Talk 3
Braid Groups and the $K(\pi, 1)$ Property

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1 Defining Ordered and Unordered Configuration Spaces

Definition 1.1 (Ordered Configuration Spaces, [2])**.** Let X be a topological space. The (ordered) configuration space of n points on X is given by

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for each } i \neq j\}.$$

The space is topologized by the subspace topology on X^n.

- $\mathrm{Conf}_n(X)$ can be viewed as a collection of distinct configurations (i.e. n distinctly *labeled* points in the space X).
- Each configuration represents a single point in $\mathrm{Conf}_n(X)$.
- Forgetting the $(n + 1)$-st point yields a map yields a map $\mathrm{Conf}_{n+1}(X) \to \mathrm{Conf}_n(X)$.

If we forget the labels on n points in $\mathrm{Conf}_n(X)$, we obtain the *unordered* configuration space.

Definition 1.2 (Unordered Configuration Spaces, [2])**.** Let X be a topological space. The unordered configuration space of n points on X is given by

$$\overline{\mathrm{Conf}}_n(X) := \mathrm{Conf}_n(X)/\Sigma_n.$$

1.1 Why Do We Care?

The following examples illustrate the significance of configuration spaces as *homeomorphism* invariants, which are more sensitive than homotopy invariants.

Example 1.3 (Lens Spaces, [2], [3]). Consider the 3-dimensional lens space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$. For $S^3 \subset \mathbb{C}^2$, the generator of $\mathbb{Z}/p\mathbb{Z}$ acts by

$$(z_1, z_2) \mapsto \left(e^{2\pi i/p} z_1,\ e^{2\pi i q/p} z_2\right).$$

A famous example is a pair of homotopy equivalent spaces that are not homeomorphic:

$$L(7,1) \simeq L(7,2) \quad \text{while} \quad L(7,1) \not\cong L(7,2).$$

Usual homotopy invariants (such as π_1, homology, etc.) fail to distinguish these spaces. However, their configuration spaces can be distinguished: Longoni–Salvatore showed that the universal covers of these configuration spaces have different Massey products, and hence are not homotopy equivalent [3].

1.2 Computing $\mathrm{Conf}_n(X)$ and $\overline{\mathrm{Conf}}_n(X)$

Example 1.4 ($\mathbb{R}$, [2]). Since $\mathbb{R} \cong (0,1)$, we analyze the configuration space of k points in the interval $(0,1)$.

- $\mathrm{Conf}_k((0,1))$. By definition,

  $$\mathrm{Conf}_k((0,1)) = \{(x_1, \dots, x_k) \in (0,1)^k \mid x_i \neq x_j \text{ for } i \neq j\}.$$

  Since $(0,1)$ is linearly ordered, each configuration determines a unique permutation

  $$\sigma \in \Sigma_k \quad \text{such that} \quad x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(k)}.$$

  This assignment is locally constant, so each permutation determines a connected component. Hence,

  $$\pi_0(\mathrm{Conf}_k((0,1))) \cong \Sigma_k.$$

  Each connected component is given by

  $$\{(x_1, \dots, x_k) \mid x_{\sigma(1)} < \cdots < x_{\sigma(k)}\},$$

  which is homeomorphic to the open simplex

  $$\Delta_k^\circ = \left\{(t_1, \dots, t_k) \in \mathbb{R}^k \mid t_i > 0,\ \sum t_i < 1\right\}.$$

  Thus,

  $$\mathrm{Conf}_k((0,1)) \cong \bigsqcup_{\sigma \in \Sigma_k} \Delta_k^\circ.$$

- $\overline{\mathrm{Conf}}_k((0,1))$. The unordered configuration space is defined by

  $$\overline{\mathrm{Conf}}_k((0,1)) = \mathrm{Conf}_k((0,1))/\Sigma_k.$$

The symmetric group Σ_k acts by permuting coordinates, and this action simply permutes the connected components. Therefore, after taking the quotient, all components are identified, and we obtain

$$\overline{\mathrm{Conf}}_k((0,1)) \cong \Delta_k^\circ.$$

Since the open simplex Δ_k° is contractible, we conclude that

$$\overline{\mathrm{Conf}}_k((0,1)) \simeq \mathrm{pt}.$$

Summary. The topology of configurations in $(0,1)$ is entirely determined by the ordering of points. After forgetting labels, no topological information remains, and the space becomes contractible.

Example 1.5 ($\mathrm{Conf}_2(\mathbb{R}^n)$, [2])**.** Each configuration is determined by three pieces of information — the direction, the distance, and the center of mass — which defines a homeomorphism

$$\mathrm{Conf}_2(\mathbb{R}^n) \xrightarrow{\cong} S^{n-1} \times \mathbb{R}_{>0} \times \mathbb{R}^n,$$

$$(x_1, x_2) \mapsto \left(\frac{x_2 - x_1}{\|x_2 - x_1\|}, \|x_2 - x_1\|, \frac{x_1 + x_2}{2} \right).$$

Since $\mathbb{R}_{>0}$ and $\mathbb{R}^n$ are contractible, projection onto the first factor (i.e. direction)

$$\mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$$

is a homotopy equivalence. Permuting the points $(x_1, x_2) \mapsto (x_2, x_1)$ acts on S^{n-1} by the antipodal map $v \mapsto -v$. Thus,

$$\overline{\mathrm{Conf}}_2(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}_{>0} \times \mathbb{RP}^{n-1},$$

and in particular,

$$\overline{\mathrm{Conf}}_2(\mathbb{R}^n) \simeq \mathbb{RP}^{n-1}.$$

Example 1.6 (S^1, [2])**.**

- $\mathrm{Conf}_2(S^1)$ and $\overline{\mathrm{Conf}}_2(S^1)$

  We consider

  $$\mathrm{Conf}_2(S^1) := \{(x_1, x_2) \in (S^1)^2 \mid x_1 \neq x_2\}.$$

  Identifying $S^1 \cong \mathbb{R}/\mathbb{Z}$, we may write

  $$\mathrm{Conf}_2(S^1) = \{(\theta_1, \theta_2) \in [0,1)^2 \mid \theta_1 \neq \theta_2\}.$$

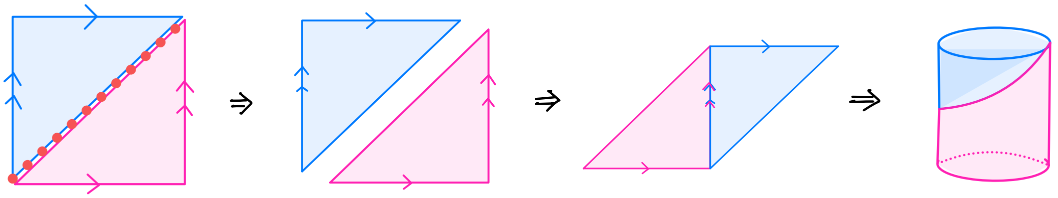
  Geometrically, this is the unit square with the diagonal $\theta_1 = \theta_2$ removed. Thus it decomposes into two open regions:

  $$\theta_1 < \theta_2 \quad \text{and} \quad \theta_1 > \theta_2,$$

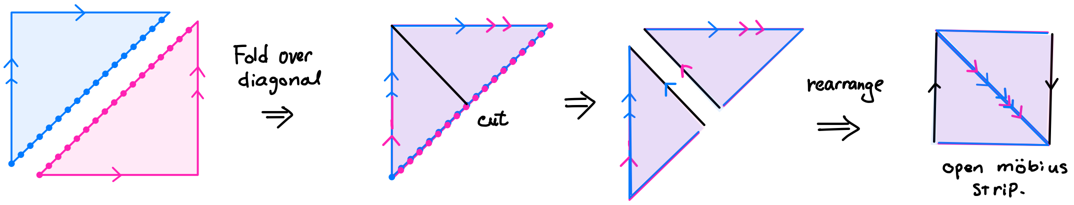
  each of which is contractible.

  The square model of $S^1 \times S^1$ is obtained by identifying opposite edges of the unit square. The topology of $\mathrm{Conf}_2(S^1)$ is therefore determined by how these two regions are glued together along their boundaries under these identifications.

  - In the *ordered case*, the two regions remain distinct, and the edge identifications preserve orientation. Gluing the boundaries accordingly produces a *cylinder*.



  - In the *unordered case*, we further quotient by the action of the symmetric group, identifying $(x_1, x_2) \sim (x_2, x_1)$. Geometrically, this corresponds to exchanging the two regions, which reverses orientation. The resulting space is a *Möbius strip*.



- $\mathrm{Conf}_k(S^1)$

  Let us identify $S^1 = \mathbb{R}/\mathbb{Z}$. Consider the subspace $A \subset \mathrm{Conf}_k(S^1)$ consisting of configurations $(x_1, \ldots, x_k)$ whose ordering agrees with the cyclic order induced by the orientation of S^1. That is,

  $$A = \left\{(x_1, \ldots, x_k) \in \mathrm{Conf}_k(S^1) \;\middle|\; (x_1, \ldots, x_k) \text{ appear in this order on } S^1\right\}.$$

  For such configurations, define

  $$t_i = x_{i+1} - x_i \in (0,1), \quad 1 \leq i \leq k,$$

  where we set

  $$t_k = (1 + x_1) - x_k.$$

  Intuitively, each t_i indicates the distance (spacing) between neighboring points. Then, since all k points are on a circle whose circumference corresponds to the unit interval $[0,1]$,

  $$\sum_{i=1}^{k} t_i = 1.$$

  Thus the collection $(t_1, \ldots, t_k)$ defines a point in the open simplex

  $$\Delta_\circ^{k-1} = \{t_i > 0 \mid \sum t_i = 1\}.$$

  Together with the choice of $x_1 \in S^1$, this determines the configuration, giving a homeomorphism

  $$A \cong S^1 \times \Delta_\circ^{k-1}.$$

In the subspace $A \times \Sigma_k$, a configuration is described by a starting point on S^1, a cyclic ordering of the labels, and spacing parameters $t_1, \ldots, t_k \in \Delta_\circ^{k-1}$. However, the choice of starting point is not intrinsic: shifting the starting point cyclically produces the same configuration.

Thus, configurations that differ by a cyclic rotation should be identified. To account for this redundancy, we mod out by the cyclic group $C_k := \langle (1\,2\,\cdots\,k) \rangle$:

$$\mathrm{Conf}_k(S^1) \cong (S^1 \times \Delta_\circ^{k-1}) \times_{C_k} \Sigma_k.$$

- $\overline{\mathrm{Conf}}_k(S^1)$

  By the above discussion, we have

  $$\overline{\mathrm{Conf}}_k(S^1) = \mathrm{Conf}_k(S^1)/\Sigma_k \cong (S^1 \times \Delta_\circ^{k-1}) \simeq S^1.$$

1.3 Forgetting a Point

Let M be a manifold. Natural projections from the product factor through the configuration space:

$$\mathrm{Conf}_k(M) \longrightarrow \mathrm{Conf}_{k-1}(M) \xrightarrow{\pi_i} M,$$

where $\pi_i(x_1, x_2, \ldots, x_{k-1}) = x_i$. In fact, the following results are due to Fadell and Neuwrith [1]. In their paper, they use the

$$F_{m,n}(M) := \mathrm{Conf}_n(M \setminus \{m \text{ points}\}).$$

Here we consider the case where $m = 0$, so that $F_{m,n}(M) = \mathrm{Conf}_n(M)$, and interpret their results showing that the map forgetting a point $\mathrm{Conf}_{k+1}(M) \to \mathrm{Conf}_k(M)$ defined by

$$(x_1, \ldots, x_{k+1}) \longrightarrow (x_1, \ldots, x_k)$$

is a fiber bundle.

Theorem 1.7 ([1]). *$\pi : \mathrm{Conf}_n(M) \to M$ is a locally trivial fiber bundle with fiber $\mathrm{Conf}_{n-1}(M \setminus \{y\})$, where $y \in M$.*

Proof. Let $y \in M$ be a chosen point, and fix some other point $x_0 \in M$. Let $\alpha : M \to M$ be a homeomorphism such that $\alpha(y) = x_0$. Let U be a Euclidean neighborhood of x_0. Let $\theta : U \times \overline{U} \to \overline{U}$ denote a map with the following properties:

1. For each $x \in U$, $\theta(x, -) : \overline{U} \to \overline{U}$ is a homeomorphism that fixes ∂U.
2. For each $x \in U$, $\theta(x, x) = x_0$.

We can extend θ to $U \times M \to M$ by setting $\theta(x, y) = y$ for each $y \notin U$.

The goal is to construct a homeomorphism

$$\varphi : \pi^{-1}(U) \to U \times \mathrm{Conf}_{n-1}(M \setminus \{x_0\}).$$

Define

$$\varphi(x, p_2, \ldots, p_n) = \left(x,\ \theta^{-1}(x, -)(\alpha(p_2)), \ldots, \theta^{-1}(x, -)(\alpha(p_n))\right).$$

Then the inverse is given by

$$\varphi^{-1}(x, p_2, \ldots, p_n) = \left(x,\ \alpha^{-1}(\theta(x, p_2)), \ldots, \alpha^{-1}(\theta(x, p_n))\right),$$

and we have that $\varphi\varphi^{-1} = \varphi^{-1}\varphi$ as we wanted. $\square$

The proof of Theorem 1.7 can be extended to obtain the following theorem:

Theorem 1.8 ([1])**.** *$\pi : \mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ defined by*

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$$

is a locally trivial fiber bundle with fiber

$$\mathrm{Conf}_1(M \setminus \{x_1, \dots, x_{n-1}\}) \cong M \setminus \{x_1, \dots, x_{n-1}\}.$$

1.4 Forgetting the Ordering

So far, we have seen that there is a natural map

$$\mathrm{Conf}_n(M) \to \overline{\mathrm{Conf}}_n(M)$$

obtained by forgetting the ordering of the points. For each unordered configuration of n distinct points, the fiber consists of all possible orderings of these points. Therefore, each fiber is naturally identified with the symmetric group Σ_n, which is a discrete space consisting of $n!$ elements.

Moreover, this map is locally trivial: around each unordered configuration, there exists a neighborhood U such that

$$\pi^{-1}(U) \cong U \times \Sigma_n.$$

Hence, the projection

$$\mathrm{Conf}_n(M) \to \overline{\mathrm{Conf}}_n(M)$$

is a fiber bundle with discrete fiber Σ_n. Since the fibers are discrete, this map is also an $n!$-sheeted covering map.

The following corollary illustrates how this covering space structure can be used to compute homotopy invariants, such as the fundamental group.

Corollary 1.9 ([2]). *Suppose M is a simply-connected n-dimensional manifold with $n \geq 3$. Then $\mathrm{Conf}_k(M)$ is simply connected for all $k \geq 0$. In particular, $\pi_1(\overline{\mathrm{Conf}}_k(M)) \cong \Sigma_k$.*

Proof. We proceed in two steps.

Step 1: $\mathrm{Conf}_k(M)$ **is simply connected.**

We induct on k. The cases $k = 0$ and $k = 1$ are immediate: the former is trivial, and the latter $\mathrm{Conf}_1(M) \cong M$ is true by assumption.

For the inductive step, assume that $\mathrm{Conf}_{k-1}(M)$ is simply connected. Recall from Theorem 1.8 that the map forgetting the last point,

$$\pi \colon \mathrm{Conf}_k(M) \to \mathrm{Conf}_{k-1}(M),$$

is a locally trivial fiber bundle with fiber $F = M \setminus \{x_1, \dots, x_{k-1}\}$. Thus, the long exact sequence of homotopy groups gives

$$\cdots \longrightarrow \pi_1(F) \longrightarrow \pi_1(\mathrm{Conf}_k(M)) \longrightarrow \pi_1(\mathrm{Conf}_{k-1}(M)) \longrightarrow \dots$$

By the inductive hypothesis, $\pi_1(\mathrm{Conf}_{k-1}(M)) = 0$. It therefore suffices to show that $\pi_1(F) = 0$.

To see this, fix a point $p \in M$ and write $M = (M \setminus \{p\}) \cup U$, where $U \cong \mathbb{R}^n$ is a Euclidean neighborhood of p. Then $\pi_1(U) = 0$, and the intersection satisfies $U \setminus \{p\} \simeq S^{n-1}$ with $n \geq 3$, so the Van Kampen theorem gives

$$\pi_1(M) \cong \pi_1(M \setminus \{p\}) *_{\pi_1(S^{n-1})} \pi_1(U).$$

Since $\pi_1(U) = 0$ and $\pi_1(S^{n-1}) = 0$ (since $n \geq 3$) and $\pi_1(M) = 0$ by assumption, we conclude $\pi_1(M \setminus \{\mathrm{pt}\}) = 0$. We can repeat this for $k-1$ points, so $\pi_1(F) = 0$.

Hence $\pi_1(\mathrm{Conf}_k(M)) = 0$, completing the induction.

Step 2: $\pi_1(\overline{\mathrm{Conf}}_k(M)) \cong \Sigma_k$.

The symmetric group Σ_k acts freely on $\mathrm{Conf}_k(M)$ by permuting coordinates, and the quotient is precisely $\overline{\mathrm{Conf}}_k(M)$. That is,

$$\mathrm{Conf}_k(M) \longrightarrow \overline{\mathrm{Conf}}_k(M)$$

is a regular Σ_k-covering. Since $\mathrm{Conf}_k(M)$ is simply connected by Step 1, this covering is in fact the *universal cover* of $\overline{\mathrm{Conf}}_k(M)$. The deck transformation group of the universal cover equals π_1 of the base, so

$$\pi_1(\overline{\mathrm{Conf}}_k(M)) \cong \Sigma_k.$$

□

Moreover, when $\dim(M) \geq 3$,

$$\pi_1(\mathrm{Conf}_k(M)) \cong \pi_1(M)^k,$$

so $\pi_1(\overline{\mathrm{Conf}}_k(M))$ is a semidirect product of $\pi_1(M)^k$ and Σ_k.

1.5 $K(\pi, 1)$-Spaces

We saw that when M is a simply-connected manifold of dimension $n \geq 3$, the configuration space $\mathrm{Conf}_k(M)$ is also simply connected. The situation changes for surfaces, where configuration spaces are closely related to braid groups and $K(\pi, 1)$-spaces.

Corollary 1.10 ([2]). *Let M be a connected surface other than S^2 or $\mathbb{RP}^2$. Then $\mathrm{Conf}_k(M)$ is aspherical (i.e. $\pi_i = 0$ for all $i \geq 2$) for every $k \geq 0$. In particular, $\overline{\mathrm{Conf}}_k(M)$ is aspherical.*

Proof. We proceed in two steps.

Step 1: $\mathrm{Conf}_k(M)$ is aspherical.

We induct on k. The case $k = 0$ is trivial.

The case $k = 1$ holds by assumption, since $\mathrm{Conf}_1(M) \cong M$ itself is aspherical (any connected surface other than S^2 or $\mathbb{RP}^2$ is a $K(\pi, 1)$).

For the inductive step, assume that $\pi_i(\mathrm{Conf}_{k-1}(M))$ is asperical. Recall from Theorem 1.7 that

$$\pi \colon \mathrm{Conf}_k(M) \to \mathrm{Conf}_{k-1}(M)$$

is a locally trivial fiber bundle with fiber $F = M \setminus \{x_1, \ldots, x_{k-1}\}$. The long exact sequence in homotopy associated to the fibration gives

$$\cdots \to \pi_i(F) \to \pi_i(\mathrm{Conf}_k(M)) \to \pi_i(\mathrm{Conf}_{k-1}(M)) \to \pi_{i-1}(F) \to \cdots$$

for each $i \geq 2$. By the inductive hypothesis, $\pi_i(\mathrm{Conf}_{k-1}(M)) = 0$. It therefore suffices to show that $\pi_i(F) = 0$ for all $i \geq 2$.

Since M is a surface different from S^2 or $\mathbb{RP}^2$, removing finitely many points gives a non-compact surface $M \setminus \{\mathrm{pt}\}$, which deformation retracts onto a wedge of circles. Therfore, F is a $K(\pi, 1)$. Thus, $\pi_i(F) = 0$ for all $i \geq 2$.

Hence $\pi_i(\mathrm{Conf}_k(M)) = 0$ for all $i \geq 2$, completing the induction.

Step 2: $\overline{\mathrm{Conf}}_k(M)$ is aspherical.
The covering map $\pi\colon \mathrm{Conf}_k(M) \to \overline{\mathrm{Conf}}_k(M)$ induces isomorphisms

$$\pi_i(\mathrm{Conf}_k(M)) \cong \pi_i(\overline{\mathrm{Conf}}_k(M)) \quad \text{for all } i \geq 2,$$

since covering maps are π_i-isomorphisms for $i \geq 2$. As $\mathrm{Conf}_k(M)$ is aspherical by Step 1, it follows that $\overline{\mathrm{Conf}}_k(M)$ is aspherical as well. □

Recall that a space X is called a $K(\pi, 1)$-space if

$$\pi_i(X) = 0 \qquad (i \geq 2).$$

In particular,

$$\mathrm{Conf}_k(M) \simeq K(PB_k(M), 1), \qquad \overline{\mathrm{Conf}}_k(M) \simeq K(B_k(M), 1).$$

We will take a closer look at these in the following section.

2 Braid Groups

In the previous section, we saw that configuration spaces of connected surfaces are aspherical. Their topology is therefore described by their fundamental groups, which are called *braid groups* and *pure braid groups.*

Definition 2.1 (Braid Groups, Pure Braid Groups)**.** The braid groups $B_n(M)$ of a manifold M are the groups

$$B_n(M) := \pi_1\big(\overline{\mathrm{Conf}}_n(M)\big).$$

The pure braid groups $PB_n(M)$ are the groups

$$PB_n(M) := \pi_1\big(\mathrm{Conf}_n(M)\big).$$

2.1 $M = \mathbb{R}^2$ and Artin's Braid Group

Let's compute

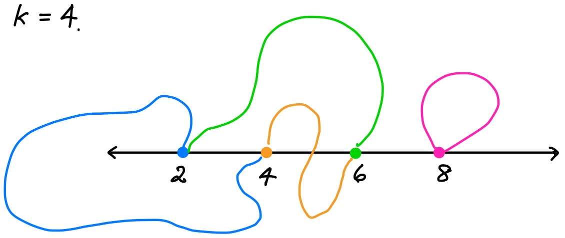
$$\pi_1(\overline{\mathrm{Conf}}_k(\mathbb{R}^2)).$$

- Fix a basepoint

  $$\{(2i, 0)\}_{i=1}^{k}.$$

  An element of $\pi_1(\overline{\mathrm{Conf}}_k(\mathbb{R}^2))$ is a path in $\overline{\mathrm{Conf}}_k(\mathbb{R}^2)$ starting and ending at this basepoint. Since we are in the *unordered* configuration space, the order of the points may change along the path.

  eg. when $k = 4$.



The starting point and the endpoint of a path may differ in the unordered configuration space in $\mathbb{R}^2$.

- Thus, an element of $\pi_1(\overline{\mathrm{Conf}}_k(\mathbb{R}^2))$ is determined by the following data:

  1. A permutation $\tau \in \Sigma_k$,
  2. A path
  $$p : [0,1] \to (\mathbb{R}^2)^k$$
  such that:
  $$\begin{aligned}
  &(1) \quad p(0)_r = (2r, 0) \quad \text{for } 1 \le r \le k \quad (\textit{where the } r\textit{-th strand starts}),\\
  &(2) \quad p(1)_r = (2\tau(r), 0) \quad \text{for } 1 \le r \le k \quad (\textit{where the } r\textit{-th strand ends}),\\
  &(3) \quad p(t)_r \ne p(t)_{r'} \quad \text{for } r \ne r',\ t \in [0,1] \quad (\textit{distinct strands never collide}).
  \end{aligned}$$

- This gives a canonical group homomorphism
  $$\pi_1(\overline{\mathrm{Conf}}_k(\mathbb{R}^2)) \longrightarrow \Sigma_k,$$
  which is surjective (we will show this later 2.2).

The **kernel** of the map above consists of *those paths which start and end at the same ordered configuration*, i.e., those with trivial permutation. This is precisely

$$\pi_1(\mathrm{Conf}_k(\mathbb{R}^2)) = PB_k,$$

the *pure braid group.*

Therefore, we obtain a short exact sequence:

$$1 \longrightarrow PB_k \longrightarrow B_k \longrightarrow \Sigma_k \longrightarrow 1.$$

In particular, PB_k is a subgroup of index $k!$ in B_k, corresponding to the fact that

$$\mathrm{Conf}_k(\mathbb{R}^2) \to \overline{\mathrm{Conf}}_k(\mathbb{R}^2)$$

is a $k!$-sheeted covering space.

Claim 2.2 ([2]). *$B_k \to \Sigma_k$ is surjective.*

Proof. Since this map is a group homomorphism, it suffices to show that generators can be attained. That is, we need to show there exist elements $\sigma_i \in \pi_1(\overline{\mathrm{Conf}}_k(\mathbb{R}^2))$ lifting the transpositions $\tau_i = (i\ i{+}1)$.

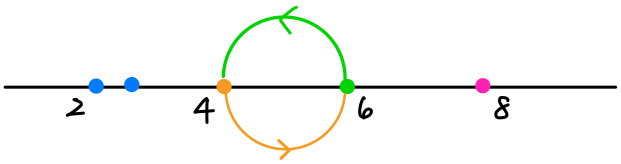
For $1 \le i \le k-1$, define a path $p_i \colon [0,\pi] \to (\mathbb{R}^2)^k$ by

$$p_i(t)_r = \begin{cases} (2r, 0) & r \notin \{i, i+1\} \\ c_{2i+1,\,1}(t+\pi) & r = i \\ c_{2i+1,\,1}(t) & r = i+1, \end{cases}$$

where $c_{a,b}(t) = (a + b\cos t,\ b\sin t)$ is the standard parametrization of the circle of radius b centered at $(a, 0)$.

In other words, all points except the i-th and $(i{+}1)$-th remain fixed at $(2r, 0)$, while those two trace opposite semicircles of radius 1 centered at $(2i+1, 0)$, swapping their positions at $t = \pi$.

eg. When $k=4, i=2$, we have $p_2(t)_r = \begin{cases} (2r,0) & r \notin \{2,3\}, \\ \big(5+\cos(t+\pi), \sin(t+\pi)\big) & r=2, \\ \big(5+\cos(t), \sin(t)\big) & r=3. \end{cases}$



Notice that

$$c_{a,b}(t) = (a + b\cos t, b\sin t) \quad \text{and} \quad c_{a,b}(t+\pi) = (a - b\cos t, -b\sin t).$$

If these were equal, then $\sin t = 0$, so $t \in \{0, \pi\}$. But in these cases,

$$b\cos t \neq -b\cos t.$$

Hence $c_{a,b}(t) \neq c_{a,b}(t+\pi)$ for all $t \in [0,\pi]$, so the path is well-defined in $\mathrm{Conf}_k(\mathbb{R}^2)$. So, the dashed factorization exists in the commuting diagram.

$$\begin{tikzcd}
{[0,\pi]} \arrow[r] \arrow[d, "p_i"'] \arrow[dr, dashed] & \overline{\mathrm{Conf}}_k(\mathbb{R}^2) \\
(\mathbb{R}^2)^k & \mathrm{Conf}_k(\mathbb{R}^2) \arrow[l] \arrow[u, "\pi"]
\end{tikzcd}$$

We write σ_i for the homotopy class of the top horizontal map relative to $\{0,\pi\}$. Since $p_i(0) = p_i(\pi)$ in $\overline{\mathrm{Conf}}_k(\mathbb{R}^2)$, the class σ_i defines an element of $\pi_1(\overline{\mathrm{Conf}}_k(\mathbb{R}^2))$ lifting τ_i, as required.

$\square$

Lemma 2.3 ([2, 2.4.3])**.** *If* $|i-j| > 1$, *then* $\sigma_i\sigma_j = \sigma_j\sigma_i$.

Proof. Without loss of generality, assume $j > i$. Define a homotopy $H\colon [0,\pi]\times[0,\pi] \to (\mathbb{R}^2)^k$ by

$$H(s,t)_r = \begin{cases} (2r,0) & r \notin \{i,i+1,j,j+1\} \\ (p_i(s))_r & r \in \{i,i+1\} \\ (p_j(t))_r & r \in \{j,j+1\}. \end{cases}$$

To verify that H is a homotopy between $p_i \cdot p_j$ and $p_j \cdot p_i$, we check the boundary values:

$$H(s,0)_r = \begin{cases} (2r,0) & r \notin \{i,i+1\} \\ (p_i(s))_r & r \in \{i,i+1\}, \end{cases}$$

$$H(\pi,t)_r = \begin{cases} (2r,0) & r \notin \{i,i+1,j,j+1\} \\ \tau_i(r) & r \in \{i,i+1\} \\ (p_j(t))_r & r \in \{j,j+1\}. \end{cases}$$

Hence H defines a homotopy between $p_i \cdot p_j$ and $p_j \cdot p_i$. The key point is that since $|i-j|>1$, the index sets $\{i,i+1\}$ and $\{j,j+1\}$ are disjoint, so the two strands move independently. $\square$

Remark 2.4. In contrast, when $|i-j|=1$, the strands are *adjacent* and the commutativity relation breaks down. Instead, one obtains the *braid relation*

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

which can be seen geometrically by imagining pulling the strands taut (i.e. straightening them as physical strings). Finally, inverses are obtained by reflecting across $\mathbb{R}^2 \times \{1\}$.

Lemma 2.5 ([2]). *For each* $1 \leq i \leq k-1$,

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

The proof of this lemma is omitted.

Lemmas 2.3 and 2.5 together give us *Artin's braid group*:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1 \rangle .$$

We can make an analogous observation for pure braid groups. The generators are T_{ij} for $1 \leq i < j \leq n$, where T_{ij} is the element in which strand i wraps around strand j once counterclockwise while all other strands remain fixed.

2.2 Mapping Class Groups

Definition 2.6. Let M be a smooth oriented manifold. Define

$$\mathrm{Diff}^+(M, \partial M) = \{\text{orientation-preserving diffeomorphisms } f : M \to M \mid f|_{\partial M} = \mathrm{id}\} .$$

Equip $\mathrm{Diff}^+(M, \partial M)$ with the compact-open topology. Path components correspond to isotopy classes: we write $f \sim g$ if there exists a homotopy

$$H : M \times [0,1] \to M$$

such that:

- $H(-,0) = f$ and $H(-,1) = g$,
- each $H(-,t)$ is a diffeomorphism,
- $H(-,t)|_{\partial M} = \mathrm{id}$ for all t.

The *mapping class group* is

$$\mathrm{Mod}(M) = \mathrm{MCG}(M) := \mathrm{Diff}^+(M, \partial M)/\sim \;\cong\; \pi_0(\mathrm{Diff}^+(M, \partial M)).$$

Fact (Alexander Lemma).

$$\mathrm{MCG}(D^2) \cong \{e\}.$$

2.2.1 Braids and Mapping Class Groups

Let D_n denote the disk D^2 with n marked points, and define

$$\mathrm{Mod}(D_n) := \left\{ \text{homeomorphisms of } (D^2, \partial D^2) \text{ preserving the set of marked points} \right\} \Big/ \text{isotopy}.$$

Let ϕ be a homeomorphism of $(D^2, \partial D^2)$ preserving the marked points. Since $\mathrm{MCG}(D^2)$ is trivial, we have

$$\phi \sim \mathrm{id}.$$

During this isotopy, the marked points move around the disk and return to themselves as a set. Tracking the trajectories of these points through the isotopy produces a geometric braid.
Hence,

$$B_n \cong \mathrm{Mod}(D_n).$$

Under this correspondence, the standard generator $\sigma_i \in B_n$ corresponds to the half-twist exchanging the i-th and $(i+1)$-st marked points.

References

[1] E. Fadell and L. Neuwirth, *Configuration spaces*, Mathematica Scandinavica **10** (1962), 111–118.

[2] B. Knudsen, *Configuration Spaces in Algebraic Topology*, arXiv:1803.11165, 2018.

[3] R. Longoni and P. Salvatore, *Configuration spaces are not homotopy invariant*, Topology, Volume 44, Issue 2, 2005, pp. 375–380.