Fibrations

Expository notes for the Configuration Spaces Summer School

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Covering spaces

Definition 1. A covering space of a space B is a space E together with a map $p\colon E \to B$ satisfying the following condition:

> Each point $b \in B$ has an open neighborhood U in B such that $p^{-1}(U)$ is a union of disjoint open sets in E, each of which is mapped homeomorphically onto U by p.

Such a U is called evenly covered and the disjoint open sets in E that project homeomorphically to U by p are called sheets of E over U. If U is connected these sheets are the connected components of $p^{-1}(U)$.

Example 2. Let $p\colon \mathbb{R} \to S^1$ be given by

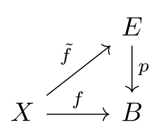
$$p(t) = (\cos 2\pi t, \sin 2\pi t).$$

Any open arc in S^1 is evenly covered. For example, if $U = S^1 \setminus \{p(0)\}$, then

$$p^{-1}(U) = \cup_{n\in\mathbb{Z}}(n, n+1) \cong \sqcup_{n\in\mathbb{Z}}(0,1) \cong \sqcup_{n\in\mathbb{Z}} U.$$

From the viewpoint of algebraic topology, the distinctive feature of covering spaces is their behavior with respect to lifting maps.

Definition 3. Let $p\colon E \to B$ be a covering space. A lift of a map $f\colon X \to B$ is a map $\tilde{f}\colon X \to E$ such that the diagram



commutes.

Proposition 4 ([2, §1.3]). *Given a covering space $p\colon E \to B$, a homotopy $f_t\colon X \to B$, and a map $\tilde{f}_0\colon X \to E$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t\colon X \to E$ of $\tilde{f}_0$ that lifts f_t.*

Taking X to be a point gives the path lifting property for a covering space $p\colon E \to B$, which says that for each path $f\colon I \to X$ starting at x_0 and each $e_0 \in p^{-1}(x_0)$ there is a unique path $\tilde{f}\colon I \to E$ lifting f starting at e_0.

Fiber bundles

Definition 5. A fiber bundle structure on a space E, with fiber F, consists of a projection map $p\colon E \to B$ such that each point $b \in B$ has an open neighborhood U for which there is a homeomorphism $\varphi\colon p^{-1}(U) \to U \times F$ making the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

where pr_1 is projection onto the first factor. Commutativity of the diagram implies that φ carries each fiber $p^{-1}(b)$ homeomorphically onto the copy $\{b\} \times F$ of F.

Given a fiber bundle $p\colon E \to B$ with fiber F, the space B is called the base space and the space E is called the total space. We sometimes denote a fiber bundle by a "short exact sequence of spaces" $F \to E \to B$.

Example 6. Let $E = B \times F$ and $p = \mathrm{pr}_1\colon E \to B$. Then p is a fiber bundle over B with fiber F. Any such fiber bundle is called a trivial bundle.

Example 7. A fiber bundle with fiber a discrete space is a covering space. On the other hand, a covering space whose fibers all have the same cardinality, for example a covering space over a connected space B, is a fiber bundle with discrete fiber.

Although fiber bundles have a lifting property similar to that of Proposition 4 (as we will show later), the lifts are no longer necessarily unique, as the following example shows.

Example 8. Consider the Möbius band $E = I/\!\sim$ where $(0, y) \sim (1, 1 - y)$ for all $y \in I$. Let $B = \{(x, \frac{1}{2}) \in E\} \cong S^1$. The map

$$p\colon E \to B \qquad (x, y) \mapsto (x, \tfrac{1}{2})$$

is a fiber bundle with fiber I. Now, to see that there are two lifts of the same map, consider the paths α and β in E given by

$$t \mapsto (t, \tfrac{1}{2}) \quad \text{and} \quad t \mapsto (t, \tfrac{1}{2} + \tfrac{1}{2}\sin(2\pi t)),$$

respectively. Clearly, $p \circ \alpha = p \circ \beta$ but $\alpha \neq \beta$, that is to say, lifts are no longer unique.

Example 9. Let M be a smooth manifold with or without boundary, and let p be a point in M. A linear map $v\colon C^\infty(M) \to \mathbb{R}$ is called a derivation at p if it satisfies

$$v(fg) = f(p)vg + g(p)vf$$

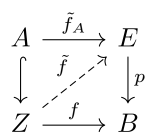
for all $f, g \in C^\infty(M)$. The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the tangent space to M at p. We define the tangent bundle, denoted by TM, to be the disjoint union of the tangent spaces at all points of M, that is,

$$TM = \sqcup_{p\in M} T_pM.$$

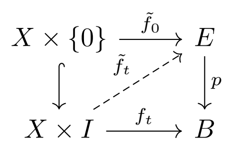
It comes equipped with a projection map $\pi\colon TM \to M$ given by $\pi(p, v) = p$. Moreover, if M is n-dimensional, then the fibers of π are homeomorphic to $\mathbb{R}^n$.

Hurewicz fibrations

Definition 10. A map $p\colon E \to B$ has the lift extension property for a pair (Z, A) if every map $f\colon Z \to B$ has a lift $\tilde{f}\colon Z \to E$ extending a given lift $\tilde{f}_A$ defined on a subspace $A \subset Z$. The following commutative diagram shows the situation.



Definition 11. A map $p\colon E \to B$ has the homotopy lifting property for a space X if it has the lift extension property for the pair $(X \times I, X \times \{0\})$. In this case, the commutative diagram looks as follows.



Definition 12. A Hurewicz fibration, or simply a fibration, is a map $p\colon E \to B$ having the homotopy lifting property for all spaces X.

First, we show some examples, and then we show a few interesting results.

Example 13. A trivial bundle $\mathrm{pr}_1\colon B \times F \to B$ is a fibration. Given a homotopy $f_t\colon X \times I \to B$ and a map $\tilde{f}_0 = (f_0, g)\colon X \times \{0\} \to B \times F$ lifting f_0, we can choose a lift $\tilde{f}_t\colon X \times I \to B \times F$ of the form

$$\tilde{f}_t(x) = (f_t(x), g(x)).$$

Example 14. According to Proposition 4, covering spaces have a property stronger than the homotopy lifting property for all spaces X. It follows that covering spaces are fibrations.

As in the previous example, we would like fiber bundles to be fibrations. Unless we add some assumptions, this is not always true.

Proposition 15 ([5, §2.7])**.** *Fiber bundles over paracompact Hausdorff spaces are fibrations.*

By definition, the fibers $p^{-1}(b)$ of a fiber bundle $F \to E \to B$ are all homeomorphic to F. We have a weaker result for fibrations.

Proposition 16 ([2, §4.3])**.** *For a fibration $p\colon E \to B$, the fibers $p^{-1}(b)$ over each path component of B are all homotopy equivalent.*

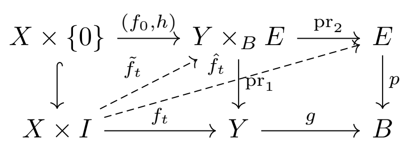
We can build new examples from the old ones using the following proposition.

Proposition 17. *Let $p\colon E \to B$ be a map having the homotopy lifting property for a space X, and $g\colon Y \to B$ be any other map. Then the map $Y \times_B E \to Y$ given by projection onto the first factor has the homotopy lifting property for X where*

$$Y \times_B E = \{(y, e) \in Y \times E\colon p(e) = g(y)\}.$$

This map is called the pullback of p along g.

Proof. Consider the following diagram



Since $p\colon E \to B$ has the homotopy lifting property for X, there is a homotopy $\hat{f}_t\colon X \times I \to E$ making the corresponding triangles commute. Then, the universal property of the pullback gives us the desired homotopy $\tilde{f}_t$. □

In particular, if we apply Proposition 17 for all spaces X, then we get that the pullback of p along g is a fibration whenever p is a fibration.

Example 18 ([1, §5.5])**.** The map $(\mathrm{ev}_0, \mathrm{ev}_1)\colon B^I \to B \times B$ given by

$$\gamma \mapsto (\gamma(0), \gamma(1))$$

is a fibration.

Proposition 19. *Any map $f\colon A \to B$ can be factored as a homotopy equivalence followed by a fibration.*

Proof. Let $E_f = \{(a, \gamma) \in A \times B^I\colon \gamma(0) = f(a)\}$ be the mapping space of f. Since the pullback of $(\mathrm{ev}_0, \mathrm{ev}_1)$ along $f \times \mathrm{id}_B$ is a fibration, the map

$$\mathrm{pr}_1\colon (A \times B) \times_B B^I \to A \times B \qquad (a, b, \gamma) \mapsto (a, b) = (a, \gamma(1))$$

is a fibration. Moreover, since $E_f \cong (A \times B) \times_B B^I$ and the map $\mathrm{pr}_2\colon A \times B \to B$ is a fibration, we have that the composite

$$E_f \to A \times B \to B \qquad (a, \gamma) \mapsto \gamma(1)$$

is a fibration. Therefore, f can be factored as

$$A \xhookrightarrow{\ \simeq\ } E_f \xrightarrow{\text{fibration}} B$$

where the first inclusion given by $a \mapsto (a, \gamma_{f(a)})$ is a homotopy equivalence. □

Proposition 20 ([2, §4.3])**.** *If $p\colon E \to B$ is a fibration, then the inclusion $E \hookrightarrow E_p$ is a fiber homotopy equivalence. In particular, the fibers of the fibrant replacement of p are homotopy equivalent to the actual fibers.*

Example 21. Let $f\colon \{*\} \to X$ be given by $f(*) = x_0$. We have that

$$\begin{aligned} E_f &= \{(*, \gamma) \in \{*\} \times X^I \colon \gamma(0) = f(*) = x_0\} \\ &\cong \{\gamma \in X^I \colon \gamma(0) = x_0\} = PX, \end{aligned}$$

the based path space of the pointed space (X, x_0). In this case, the map $\mathrm{ev}_1\colon PX \to X$ is a fibration, and its fiber over x_0 is the loop space ΩX.

Before we move to the next section, let us show one more interesting example: a fibration that is not a fiber bundle.

Example 22. Let

$$\Delta = \{(x, y) \in \mathbb{R}^2 \colon y - 1 \leq x \leq 1 - y \text{ and } y \geq 0\}.$$

We claim that the map $\mathrm{pr}_1\colon \Delta \to [-1, 1]$ is a fibration but not a fiber bundle. Suppose the square

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{(f_0, g)} & \Delta \\ \big\downarrow & & \big\downarrow \mathrm{pr}_1 \\ X \times I & \xrightarrow{f_t} & [-1, 1] \end{array}$$

commutes. Define $\tilde{f}_t\colon X \times I \to \Delta$ by $\tilde{f}_t(x) = (f_t(x), \alpha(f_t(x), g(x)))$ where $\alpha\colon [-1, 1] \times I \to I$ is given by

$$\alpha(x, y) = \begin{cases} x + 1 & \text{if } -1 \leq x \leq y - 1 \\ y & \text{if } y - 1 \leq x \leq 1 - y \\ 1 - x & \text{if } 1 - y \leq x \leq 1. \end{cases}$$

Clearly, $f_0 = \mathrm{pr}_1 \circ (f_0, g)$. On the other hand,

$$\mathrm{pr}_1 \circ \tilde{f}_0 = f_0 \quad \text{and} \quad (\mathrm{pr}_2 \circ \tilde{f}_0)(x) = \alpha(f_0(x), g(x)) = g(x)$$

because $(f_0(x), g(x)) \in \Delta$, showing that $\tilde{f}_0 = (f_0, g)$. The map is not a fiber bundle because the fibers are not homeomorphic.

Serre fibrations

There is value in a map having the homotopy lifting property for a smaller set of spaces, for example, the disks D^n. Although it is a weaker notion, we can prove several results about it.

Definition 23. A Serre fibration is a map $p\colon E \to B$ having the homotopy lifting property for all disks D^n.

Example 24. Fibrations are Serre fibrations.

Proposition 25. *Suppose $p\colon E \to B$ is a Serre fibration. Let $b \in B$ and $e \in p^{-1}(b) \subset E$. Then the map $p_*\colon \pi_n(E, p^{-1}(b), e) \to \pi_n(B, b)$ is an isomorphism for all $n \geq 1$.*

In contrast to the earlier result that fiber bundles are fibrations, no assumptions are required to show they are Serre fibrations.

Proposition 26. *Fiber bundles are Serre fibrations.*

Quasi-fibrations

Definition 27. Given a map $f\colon A \to B$, the homotopy fiber of f over $b \in B$, denoted by $F(f, b)$, is the fiber of the fibrant replacement. In other words, $F(f, b) = \mathrm{ev}_1^{-1}(b)$ where

$$\mathrm{ev}_1\colon E_f \to B \qquad (a, \gamma) \mapsto \gamma(1)$$

is the fibrant replacement (see Proposition 19). It follows that

$$F(f, b) = \{(a, \gamma) \in A \times B^I \colon \gamma(0) = f(a) \text{ and } \gamma(1) = b\}.$$

If two points of B are in the same path-connected component, then their homotopy fibers are homotopy equivalent (see Proposition 16).

Definition 28. A surjective map $p\colon E \to B$ is a quasi-fibration if the inclusions

$$p^{-1}(b) \hookrightarrow F(p, b) \qquad e \mapsto (e, \gamma_b)$$

are weak equivalences for all $b \in B$.

Example 29. Surjective fibrations are quasi-fibrations (see Proposition 20).

Alternatively, quasi-fibrations may be defined as follows.

Definition 30. A surjective map $p\colon E \to B$ is a quasi-fibration if it induces isomorphisms

$$p_*\colon \pi_n(E, p^{-1}(b), e) \to \pi_n(B, b)$$

for all $b \in B$, $e \in p^{-1}(b)$ and $n \geq 0$ (bijections when $n = 0, 1$) where $\pi_0(E, p^{-1}(b), e)$ is the quotient set $\pi_0(E, e)/i_*(\pi_0(p^{-1}(b), e))$.

Example 31. Serre fibrations satisfy the quasi-fibration condition for all $n \geq 1$ (see Proposition 25). Assuming in addition that B is path-connected, we can show that

$$\pi_0(E, e)/i_*(\pi_0(p^{-1}(b), e)) \to \underbrace{\pi_0(B, b)}_{\cong 0}$$

is a bijection. Equivalently, the map $i_*\colon \pi_0(p^{-1}(b), e) \to \pi_0(E, e)$ is surjective. Let $x \in E$ be an arbitrary point and $\alpha\colon I \to B$ be a path from $p(x)$ to b. The homotopy lifting property for the diagram

$$\begin{array}{ccc} \{0\} & \xrightarrow{x} & E \\ \downarrow & & \downarrow p \\ I & \xrightarrow{\alpha} & B \end{array}$$

implies that there is a lift $\tilde{\alpha}\colon I \to E$, that is, $\tilde{\alpha}$ is a path from x to an element in $e' \in p^{-1}(b)$. It follows that $i_*([e']) = [x]$, as desired. Thus, surjective Serre fibrations over path-connected spaces are quasi-fibrations.

Proposition 32. *Quasi-fibrations have associated long exact sequences of homotopy groups.*

Proof. Let $p\colon E \to B$ be a quasi-fibration. Fix $e \in p^{-1}(b) \subset E$. We know that the relative groups $\pi_n(E, p^{-1}(b), e)$ fit into a long exact sequence

$$\cdots \to \pi_n(p^{-1}(b), e) \to \pi_n(E, e) \to \pi_n(E, p^{-1}(b), e) \to \cdots \to \pi_0(E, p^{-1}(b), e) \to 0$$

It follows that we have a long exact sequence as follows

$$\cdots \to \pi_n(p^{-1}(b), e) \to \pi_n(E, e) \to \pi_n(B, b) \to \cdots \to \pi_0(B, b) \to 0. \qquad \square$$

Remark. If B is path-connected then any covering space, fiber bundle, fibration or Serre fibration is a quasi-fibration. It follows that all these types of maps have associated long exact sequences of homotopy groups.

Next, we present an example of a quasi-fibration that is not a Serre fibration.

Example 33. Consider the L shaped space

$$L = \underbrace{\{(x, 0) \in \mathbb{R}^2 \colon x \in I\}}_{A} \cup \underbrace{\{(0, y) \in \mathbb{R}^2 \colon y \in I\}}_{B}.$$

We claim that the map $\mathrm{pr}_1\colon L \to I$ is a quasi-fibration that is not a Serre fibration. Clearly, we have a commuting square

$$\begin{array}{ccc} \{0\} & \xrightarrow{p} & L \\ \downarrow & & \downarrow \mathrm{pr}_1 \\ I & \xrightarrow{\alpha} & I \end{array}$$

where $\alpha = \mathrm{id}_I$ and p is choosing the point $(0, 1)$ in L. Suposse, by contraction, that there is a lift $\tilde{\alpha}\colon I \to L$. We have that $\tilde{\alpha}(0) = (0, 1)$. On the other hand, $\mathrm{pr}_1 \circ \tilde{\alpha} = \alpha = \mathrm{id}_I$ implies that $\tilde{\alpha}(t) = (t, 0)$ for all $t > 0$ because $\mathrm{pr}_1^{-1}(t) = \{(t, 0)\}$ for $t > 0$. It follows that $\lim_{t\to 0} \tilde{\alpha}(t) = (0, 0)$, a contradiction because $\tilde{\alpha}(0) = (0, 1)$. Thus, pr_1 is not a Serre fibration.

Now, let us show that pr_1 is a quasi-fibration. Let $b \in I$ and $e \in \mathrm{pr}_1^{-1}(b)$. We want to show that

$$(\mathrm{pr}_1)_* \colon \pi_n(L, \mathrm{pr}_1^{-1}(b), e) \to \underbrace{\pi_n(I, b)}_{\cong 0}$$

is an isomorphism for all $n \geq 0$. Since L, $\mathrm{pr}_1^{-1}(0) = B$ and $\mathrm{pr}_1^{-1}(b) = \{(b, 0)\}$ (for all $b > 0$) are contractible spaces, the long exact sequence for the pair $(L, \mathrm{pr}_1^{-1}(b))$ implies that

$$\pi_n(L, \mathrm{pr}_1^{-1}(b), e) = 0$$

for all $n \geq 0$, showing that $(\mathrm{pr}_1)_*$ is an isomorphism for all $n \geq 0$ and choices of basepoints. Therefore, pr_1 is a quasi-fibration.

Example 34. Let $\Delta \colon X \to X \times X$ be the diagonal map, that is, $\Delta(x) = (x, x)$. Let $p \colon E_\Delta \to X \times X$ given by $(x, \gamma) \mapsto \gamma(1)$ be a fibrant replacement where

$$E_\Delta = \{(x, \gamma) \in X \times (X \times X)^I \colon \gamma(0) = \Delta(x) = (x, x)\}$$

(recall that $X \hookrightarrow E_\Delta$ is a homotopy equivalence). Fix $(x_0, x_0) \in X \times X$. We have that

$$F(\Delta, x_0) = \{(x, \gamma) \in X \times (X \times X)^I \colon \gamma(0) = (x, x) \text{ and } \gamma(1) = (x_0, x_0)\}.$$

In other words, an element in the homotopy fiber is a point $x \in X$ and a pair of paths $\alpha, \beta \colon I \to X$ from x to x_0. We claim that the map

$$F(\Delta, x_0) \to \Omega X \qquad (x, \gamma) \mapsto \alpha^{-1} * \beta$$

is a homeomorphism where ΩX is the space of loops based on x_0. We have a long exact sequence of homotopy groups of the form

$$\cdots \to \underbrace{\pi_n(E_\Delta)}_{\cong \pi_n(X)} \xrightarrow{\Delta_*} \pi_n(X \times X) \xrightarrow{\delta} \underbrace{\pi_{n-1}(\Omega X)}_{\cong \pi_n(X)} \to \cdots.$$

Since the map $\Delta_* \colon \pi_n(X) \to \pi_n(X \times X)$ is injective, we get short exact sequences of the form

$$0 \to \pi_n(X) \xrightarrow{\Delta_*} \pi_n(X \times X) \xrightarrow{\delta} \pi_n(X) \to 0$$

that is split exact. Thus, $\pi_n(X \times X) \cong \pi_n(X) \oplus \pi_n(X)$ for $n \geq 1$.

Homology fibrations

Definition 35 ([3, §5])**.** A map $p \colon E \to B$ is a homology fibration if the inclusions

$$p^{-1}(b) \hookrightarrow F(p, b)$$

induce isomorphisms on homology for all $b \in B$, that is to say,

$$i_* \colon H_n(p^{-1}(b)) \to H_n(F(p, b))$$

is an isomorphism for all $n \geq 0$ and all $b \in B$.

Example 36. Quasi-fibrations are homology fibrations because weak equivalences induce homology equivalences.

Example 37. Let A be an acyclic space. Let CA be the pushout

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & \{*\} \\ \big\downarrow & & \big\downarrow \\ A \times I & \longrightarrow & CA \end{array}$$

that is, $CA = (A \times I)/\sim$ where $(a, 0) \sim (a', 0)$ for all $a, a' \in A$. We claim that the map $p\colon CA \to I$ defined by $(a, b) \mapsto b$ is a homology fibration. We have that $p^{-1}(0) \cong \{*\}$ and $p^{-1}(b) \cong A$, an acyclic space, for all $b > 0$. Therefore,

$$H_n(p^{-1}(b)) \cong H_n(\{*\})$$

for all $n \geq 0$. On the other hand,

$$F(p, 0) = \{(a, b, \gamma) \in CA \times I^I : \gamma(0) = b \text{ and } \gamma(1) = 0\}.$$

Let $D = \{(a, 0, c_0)\} \subseteq F(p, 0)$ where c_0 is the constant path at 0 (recall that $(a, 0) \sim (a', 0)$ for all $a, a' \in A$). On one hand, $D \cong \{*\}$. On the other hand, consider the homotopy

$$f_t\colon F(p, 0) \to F(p, 0) \qquad (a, b, \gamma) \mapsto (a, \gamma_\ell(0), \gamma_\ell)$$

where $\gamma_\ell\colon I \to I$ is given by $\gamma_\ell(t) = \gamma(\ell + (1 - \ell)t)$. We have that $\gamma_0 = \gamma$ and $\gamma_1 = c_0$. It follows that

$$f_0(a, b, \gamma) = (a, \gamma_0(0), \gamma_0) = (a, \gamma(0), \gamma) = (a, b, \gamma),$$

that is, $f_0 = \mathrm{id}_{F(p,0)}$. Additionally,

$$f_1(a, b, \gamma) = a, \gamma_1(0), \gamma_1) = (a, c_0(0), c_0) = (a, 0, c_0) \in D.$$

We conclude that f_t is a deformation retraction of $F(g, b)$ onto D. Thus,

$$H_n(F(g, b)) \cong H_n(D) \cong H_n(\{*\}),$$

showing that p is a homology fibration. Now, suppose that $\pi_k(A)$ is nonzero for some k. We have that

$$\pi_k(p^{-1}(b)) \cong \pi_k(A) \ncong \pi_k(\{*\}) \cong \pi_k(F(p, b)),$$

showing that f is not a quasi-fibration.

Serre spectral sequence

Theorem 38 ([4, §8]). *Let $f\colon E \to B$ be a Serre fibration. Then, there is a first quadrant spectral sequence $\{E^r, d^r\}_{r\geq 2}$, called the Serre spectral sequence, with second page $E^2_{p,q} = H_p(B; \underline{H_q}(F))$, the singular homology with coefficients in the local system induced by the fibers of f. The spectral sequence converges to the singular homology of the space E. This is denoted by*

$$E^2_{p,q} = H_p(B; \underline{H_q}(F)) \implies H_{p+q}(E).$$

Remark. If B is simply connected, then any local system on B is isomorphic to a constant local system on B. In this case, $E^2_{p,q} = H_p(B; H_q(F))$ where $F = f^{-1}(b)$ for any $b \in B$.

Remark. If $f\colon E \to B$ is a Serre fibration, then the map $b \mapsto H_q(f^{-1}(b))$ defines a local system of abelian groups. Now suppose f is only a homology fibration. In this case, the homology of the actual fiber at b agrees with the homology of the homotopy fiber at b, showing that the map $b \mapsto H_q(f^{-1}(b))$ still gives a local system of abelian groups. Therefore, we can take a fibrant replacement of f, form its Serre spectral sequence, and call that the Serre spectral sequence of the homology fibration. Note that the E^2 page is still expressed in terms of the actual fibers.

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