

WRITE UP FOR TREVOR KARN'S TALK 1

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1. HOMOLOGICAL STABILITY

These notes are a write up of Trevor Karn's talk at the Summer school in configuration space. The goal for the first talk and these notes is to:

- (1) Establish notation - make sure that we have consistent notation when talking about configuration space
- (2) Provide background for Segal's homological stability theorem.

We will assume basic facts about manifolds, about homology and cohomology, and about relative homology. All cohomology and homology groups that appear will be with coefficients in \mathbb{Z} , and as such, in these notes we will suppress the coefficients to make notation slightly neater.

Given a topological space X , homology is a nice (functorial) way to assign for each n a \mathbb{Z} -module $H_n(X)$. Our goal is to understand what happens to the homology when we have a sequence of spaces as we vary the spaces in the sequence.

Example 1.1. Consider the infinite family of spaces $\{S^n\}_{n \geq 1}$ of n -dimensional spheres. We have that

$$H_d(S^n) = \begin{cases} 0, & \text{if } d \neq 0, n \\ \mathbb{Z}, & \text{if } d = 0, n \end{cases}$$

We have the following collection of data:

$$\begin{aligned} H_0(S^1) &= H_0(S^2) = \dots = \mathbb{Z} \\ H_1(S^1) &= \mathbb{Z}, H_1(S^2) = H_1(S^3) = \dots = 0 \\ H_2(S^1) &= 0, H_2(S^2) = \mathbb{Z}, H_2(S^3) = H_2(S^4) = \dots = 0 \end{aligned}$$

In general, if we fix d , we see the d th homology eventually stops changing. More precisely, for any d and $n \geq d + 1$, we have an isomorphism $H_d(S^n) \cong H_d(S^{n+1})$. This general phenomenon that eventually we have an isomorphism on homology is known as homological stability.

Definition 1.2. We say a sequence $\{X_n\}$ of topological spaces along with maps $f_n : X_n \rightarrow X_{n+1}$ is a *homologically stable sequence* if for all d and for all n large enough depending on d , f_n induces an isomorphism on homology $H_d(X_n) \cong H_d(X_{n+1})$. If n is chosen to be large enough relative to d as in this definition, we say that n is in the *stable range*.

We note that in example 1.1, we didn't explicitly provide maps between S^n and S^{n+1} . Because in the stable range, both homologies are 0, the choice of map doesn't matter in that particular example. However, in general, we will have to find maps f_n that provide the desired isomorphisms on homology in the stable range.

2. CONFIGURATION SPACES

The main family of spaces we will care about for the remainder of this talk and also for future talks in this Summer school are unordered configuration spaces.

Definition 2.1. Let M be a manifold. Then pure or *ordered configuration space* on M is

$$\text{PConf}_n(M) := \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

The condition that $x_i = x_j$ is a closed condition on the cartesian product M^n , so $\text{PConf}_n(M)$ is an open subset of M^n and therefore also a manifold. We have an action of the symmetric group \mathfrak{S}_n on $\text{PConf}_n(M)$ by permuting coordinates.

Definition 2.2. *Unordered configuration space* is

$$\text{UConf}_n(M) := \text{PConf}_n(M)/\mathfrak{S}_n$$

Ordered and unordered configuration spaces are both tools to keep track of all ways to list n distinct points on a manifold M . Ordered configuration space remembers the order of these n points while unordered configuration space only cares about the points themselves but forgets what order they were listed in.

Example 2.3. When $M = \{z \in \mathbb{C} : |z| < 1\}$, then we can define a map $\text{UConf}_n(M) \rightarrow \text{UConf}_{n+1}(M)$ as follows: We map an element $\{z_1, \dots, z_n\} \in \text{UConf}_n(M)$ to $\{\frac{z_1}{2}, \dots, \frac{z_n}{2}, \frac{3}{4}\}$. We divide all of our points by 2 so that we have n points in M all with magnitude $< \frac{1}{2}$. Therefore, when we choose a point with magnitude $\geq \frac{1}{2}$, it will be a distinct point from the first n points.

In summary, to get a map $\text{UConf}_n(M) \rightarrow \text{UConf}_{n+1}(M)$ we needed a way of choosing an $n+1$ st point sufficiently far from the first n points. We call this process of choosing such an $n+1$ st point "adding a point at ∞ ," and while we did it very explicitly for $M = \{z \in \mathbb{C} : |z| < 1\}$, this process of "adding a point at ∞ " can be done for any noncompact connected manifold M .

Now that we have maps $f_n : \text{UConf}_n(M) \rightarrow \text{UConf}_{n+1}(M)$, we can ask when do we have homological stability?

Theorem 2.4. *Arnol'd, 1969.* If $M = \mathbb{C}$, then have homological stability $H_d(\text{UConf}_n(\mathbb{C})) \cong H_d(\text{UConf}_{n+1}(\mathbb{C}))$ whenever

$$\lfloor \frac{n}{2} \rfloor \geq d$$

Theorem 2.5. *Segal, 1979. If M is a noncompact connected manifold then we have homological stability $H_d(UConf_n(M)) \cong H_d(UConf_{n+1}(M))$ whenever*

$$\left\lfloor \frac{n}{2} \right\rfloor \geq d$$

3. COHOMOLOGY WITH COMPACT SUPPORT

Let X be a locally compact Hausdorff topological space. We can add a point to X to obtain a new space $X^+ = X \cup \{\infty\}$. X^+ is a topological space where $U \subset X^+$ is an open set if either $U \subset X$ and U is open in X or $U = X \setminus C \cup \{\infty\}$ where C is a compact subset of X . We call X^+ the *one point compactification* of X , and we note that X^+ is compact.

We give three definitions of cohomology with compact support.

Definition 3.1.

- (1) Let $C_c^i(X)$ be the collection of cochains $\varphi : C_i(X) \rightarrow \mathbb{Z}$ such that there exists a compact set K where φ is zero on all chains contained in $X \setminus K$. The *cohomology with compact support* is $H_c^i(X)$ is the cohomology of this cochain complex.
- (2) For each compact set $K \subset X$ we can look at the relative cohomology $H^i(X, X \setminus K)$. If $K \subset L$, we have an induced map

$$H^i(X, X \setminus K) \rightarrow H^i(X, X \setminus L).$$

If we partially order compact subsets of X by inclusion, the cohomology with compact support of X is the limit

$$H_c^i(X) = \varinjlim H^i(X, X \setminus K)$$

- (3) $H_c^i(X) := H_i(X^+, \infty)$

The equivalence of the first two definitions is found on page 244 of Hatcher, and the equivalence of the third definition with the first two definitions is exercise 21 of chapter 3 of Hatcher or follows from proposition 4.3 of these notes.

4. POINCARÉ DUALITY (WHY DO WE CARE ABOUT COHOMOLOGY WITH COMPACT SUPPORT)

Theorem 4.1. *Poincare Duality Version 1 [1, Hatcher 3.30] Let M be a closed orientable d manifold. Then there exists an isomorphism $H^i(M) \cong H_{d-i}(M)$*

When M is compact, $H_c^i(M) \cong H^i(M)$. This follows for example from definition two of $H_c^i(X)$ because the directed system of compact subspaces of X has a unique maximal element $K = X$.

Theorem 4.2. *Poincare Duality Version 2 [1, Hatcher 3.35] Let M be an orientable d manifold. Then we have an isomorphism $H_c^i(M) \cong H_{d-i}(M)$*

The following proposition is a collection of useful facts about $H_c^i(X)$.

Proposition 4.3.

- (1) Let Y be compact and $X \subset Y$ be open. Then $H_c^i(X) = H^i(Y, Y \setminus X)$.
- (2) If $U \rightarrow X$ is an open embedding we get maps $H_c^i(U) \rightarrow H_c^i(X)$.
- (3) If $Z \rightarrow X$ is a proper map meaning the preimages of compact sets is compact, then it induces a map $H_c^i(X) \rightarrow H_c^i(Z)$.

Proof of (2) and (3). We use our first definition of cohomology with compact support. If $U \subset X$ is open, and $\varphi \in C_c^i(U)$, then we can extend φ by 0 to a map on $C_i(X)$. In other words we set $\varphi(\sigma) = 0$ for every new simplex $\sigma \in C_i(X) \setminus C_i(U)$. This map $C_c^i(U) \rightarrow C_c^i(X)$ induces the map $H_c^i(U) \rightarrow H_c^i(X)$.

For (3), let $f : Z \rightarrow X$ be a proper map. Then we have $f^* : C^i(X) \rightarrow C^i(Z)$. If $\varphi \in C_c^i(X)$, then by definition, φ is zero on all simplices contained in $X \setminus K$ for some compact set K . $f^*(\varphi)$ is nonzero on all simplices contained in $Z \setminus f^{-1}(K)$ so since $f^{-1}(K)$ is compact, $f^*(\varphi) \in C_c^i(Z)$. In particular $f^* : C_c^i(X) \rightarrow C_c^i(Z)$ induces the map $H_c^i(X) \rightarrow H_c^i(Z)$. □

Using the fact that closed embeddings are proper, for any topological space X with $C \subset X$ closed, and $U = X \setminus C$ parts (2) and (3) of the previous proposition give us maps $H_c^*(U) \rightarrow H_c^*(X)$ and $H_c^*(X) \rightarrow H_c^*(C)$.

Proposition 4.4. *There exists a long exact sequence on cohomology*

$$\dots \rightarrow H_c^{d-1}(C) \rightarrow H_c^d(U) \rightarrow H_c^d(X) \rightarrow H_c^d(C) \rightarrow H_c^{d+1}(U) \rightarrow \dots$$

Proof. Consider the inclusion of pairs $(C^+, \infty) \rightarrow (X^+, \infty)$. This map induces a long exact sequence of pairs

$$\begin{aligned} \dots &\rightarrow H^{d-1}(C^+, \infty) \\ &\rightarrow H^d(X^+, C^+) \rightarrow H^d(X^+, \infty) \rightarrow H^d(C^+, \infty) \rightarrow H^{d+1}(X^+, C^+) \rightarrow \dots \end{aligned}$$

The statement follows since $H^d(X^+, C^+) = H_c^d(U)$ by proposition 4.3. □

We should think of this long exact sequence as some sort of analogue for the Mayer-Vietoris sequence. The Mayer-Vietoris sequence is a tool to help us understand the cohomology of a space X provided we can understand the cohomology of A , B and $A \cap B$ where $X = A \cup B$ for A and B open. Where the Mayer-Vietoris sequence is a long exact sequence containing information about the cohomologies of X , A , B , and $A \cap B$, the previous proposition gives us a similar long exact sequence involving the cohomologies with compact support of X , C , and U where $C \subset X$ is closed and U is its complement.

5. SEGAL MAP

Unordered configuration spaces are spaces where we choose n points and require all n points to be distinct. Define the n -fold symmetric product of M by $A_n(M) = M^n / \mathfrak{S}_n$. This is analogous to $\text{UConf}_n(M)$ except here we allow

the points to repeat. Segal introduces a filtration of spaces $A_{n,0} \supset A_{n,1} \supset \dots$ where $A_{n,k} = \emptyset$ for $2k \geq n$. This filtration allows Segal to use downward induction on k to help prove Theorem 2.5. The details of this filtration will be defined in a later talk, so this paragraph will be our only brief mention of this filtration. We mention that $A_{n,0} = A_n$ and $\text{UConf}_n(M) = A_{n,0} \setminus A_{n,1}$, and that for much of what we say in the remainder of this section about $\text{UConf}_n(M)$, a similar statement can also be made about $A_{n,k}$. Notably, the construction of “adding a point at ∞ ” carries through with little change for $A_{n,k}$.

We would like to study the map $\text{UConf}_n(M) \rightarrow \text{UConf}_{n+1}(M)$ by “adding a point at ∞ .” Unfortunately this map is neither open nor proper, so in order to use the machinery of cohomology with compact support, it will be helpful to replace our map with an open map. If we let $q = \dim M$, as in the figure below, we obtain the map $\text{UConf}_n(M) \times \mathbb{R}^q \rightarrow \text{UConf}_{n+1}(M)$ where \mathbb{R}^q is a neighborhood around the point at ∞ . This map is open and therefore induces a map on cohomology with compact support.

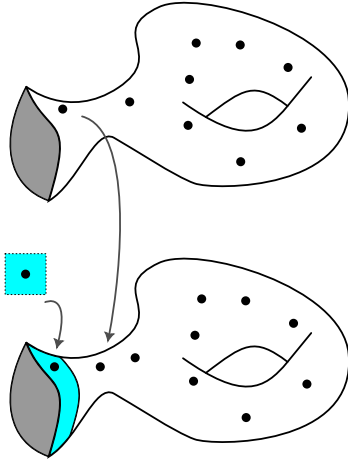


FIGURE 1. An embedding $\mathbb{R}^2 \times \text{UConf}_9(M) \rightarrow \text{UConf}_{10}(M)$, for an infinite genus surface M .

Remark 5.1. Cohomology with compact support is not homotopy invariant, and it would be false to think that $\text{UConf}_n(M) \times \mathbb{R}^q$ and $\text{UConf}_n(M)$ have the same cohomology with compact support, because even though \mathbb{R}^q is contractible, its one point compactification S^q is not. We instead get an equivalence $H_c^i(\text{UConf}_n(M) \times \mathbb{R}^q) \simeq H_c^{i+q}(\text{UConf}_n M)$. This relation of cohomology with compact support is the content of exercise 22 in chapter 3.3 of Hatcher.

Supposing M is orientable, since $\text{UConf}_n(M) \times \mathbb{R}^q$ has dimension $nq + q = (n + 1)q$, we can use Poincaré duality to study

$$H_c^i(\text{UConf}_n(M) \times \mathbb{R}^q) = H_{(n+1)q-i}(\text{UConf}_n(M)).$$

Instead of showing an equivalence on homology for configuration space, Segal instead shows the map $\mathbb{R}^q \times \text{UConf}_n(M) \rightarrow \text{UConf}_{n+1}(M)$ induces an equivalence on cohomology with compact support above dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor \geq d$. It turns out that even in the nonorientable case, this equivalence of compactly supported cohomology is one of the main ingredients to prove Segal's theorem on homological stability of $\text{UConf}_n(M)$.

REFERENCES

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