm{SP}_{n-k+1}(U)$$

is an H_c^*-equivalence above dimensions $\geq (n+1-k)q - \lceil (n-k)/2 \rceil$, then

$$\mathbb{R}^q \times \mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M) \longrightarrow \mathrm{SP}_{n+1}^k(M) \backslash \mathrm{SP}_{n+1}^{k+1}(M)$$

is an H_c^*-equivalence above dimensions $\geq (n+1)q - \lceil n/2 \rceil$.

Proof. Previously, we got that

$$\mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M) = \mathrm{SP}_k(M_0) \times \mathrm{SP}_{n-k}(U)$$

and M_0 has cells of dimension strictly less than q. Using it, the map

$$\mathbb{R}^q \times \mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M) \xrightarrow{t_*} \mathrm{SP}_{n+1}^k(M) \backslash \mathrm{SP}_{n+1}^{k+1}(M)$$

can be viewed as

$$\mathrm{SP}_k(M_0) \times \big(\mathbb{R}^q \times \mathrm{SP}_{n-k}(U)\big) \xrightarrow{\mathrm{id} \times t_*} \mathrm{SP}_k(M_0) \times \mathrm{SP}_{n+1-k}(U)$$

This is a special case of the following claim where $B = \mathrm{SP}_k(M_0)$, $X = \mathbb{R}^q \times \mathrm{SP}_{n-k}(U)$ and $Y = \mathrm{SP}_{n-k+1}(U)$ with $\dim\big(\mathrm{SP}_k(M_0)\big) = k(q-1)$, here it is important that $\dim(M_0) = q-1$,

$$k(q-1) + \big((n-k+1)q - \lceil (n-k)/2 \rceil\big) = (n+1)q - k - \lceil (n-k)/2 \rceil \leq (n+1)q - \lceil n/2 \rceil.$$

Claim. *Let X, Y be locally compact Hausdorff spaces and $f\colon X \to Y$ an open embedding inducing an H_c^*-equivalence above dimension n. Let B be a finite CW complex of dimension m. Then $\mathrm{id}_B \times f$ induces an H_c^*-equivalence above dimension $m+n$.*

To prove this claim we use induction on the dimension of B. If the dimension of B is zero, B is a finite collection of points. Thus the map

$$\mathrm{id}_B\colon B \times X \longrightarrow B \times Y$$

can be viewed as a map between disjoint unions

$$\coprod_{pt\in B} X \longrightarrow \coprod_{pt \in B} Y.$$

As H_c^* turns disjoint unions into direct sums, and f induces an isomorphism on each component of the direct sum, the claim for the 0-skeleton is true.

For the induction step, let $m \geq 1$ and assume that

$$\mathrm{id}_{B'} \times f\colon B' \times X \longrightarrow B' \times Y$$

induces an H_c^*-equivalence above dimension $n + (m-1)$ for all finite CW-complexes of dimension $m-1$. Let us observe that $B^{(m-1)}$ is a closed subspace of B and the complement $B \backslash B^{(m-1)}$ is a finite disjoint union of open m-discs. By multiplying by X and Y, we obtain two long exact sequences on compact cohomology, for all $* \geq n+m$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^*(\coprod_i(\mathbb{R}^m \times X)) \arrow[r] \arrow[d, "\cong"] & H_c^*(B \times X) \arrow[r] \arrow[d] & H_c^*(B^{(m-1)} \times X) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_c^*(\coprod_i(\mathbb{R}^m \times Y)) \arrow[r] & H_c^*(B \times Y) \arrow[r] & H_c^*(B^{(m-1)} \times Y) \arrow[r] & \cdots
\end{tikzcd}$$

The right isomorphism is by induction, while the left one is induced by the map $H_c^*(\coprod_i(\mathbb{R}^m \times X)) \to H_c^*(\coprod_i(\mathbb{R}^m \times Y))$, for each factor:

$$H_c^*(\mathbb{R}^m \times X) \cong \tilde{H}^*((\mathbb{R}^m \times X)^+) \cong \tilde{H}^*(\Sigma^m X^+) \cong \tilde{H}^{*-m}(X^+) \cong H_c^{*-m}(X)$$

So f induces an equivalence on each summand. Now we can apply the Five Lemma and in the case for $* = n+m$, we can use the Four Lemma instead. This proves the claim and thus the lemma. □

Lemma 2.7. *If*

$$\mathbb{R}^q \times \mathrm{SP}_n^k(M) \backslash \mathrm{SP}_n^{k+1}(M) \longrightarrow \mathrm{SP}_{n+1}^k(M) \backslash \mathrm{SP}_{n+1}^{k+1}(M)$$

is an H_c^*-equivalence above dimensions $\geq (n+1)q - \lceil n/2 \rceil$ and

$$\mathbb{R}^q \times \mathrm{SP}_n(M) \backslash \mathrm{SP}_n^k(M) \longrightarrow \mathrm{SP}_{n+1}(M) \backslash \mathrm{SP}_{n+1}^k(M)$$

is an H_c^*-equivalence above dimensions $\geq (n+1)q - \lceil n/2 \rceil$ then

$$\mathbb{R}^q \times \mathrm{SP}_n(M) \backslash \mathrm{SP}_n^{k+1}(M) \longrightarrow \mathrm{SP}_{n+1}(M) \backslash \mathrm{SP}_{n+1}^{k+1}(M)$$

is a H_c^*-equivalence above dimensions $\geq (n+1)q - \lceil n/2 \rceil$.

Proof. To this end, we use the map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^{*-q}(\mathrm{SP}_n \setminus \mathrm{SP}_n^k) \arrow[r] \arrow[d, "t_*"] & H_c^{*-q}(\mathrm{SP}_n \setminus \mathrm{SP}_n^{k+1}) \arrow[r] \arrow[d, "t_*"] & H_c^{*-q}(\mathrm{SP}_n^k \setminus \mathrm{SP}_n^{k+1}) \arrow[r] \arrow[d, "t_*"] & H_c^{*-q+1}(\mathrm{SP}_n \setminus \mathrm{SP}_n^k) \arrow[r] \arrow[d, "t_*"] & \cdots \\
\cdots \arrow[r] & H_c^{*}(\mathrm{SP}_{n+1} \setminus \mathrm{SP}_{n+1}^k) \arrow[r] & H_c^{*}(\mathrm{SP}_{n+1} \setminus \mathrm{SP}_{n+1}^{k+1}) \arrow[r] & H_c^{*}(\mathrm{SP}_{n+1}^k \setminus \mathrm{SP}_{n+1}^{k+1}) \arrow[r] & H_c^{*+1}(\mathrm{SP}_{n+1} \setminus \mathrm{SP}_{n+1}^k) \arrow[r] & \cdots
\end{tikzcd}$$

where SP_j^i denotes $\mathrm{SP}_j^i(M)$. Note that this uses that $\mathrm{SP}_n^{k+1}(M)$ is a closed subset of $\mathrm{SP}_n^k(M)$. Also note that the downwards maps uses the isomorphism $H_c^*(\mathbb{R}^q \times X) \cong H_c^{*-q}(X)$.

By hyphotesis,

$$t_*\colon H_c^{*-q}(\mathrm{SP}_n \setminus \mathrm{SP}_n^k) \longrightarrow H_c^*(\mathrm{SP}_{n+1} \setminus \mathrm{SP}_{n+1}^k)$$

is an isomorphism for $* > (n+1)q - \lceil n/2 \rceil$ and an epimorphism for $* = (n+1)q - \lceil n/2 \rceil$. Also by hypothesis

$$t_*\colon H_c^{*-q}(\mathrm{SP}_n^k \setminus \mathrm{SP}_n^{k+1}) \longrightarrow H_c^*(\mathrm{SP}_{n+1}^k \setminus \mathrm{SP}_{n+1}^{k+1})$$

is an isomorphism for $* > (n+1)q - \lceil n/2 \rceil$ and an epimorphism for $* = (n+1)q - \lceil n/2 \rceil$ because, as shown before

$$k(q-1) + (n-k+1)q - \lceil (n-k)/2 \rceil \leq (n+1)q - \lceil n/2 \rceil.$$

By the Four Lemma,

$$t_*\colon H_c^{*-q}(\mathrm{SP}_n \setminus \mathrm{SP}_n^{k+1}) \longrightarrow H_c^*(\mathrm{SP}_{n+1} \setminus \mathrm{SP}_{n+1}^{k+1})$$

is an epimorphism for $* \geq (n+1)q - \lceil n/2 \rceil$. (This uses that $t_*\colon H_c^{*-q+1}(\mathrm{SP}_n \setminus \mathrm{SP}_n^k) \to H_c^{*+1}(\mathrm{SP}_{n+1} \setminus \mathrm{SP}_{n+1}^k)$ is a monomorphism for $* \geq (n+1)q - \lceil n/2 \rceil$.) And also by the Four Lemma,

$$t_*\colon H_c^{*-q}(\mathrm{SP}_n \setminus \mathrm{SP}_n^{k+1}) \longrightarrow H_c^*(\mathrm{SP}_{n+1} \setminus \mathrm{SP}_{n+1}^{k+1})$$

is a monomorphism for $* > (n+1)q - \lceil n/2 \rceil$. This uses that $t_*\colon H_c^{*-q-1}(\mathrm{SP}_n^k \setminus \mathrm{SP}_n^{k+1}) \to H_c^{*-1}(\mathrm{SP}_{n+1}^k \setminus \mathrm{SP}_{n+1}^{k+1})$ is an epimorphism for $* > (n+1)q - \lceil n/2 \rceil$.) □

In summary, due to $\mathrm{SP}_n^k(M) \setminus \mathrm{SP}_n^{k+1}(M) = \mathrm{SP}_k(M_0) \times \mathrm{SP}_{n-k}(U)$ if we have homological stability for $\mathrm{SP}_n(U)$ we would have it also for $\mathrm{SP}_n(M)$. Hence it is enough to show the following proposition.

Proposition 2.8. *The map*

$$t\colon \mathbb{R}^q \times \mathrm{SP}_n(\mathbb{R}^q) \longrightarrow \mathrm{SP}_{n+1}(\mathbb{R}^q)$$

is an H_c^*-equivalence above dimension $(n+1)q - \lceil n/2 \rceil$.

Let us show first that this proposition implies Proposition 2.2.

Proof. We will prove by induction on k that

$$\mathbb{R}^q \times (\mathrm{SP}_n(M) \setminus \mathrm{SP}_n^k(M)) \longrightarrow (\mathrm{SP}_{n+1}(M) \setminus \mathrm{SP}_{n+1}^k(M))$$

is an H_c^*-equivalence in dimensions $\geq (n+1)q - \lceil n/2 \rceil$. The induction beginning $k = 0$ is trivial because $\mathrm{SP}_n(M) = \mathrm{SP}_n^0(M)$ hence both sides are empty.

Now assume the statement for k and aim to prove the statement for $k+1$, say

$$\mathbb{R}^q \times (\mathrm{SP}_n(M) \setminus \mathrm{SP}_n^k(M)) \longrightarrow (\mathrm{SP}_{n+1}(M) \setminus \mathrm{SP}_{n+1}^k(M))$$

is an H_c^*-equivalence in dimensions $\geq (n+1)q - \lceil n/2 \rceil$, by lemma 2.7 we are done.

So in the case where $k = n+2$ we get

$$\mathbb{R}^q \times \mathrm{SP}_n(M) = \mathbb{R}^q \times (\mathrm{SP}_n(M) \setminus \mathrm{SP}_n^{n+2}(M)) \longrightarrow (\mathrm{SP}_{n+1}(M) \setminus \mathrm{SP}_{n+1}^{n+2}(M)) = \mathrm{SP}_{n+1}(M)$$

is an H_c^*-equivalence in dimensions $\geq (n+1)q - \lceil n/2 \rceil$.

□

We work towards the proof of Proposition 2.8 next.

3 The case $q = 2$

Computing the compact cohomology of $\mathrm{SP}_n(\mathbb{R}^q)$ can be quite complicated as the space itself is frequently not a manifold. This is because the action of S_n on X^n is not free. In fact, the only points on which the action is free lie in $\mathrm{Conf}_n(X) \subset X^n$, for any $x \in X^n \setminus \mathrm{Conf}_n(X)$ with $x = (x_1, \ldots, x_n)$ there exists $i \neq j$ such that $x_i = x_j$, and will be fixed by the transposition action (i, j) that exchanges the i-th and j-th coordinate. We will study the particular example when $n = 2$.

Example 3.1. *Let us consider* $\mathrm{SP}_2(\mathbb{R}^q) = (\mathbb{R}^q \times \mathbb{R}^q)/S_2$. *The* S_2 *action on* $\mathbb{R}^q \times \mathbb{R}^q$ *is given by* $(x, y) \mapsto (y, x)$. *Consider the isomorphism*

$$\mathbb{R}^q \times \mathbb{R}^q \longrightarrow \mathbb{R}^q \times \mathbb{R}^q$$

given by $(x, y) \mapsto (u, v) = (x + y, x - y)$. *Now the* S_2 *action is given by* $(u, v) \mapsto (u, -v)$. *Hence*

$$\mathrm{SP}_2(\mathbb{R}^q) = (\mathbb{R}_u^q \times \mathbb{R}_v^q)/S_2 = \mathbb{R}_u^q \times (\mathbb{R}_v^q/\sim) \cong \mathbb{R}^q \times C(\mathbb{RP}^{q-1}),$$

where $\sim$ *identifies antipodal elements and where* $C(X) := X \times [0, \infty)$ *denotes the open cone of a topological space* X. *Note that for* $q > 2$, $C(\mathbb{RP}^{q-1})$ *is not a manifold.*

We can also use this homeomorphism to compute the compactly supported cohomology:

$$\begin{aligned} H_c^*(\mathrm{SP}_2(\mathbb{R}^q)) &= H_c^*((\mathbb{R}^q \times \mathbb{R}^q)/\sim) \cong H_c^{*-q}(\mathbb{R}^q/\sim) \\ &\cong \tilde{H}^{*-q}((\mathbb{R}^q/\sim)^+) \cong \tilde{H}^{*-q}(\Sigma \mathbb{RP}^{q-1}) \cong \tilde{H}^{*-q-1}(\mathbb{RP}^{q-1}) \end{aligned}$$

This cohomology usually has 2-torsion, except in the case that $q = 2$.

Proposition 3.2. *The spaces* $\mathrm{SP}_n(\mathbb{R}^2)$ *and* $\mathbb{C}^n$ *are homeomorphic.*

Proof. Consider the following sequence of maps

$$(\mathbb{R}^2)^n \cong \mathbb{C}^n \xrightarrow{q} \mathrm{SP}_n(\mathbb{C}) \xrightarrow{p} M_n(\mathbb{C}) \xrightarrow{\prod_{i=0}^{n-1} c_i} \mathbb{C}^n,$$

where

- $M_n(\mathbb{C}) = \{p \in \mathbb{C}[x] \mid \deg(p) = n,\ p \text{ is monic}\}$,
- q is the natural quotient map,
- p maps $\sum n_{x_i} x_i$ to $\prod (x - x_i)^{n_{x_i}}$, and
- c_i maps $p(x) = \sum a_i x^i$ to a_i.

The map $(\prod_{i=0}^{n-1} c_i) \circ p$ is a bijection by the fundamental theorem of algebra. To show that it is an homeomorphism, first notice that the map $c_i \circ p$ is continuous as

$$\prod_{i=1}^{n} (x - x_i) = \sum_{i=1}^{n} (-1)^i x^{k-i} e_i$$

with e_m the m-th elementary symmetric polynomial

$$e_m(x'_1, \dots, x'_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} x'_{i_1} x'_{i_2} \dots x'_{i_m}$$

evaluated at $x_1, \dots, x_n$.

To see that the inverse map is continuous, it is enough to show that $(\prod_{i=0}^{n-1} c_i) \circ p$ is proper. This follows by the fact that the roots of a polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ are bounded by $1 + \max |a_i|$. □

Using Proposition 3.2, one can see that Proposition 2.8 is true for $n = 2$ as the space $\mathrm{SP}_n(\mathbb{R}^2) \cong \mathbb{C}^n$ is contractible, hence all maps are trivially equivalences if $q = 2$.

4 The rest of the cases $q \geq 2$

First, we want to see that

$$\mathrm{SP}_n(\mathbb{R}^q) \cong \mathrm{SP}_n(S^q) \setminus \mathrm{SP}_{n-1}(S^q).$$

Consider $\mathbb{R}^q$ as a subset of $S^q \cong (\mathbb{R}^q)^+$. Then there is a homeomorphism

$$\mathrm{SP}_n(S^q) \setminus \mathrm{SP}_n(\mathbb{R}^q) \longrightarrow \mathrm{SP}_{n-1}(S^q)$$

by forgetting one of the infinite points. In other words, there is an embedding i_{n-1} of $\mathrm{SP}_{n-1}(S^q)$ into $\mathrm{SP}_n(S^q)$ given by

$$i_{n-1}(x_1+\cdots+x_{n-1}) = x_1+\cdots+x_{n-1}+\infty.$$

Second, using the previous map, we can construct the direct system

$$\big(i_n : \mathrm{SP}_n(S^q) \longrightarrow \mathrm{SP}_{n+1}(S^q)\big)$$

its direct limit is called the infinite symmetric product $\mathrm{SP}_\infty(\mathbb{S}^q)$. This is a $K(\mathbb{Z}, q)$, as by the Dold–Thom Theorem, $\pi_i(\mathrm{SP}_\infty(X)) \cong \tilde{H}_i(X)$ for every CW complex X.

Proposition 4.1. *The filtration* $\mathrm{SP}_0(S^q) \subset \mathrm{SP}_1(S^q) \subset \cdots \subset \mathrm{SP}_n(S^q)$ *gives an isomorphism*

$$H^*(\mathrm{SP}_\infty(S^q)) \cong \bigoplus_{n\geq 0} H^*_c(\mathrm{SP}_n(\mathbb{R}^q)).$$

Proof. Because of the long exact sequence

$$\cdots \longrightarrow H^{k-1}(\mathrm{SP}_{n-1}(S^q)) \xrightarrow{\delta^*} H^k_c(\mathrm{SP}_n(\mathbb{R}^q)) \xrightarrow{j^*} H^k(\mathrm{SP}_n(S^q)) \xrightarrow{i^*} H^k(\mathrm{SP}_{n-1}(S^q)) \xrightarrow{\delta^*} \cdots,$$

if we see that

$$H^k(\mathrm{SP}_n(S^q)) \to H^k(\mathrm{SP}_{n-1}(S^q))$$

splits, we get that

$$H^*(\mathrm{SP}_n(S^q)) \cong H^*_c(\mathrm{SP}_n(\mathbb{R}^q)) \oplus H^*(\mathrm{SP}_{n-1}(S^q)) \cong \bigoplus_{i=0}^{n} H^*_c(\mathrm{SP}_i(\mathbb{R}^q)).$$

The splitting follows by a general fact by Steenrod [6, Section 22], where he shows that for any based simplicial set X there is a transfer map at the homology level $\tau_* : C_*(\mathrm{SP}_n(X) \to C_*(\mathrm{SP}_{n-1}(X))$ that forgets points in all possible ways and then adds them up at the homology level

$$\tau_*(\{x_1,\ldots,x_n\}) = \sum_{i=1}^{n-1}\{x_1,\ldots,\hat{x_i},\ldots,x_n\}$$

; and proves that the map $\mathrm{SP}_{n-1}(X) \to \mathrm{SP}_n(X)$ splits on homology.

To prove the desired result, we want to show

$$H^*(\varinjlim(\mathrm{SP}_n(S^q))) \cong \varprojlim(\bigoplus_{i=0}^{n} H^*_c(\mathrm{SP}_i(\mathbb{R}^q))).$$

In order to show it, we consider the Milnor exact sequence which states that for $X_0 \subset X_1 \subset \cdots$, we have the SES:

$$0 \longrightarrow \varprojlim{}^1 H^{k-1}(X_n) \longrightarrow H^k(\varinjlim X_n) \longrightarrow \varprojlim H^k(X_n) \longrightarrow 0$$

which in our case $X_n = \mathrm{SP}_n(S^q)$ has $\varprojlim{}^1$ term vanish, this is due to the maps of the inverse system are all surjective, which makes them satisfy the Mittag–Leffler condition, this is the content of LES of the pair turning into a split exact sequence, another way to view this is that $H^{k-1}(X_n) = 0$ in our case with k even.

Finally:

$$H^*(\mathrm{SP}_\infty(S^q)) \cong H^*(\varinjlim \mathrm{SP}_n(S^q)) \cong \varprojlim(H^k(\mathrm{SP}_n(S^q))) \cong \varprojlim(\bigoplus_{i=0}^{n} H_c^k(\mathrm{SP}_i(\mathbb{R}^q)))$$
$$\cong \prod_{i\geq 0} H_c^k(\mathrm{SP}_i(\mathbb{R}^q)) \cong \bigoplus_{i\geq 0} H_c^k(\mathrm{SP}_i(\mathbb{R}^q))$$

the second to last isomorphism is due to inverse limit of finite direct products are infinite direct products and the last one is a consequence of Nakaoka's theorem [3]: there are only finitely many non-zero terms in any specific degree k, the infinite direct product in that degree is identical to a finite direct product. □

In what follows, we only want to summarize some of the reasoning. The strategy to prove the general case $q \geq 2$ is to show it with $\mathbb{F}_p$ coefficients for all $p > 0$ prime. We will only consider $p = 2$.

Proposition 4.2.

$$H^*(\mathrm{SP}_\infty(S^q)) \cong \bigoplus_{n\geq 0} H_c^*(\mathrm{SP}_n(\mathbb{R}^q))$$

is a bigraded ring and with field coefficients the multiplication is given by the transfer map

$$H_c^*(SP_n(\mathbb{R}^q);F)\otimes H_c^*(SP_n(\mathbb{R}^q);F) = H_c^*(SP_n(\mathbb{R}^q)\times SP_n(\mathbb{R}^q);F) \longrightarrow H_c^*(SP_{n+m}(\mathbb{R}^q);F).$$

The following theorem, known as the Serre-Cartan computation for Eilenberg-MacLane spaces, will allow us to make statements about $H_c^*(SP_n(\mathbb{R}^q);\mathbb{F}_2)$.

Theorem 4.3. *There is a ring isomorphism*

$$H^*(K(\mathbb{Z},q),\mathbb{F}_2) \cong \mathbb{F}_2[Sq^I\epsilon_q \mid I \text{ is admissible with } e(I) < q],$$

where Sq^I are the Steenrod squares, ϵ_q the fundamental class, and e is the excess of the sequence $I = (i_1, \dots, i_k)$ with $i_k > 1$.

This theorem is due to Serre [5] and it is generalized later by Cartan [1] to all p.

For example, when $q = 2$, we have $K(\mathbb{Z}, 2)$ is homotopy equivalent to $\mathbb{CP}^\infty$ so the theorem claims that $H^*(\mathbb{CP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[\epsilon_2]$ where the fundamental class of $\mathbb{CP}^2$ generates $H^2(\mathbb{CP}^\infty;\mathbb{F}_2)$.

The other result that we need is due to Nakaoka [3] as mentioned by Segal [4].

Theorem 4.4. *The generator* $Sq^I \epsilon_q$ *land in* $H_c^m(\mathrm{SP}_{2^k}(\mathbb{R}^q))$ *under the isomorphisms of Theorem 4.3 and Proposition 4.2 with* $m \leq 2^k(q-1)+1$.

We can finally give a proof idea for Proposition 2.8. The stabilization map is given by the multiplication map

$$H_c^*(\mathrm{SP}_1(\mathbb{R}^q) \times \mathrm{SP}_n(\mathbb{R}^q)) \longrightarrow H_c^*(\mathrm{SP}_{n+1}(\mathbb{R}^{q+1})),$$

where we simply multiply by ϵ_q. Hence, the stabiliztation map is always injective as $H^*(K(\mathbb{Z}, q); \mathbb{F}_2)$ does not have zero divisors.

The stabilization map is surjective if all monomials in its image are divisible by ϵ_q. This follows for a range because the algebra generators $Sq^I \epsilon_q$ cannot appear in arbitrarily large cohomological degree as stated in Theorem 4.4.

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