Scanning II: Proof assuming that the projection is a quasi-fibration/homology fibration (talk 7)

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1 INTRODUCTION

Let M be a smooth manifold. In these notes, $C(M)$ will denote the space of **unordered** configurations in M and $C_k(M)$ corresponds to the k-configurations for $k \geq 0$. Let $C(M, L)$ be the quotient space of $C(M)$ by the equivalence relation which identifies two finite subsets s and s' of M if $s \cap (M \setminus L) = s' \cap (M \setminus L)$. Let $\tilde{C}(M) := C(M, \partial M)$.

On the other hand, let E_M be the fibre bundle on M with fibre S^n obtained by taking the one-point compactification in each fibre of the tangent bundle associated to M. Let $\Gamma(M)$ be the space of cross-sections of E_M with compact support. Denote by $\Gamma_k(M)$ the k-degree cross-sections.

The main goal of the talk is to prove the following result

Theorem A ([McD75, Theorem 1.2]). *Let M be an open manifold. There are maps $C_k(M) \to \Gamma_k(M)$ which induce an isomorphism*

$$\lim_{k\to\infty} H_*(C_k(M)) \cong \lim_{k\to\infty} H_*(\Gamma_k(M)).$$

The key step corresponds to

Theorem B (The scanning map is an equivalence). *Let M be a compact manifold with boundary such that $\partial M \neq \emptyset$. Then, the scanning map is a homotopy equivalence*

$$\tilde{C}(M) = C(M, \partial M) \simeq \Gamma M.$$

which will be proven by using handle decomposition of manifolds.

2 REVIEW OF HANDLE DECOMPOSITION OF MANIFOLDS

Definition 1. *An n-dimensional k-**handle** is $D^k \times D^{n-k}$.*

Definition 2 (Handle attachment). *Let M be a smooth n-manifold with boundary. We may **attach** a k-handle by choosing a smooth embedding*

$$i \colon S^{k-1} \times D^{n-k} \to \partial M$$

and then taking $M' := M \cup_i \left(D^k \times D^{n-k}\right)$.

Definition 3. *Let M be an n-manifold. A **handle decomposition** of M is a finite sequence of manifolds $W_0, \dots, W_m$ such that $W_0 = \emptyset$, $W_m \cong M$, and W_i is obtained from W_{i-1} via handle attachment.*

Theorem 1 (Handle decomposition)**.** *Every compact smooth manifold admits a handle decomposition.*

Observe that this is similar to the notion of CW-decomposition in the context of CW-complexes.

Example 1.

- We can construct S^2 by attaching a 0-handle to $\emptyset$ and then a 2-handle.
- We can build T^2 by attaching a 0-handle, two 1-handles, and finally a 2-handle as shown in Figure 1.

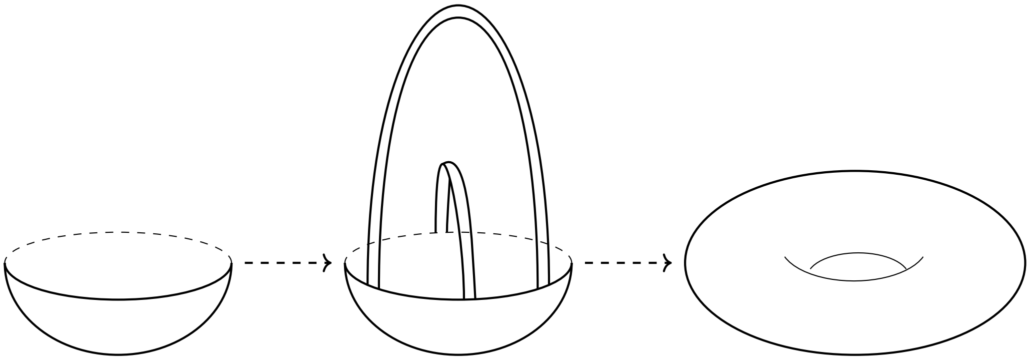

Figure 1: Handle decomposition for T^2.

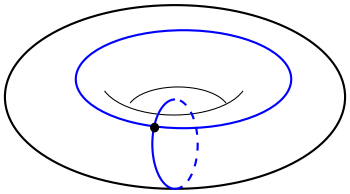

Figure 2: CW decomposition for T^2.

3 The scanning map

We want to build a map $\tilde{C}(M) \to \Gamma(M)$. Given $[s] \in \tilde{C}(M)$, let $\varepsilon > 0$ be small enough so that the elements of the configuration s are within ε of each other, and also that the exponential map is a diffeomorphism in an ε-neighborhood V of every point in M. Define the **scanning map**

$$\begin{aligned} \phi \colon \tilde{C}(M) &\to \Gamma(M) \\ [s] &\mapsto (\phi([s]) \colon M \to E_M) \end{aligned}$$

where $\phi([s])$ is defined by $x \mapsto *_x$ (where $*_x = \infty \in S_x$) whenever $d(x,s) > \varepsilon$ and $x \mapsto k_x \cdot t_x$, where $k_x \in \mathbb{R}$ and t_x corresponds to the unit tangent at x to the minimal geodesic from x to a point $y \in M$ in an ε-disk of a point in $[s]$. This geodesic exists due to the assumption on ε.

Theorem 2 (The scanning map is an equivalence)**.** *Let M be a compact manifold with boundary such that $\partial M \neq \emptyset$. Then, the scanning map is a homotopy equivalence*

$$\tilde{C}(M) = C(M, \partial M) \simeq \Gamma M.$$

Proof. First, we will consider the case $M = D^n$. Observe that $\Gamma D^n \simeq S^n$. This follows from the fact that $E \cong D^n \times S^n$, and thus, $\Gamma D^n \simeq \operatorname{Hom}(D^n, S^n) \simeq S^n$. So, it suffices to show that $\tilde{C}(D^n) \simeq S^n$.

We first claim (the stronger assertion) that $\tilde{C}_{\leq 1}(D^n) \cong S^n$. Consider the model $S^n = D^n \cup \{\infty\}$. If $[s] \in \tilde{C}_{\leq 1}$ is such that the configuration consists of a single point $x_0 \in \mathrm{Int}(D^n)$, then send $[s] \mapsto x_0$. If $[s] = \emptyset$ or consists of a single point in the boundary, map $[s] \mapsto \{\infty\}$. This yields $\tilde{C}_{\leq 1}(D^n) \cong S^n$.

Now, it suffices to show $\tilde{C}(D^n) \simeq \tilde{C}_{\leq 1}(D^n)$. One side of the equivalence is given by the inclusion $\tilde{C}_{\leq 1}(D^n) \hookrightarrow \tilde{C}(D^n)$.

The map $\Phi \colon \tilde{C}(D^n) \to \tilde{C}_{\leq 1}(D^n)$ is defined as follows. Model D^n as the unit disk in $\mathbb{R}^n$. Let $[s] \in \tilde{C}(D^n)$ and take δ as the minimum distance between points in s. Then, we may send $[s]$ to a configuration with at most one point by rescaling D^n by $\frac{1}{\delta}$.

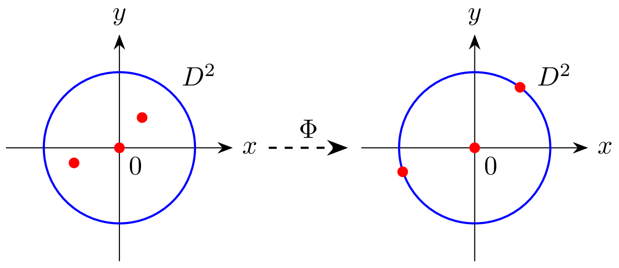

Figure 3: Map $\Phi \colon \tilde{C}(D^n) \longrightarrow \tilde{C}_{\leq 1}(D^n)$.

Thus, we have

$$\tilde{C}(D^n) \simeq \tilde{C}_{\leq 1}(D^n) \cong S^n \simeq \Gamma D^n.$$

Now, we will proceed to prove that $\tilde{C}(S^{k-1} \times D^{n-k}) \simeq \Gamma(S^{k-1} \times D^{n-k})$ by induction on k. We have already proved the base case $k = 0$. Assume that the result holds for $l < k - 1$.

Observe that

$$S^{k-1} \times D^{n-k} \cong \left(D^{k-1} \times D^{n-k}\right) \cup_{S^{k-2} \times D^{n-k}} \left(D^{k-1} \times D^{n-k}\right).$$

Thus, consider the following commutative diagram

$$\begin{tikzcd}
 & \tilde{C}(S^{k-1} \times D^{n-k}) \arrow[rr] \arrow[dl] \arrow[dd] & & \Gamma(S^{k-1} \times D^{n-k}) \arrow[dl] \arrow[dd] \\
\tilde{C}(D^{k-1} \times D^{n-k}) \arrow[rr] \arrow[dd] & & \Gamma(D^{k-1} \times D^{n-k}) \arrow[dd] & \\
 & \tilde{C}(D^{k-1} \times D^{n-k}) \arrow[rr] \arrow[dl] & & \Gamma(D^{k-1} \times D^{n-k}) \arrow[dl] \\
\tilde{C}(S^{k-2} \times D^{n-k}) \arrow[rr] & & \Gamma(S^{k-2} \times D^{n-k}) &
\end{tikzcd}$$

where the horizontal maps are all scanning maps and the maps on the sides of the cube are given by the restriction map.

The square on the left is a pullback square. Moreover, we will assume that all the maps in this square are quasi-fibrations. This will be proven in the next talk. Therefore, the left square is a homotopy pullback square.

The square on the right is a pullback along fibrations. Thus, the square on the right is a homotopy pullback square.

Now, two of the scanning maps are equivalences by the base case and the fact that the product of disks is homeomorphic to a disk. The bottom scanning map is an equivalence by our inductive assumption.

Since both squares are homotopy pullbacks, the map

$$\tilde{C}(S^{k-1} \times D^{n-k}) \to \Gamma(S^{k-1} \times D^{n-k})$$

is an equivalence as desired.

Consider the following result.

Lemma 1. *If $A \to X$ is a (Serre-)cofibration, then the induced map $\Gamma X \to \Gamma A$ is a (Serre-)fibration.*

Proof. Suppose that the following diagram commutes

$$\begin{tikzcd}
D^n \times \{0\} \arrow[r] \arrow[d] & \Gamma X \arrow[d] \\
D^n \times I \arrow[r] & \Gamma A
\end{tikzcd}$$

This commutative diagram rearranges into a map

$$X \times D^n \times \{0\} \cup_{A \times D^n \times \{0\}} A \times D^n \times I \longrightarrow E_M.$$

The inclusion of

$$X \times D^n \times \{0\} \cup A \times D^n \times I \longrightarrow X \times D^n \times I$$

is an acyclic cofibration by hypothesis.

Then, we have a commutative square

$$\begin{tikzcd}
X \times D^n \times \{0\} \cup_{A \times D^n \times I} A \times D^n \times I \arrow[r] \arrow[d] & E_M \arrow[d] \\
X \times D^n \times I \arrow[r] & X
\end{tikzcd}$$

where the left map is an acyclic cofibration and the right map is a fibration. Thus, there exists a lift. Rearranging the triangle above, we obtain the corresponding lift associated to the first diagram

$$\begin{tikzcd}
D^n \times \{0\} \arrow[r] \arrow[d] & \Gamma X \arrow[d] \\
D^n \times I \arrow[r] \arrow[ur, dashed] & \Gamma A
\end{tikzcd}$$

which shows that $\Gamma X \longrightarrow \Gamma A$ is a (Serre-)fibration as desired. ■

Now, we are ready to prove Theorem 2. Recall that every compact manifold with non-empty boundary admits a handle decomposition of handles attached to a disk. Thus, it suffices to prove the following statement

Lemma 2. *Let $M = M' \cup (D^k \times D^{n-k})$ and $\tilde{C}(M') \simeq \Gamma M'$. Then $\tilde{C}(M) \simeq \Gamma M$.*

Proof. We build a commutative diagram as follows.

$$\begin{tikzcd}
& \tilde{C}(M) \arrow[dl] \arrow[rr] \arrow[dd] & & \Gamma(M) \arrow[dl] \arrow[dd] \\
\tilde{C}(M') \arrow[rr] \arrow[dd] & & \Gamma(M') \arrow[dd] & \\
& \tilde{C}(D^k \times D^{n-k}) \arrow[rr] \arrow[dl] & & \Gamma(D^k \times D^{n-k}) \arrow[dl] \\
\tilde{C}(S^{k-1} \times D^{n-k}) \arrow[rr] & & \Gamma(S^{k-1} \times D^{n-k}) &
\end{tikzcd}$$

Again, the left square is a pullback and all of the maps in this square are quasi-fibrations by assumption. So, it is a homotopy pullback. The right square is a pullback along fibrations (by Lemma 1), so it is also a homotopy pullback.

The bottom horizontal scanning maps are equivalences by our previous argument and $\tilde{C}(M') \longrightarrow \Gamma(M')$ is a homotopy equivalence by assumption. Therefore, the map between homotopy pullbacks

$$\tilde{C}(M) \longrightarrow \Gamma M$$

is also a homotopy equivalence. ■

This concludes the proof of the lemma and the theorem. □

Remark 1. One can prove further that

(1) If M is a closed manifold then
$$H_n(C_k(M)) \cong H_n(\Gamma_k(M))$$
for $k \gg 1$.

(2) (Theorem A) If M is an open manifold, then there is an isomorphism
$$\lim_{k\to\infty} H_*(C_k(M)) \cong \lim_{k\to\infty} H_*(\Gamma_k(M)).$$

Moreover, (1) can be deduced from (2) by using homological stability.

Example 2. For $k \gg 1$, we have

$$H_n(C_k(S^1)) \cong H_n(\Gamma_k S^1) \cong H_n\left(\mathrm{Map}_k(S^1, S^1)\right) \cong H_n\left((LS^1)_k\right) \cong H_n(S^1),$$

where the last isomorphism follows from the fibration

$$\Omega_{\mathrm{b}} S^1 \longrightarrow LS^1 \longrightarrow S^1.$$

Since S^1 is a Lie group, in particular it is an H-space. Therefore, we have the splitting

$$LS^1 \simeq S^1 \times \Omega_{\mathrm{b}} S^1 \simeq S^1 \times \mathbb{Z}$$

where the last (weak) homotopy equivalence follows from the path-loop fibration. So,

$$H_n\left((LS^1)_k\right) \cong H_n\left((S^1 \times \mathbb{Z})_k\right) \cong H_n(S^1)$$

as desired.

More generally, if $TM \cong M \times \mathbb{R}^m$ is parallelizable, then for $k \gg 1$

$$H_n(C_k(M)) \cong H_n(\Gamma_k(M)) \cong H_n\left(\mathrm{Map}_k(M, S^m)\right).$$

For instance, it is possible to consider $M = T^2$, or any orientable 3-manifold, any compact Lie group, and S^7.

REFERENCES

[McD75] D. McDuff, *Configuration spaces of positive and negative particles*, Topology 14 (1975), 91–107. MR 0358766 (50 #11225)

[Böd87] C. F. Bödigheimer, *Stable splittings of mapping spaces*, Algebraic topology (Seattle, Wash., 1985), Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 174–187. MR 922926 (89c:55011)

[Seg79] G. Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), no. 1-2, 39–72. MR 533892 (81c:55013)