Homological stability 2: Arnol'd–Segal filtration (talk 4)

Configuration spaces summer school

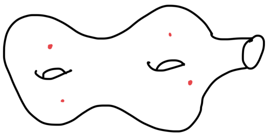
May 25-29 2026

1 Set up

Let M be a connected open manifold and $\dim(M) = q$. Let S_n denote the permutation group. Define the *Unordered configuration spaces*

$$C_n(M) = \{(x_1, \ldots, x_n) \mid x_i \in M \text{ for all } i, x_i \neq x_j \text{ for all } i \neq j\}/\, S_n.$$

The following picture represents an element in $C_4(M)$.



Remark 1.1. $C_n(M)$ is orientable if and only if one the following conditions hold:

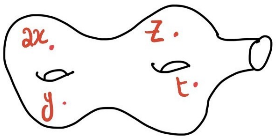
- $n = 0$,
- $\dim(M) = 1$,
- $n = 1$ and M is orientable,
- $\dim(M)$ is even and M is orientable.

For the rest of this talk, we assume that q is even and M is orientable.

Define the *Symmetric product*

$$A_n(M) = \{(x_1, \ldots, x_n) \mid x_i \in M \text{ for all } i\}/\, S_n.$$

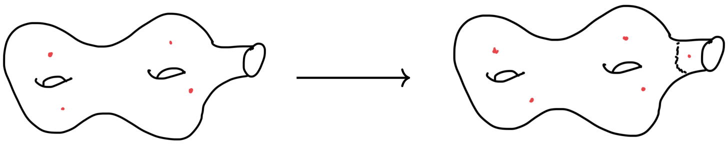
The following picture represents an element in $A_5(M)$.



Remark 1.2.

- S_n acts freely and properly discontinuously on $\{(x_1, \ldots, x_n) \mid x_i \in M \text{ for all } i, x_i \neq x_j \text{ for all } i \neq j\}$, and so $C_n(M)$ is a manifold of dimension nq.
- S_n does not act freely on $\{(x_1, \ldots, x_n) \mid x_i \in M \text{ for all } i\}$, and so $A_n(M)$ may not be a manifold. And $\mathrm{H}^i_c(A_n) = 0$ for $i > nq$.

We consider the stabilization map $C_n(M) \to C_{n+1}(M)$. Informally, this map pushes the configuration away from an end of M, then adds one new point near infinity.



The goal of this talk is to prove the following theorem.

Theorem A. *The map $C_n(M) \to C_{n+1}(M)$ is a homology equivalence up to dimension $\lfloor \frac{n}{2} \rfloor$.*

Since M is an open manifold of dimension q, there exists an embedding

$$\iota \colon \mathbb{R}^q \sqcup M \to M$$

whose image is an open subset of M, that lies entirely in the collar neighborhood. Define the open embedding

$$\begin{aligned} F_n \colon \mathbb{R}^q \times C_n(M) &\to C_{n+1}(M) \\ (r, x_1, \ldots, x_n) &\mapsto (\iota(r), \iota(x_1), \ldots, \iota(x_n)). \end{aligned}$$

Theorem A is equivalent to F_n being a homology equivalence up to dimension $\lfloor \frac{n}{2} \rfloor$, which is equivalent by Poincare duality to the following statement.

Proposition 1.3. *The map $F_n \colon \mathbb{R}^q \times C_n(M) \to C_{n+1}(M)$ is a compactly supported cohomology equivalence above dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.*

Similarly to the definition of F_n, define the open embedding

$$G_n \colon \mathbb{R}^q \times A_n(M) \to A_{n+1}(M).$$

In the proof, we will assume the following statement for symmetric products $A_n(M)$, which will be proved later in talk 6.

Proposition 1.4. *The open embedding $G_n \colon \mathbb{R}^q \times A_n(M) \to A_{n+1}(M)$ is a compactly supported cohomology equivalence above dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.*

2 Proof of Theorem A

Lemma 2.1. *Assume $f \colon Y \to Z$ is a compactly supported cohomology equivalence above the dimension m, and $\mathrm{H}^i_c(X) \cong 0$ for $i > d$. Then the map*

$$X \times Y \to X \times Z$$

is a compactly supported cohomology equivalence above the dimension $m + d$.

Proof. By Künneth formula,

$$\mathrm{H}^i_c(X \times Y) \cong \bigoplus_{a+b=i} \mathrm{H}^a_c(X) \otimes \mathrm{H}^b_c(Y), \quad \text{and}\, \mathrm{H}^i_c(X \times Z) \cong \bigoplus_{a+b=i} \mathrm{H}^a_c(X) \otimes \mathrm{H}^b_c(Z).$$

Since $\mathrm{H}^i_c(X) \cong 0$ for $i > d$, then we can assume that $a \leq d$. Thus $b = i - a \geq i - d$.

Let $i > m + d$. Then $b > m$, and so the map

$$f_* \colon \mathrm{H}^b_c(Y) \to \mathrm{H}^b_c(Z)$$

is an isomorphism.

Let $i = m + d$. Then $b = m$, and so f_* is surjective. We conclude that

$$X \times Y \to X \times Z$$

is is a compactly supported cohomology equivalence above the dimension $m + d$. □

Notation 2.2. For every element in $A_n(M)$, we write

$$(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

Remark 2.3. Let $\xi \in A_n(M)$, then

$$\xi = 2T + U$$

for some $T \in A_k(M), U \in C_{n-2k}(M)$ where U has multiplicity at most 1.

Example 2.4. $(x, x, y, y, y, z) = 2(x) + (3y + z) = 2(x + y) + (y + z)$.

Definition 2.5 (Arnol'd–Segal filtration)**.** By Remark 2.3, every $\xi \in A_n(M)$ may be written (non-uniquely) as $\xi = 2T + U$ for some $T \in A_k(M), U \in C_{n-2k}(M)$. Define

$$A_{n,k}(M) = \{\xi = 2T + U \in A_n(M) \mid \deg(T) \geq k\}.$$

Then

$$A_{n,k}(M) \supset A_{n,k+1}(M) \quad \text{for all } k \geq 0.$$

Moreover, for every $\xi = 2T + U \in A_{n,k}(M)$,

$$n = \deg(\xi) \geq 2\deg(T) \geq 2k.$$

Thus $A_{n,k}(M) = \emptyset$ for all $k > \lfloor \frac{n}{2} \rfloor$. We then obtain the descending sequence (known as the Arnol'd–Segal filtration)

$$A_n(M) = A_{n,0}(M) \supset A_{n,1}(M) \supset \cdots \supset A_{n,\lfloor \frac{n}{2} \rfloor}(M) \supset \emptyset.$$

Example 2.6. $2(x)x + (3y + z) \in A_{n,2} \subset A_{n,1}$.

Proof of Theorem A. Recall that Theorem A is equivalent to proving that the map $F_n \colon \mathbb{R}^q \times C_n(M) \to C_{n+1}(M)$ is a compactly supported cohomology equivalence above dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$. We have the following identity

$$A_{n,k}(M) - A_{n,k+1}(M) = \{\xi = 2T + U \mid \deg(T) = k\} = A_k(M) \times C_{n-2k}(M). \tag{2.1}$$

Note that $A_k(M) \times C_{n-2k}(M)$ is an open subset of $A_{n,k}(M)$. We will now show by induction on n that the map F_n is a compactly supported cohomology equivalence above the dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.

For $n = 1$, $F_1 \colon \mathbb{R}^q \times C_1(M) \to C_2(M)$. We have

$$\dim(R^q \times C_1(M)) = \dim(C_2(M)) = 2q,$$

and

$$(n+1)q - \left\lfloor \frac{n}{2} \right\rfloor = 2q - \left\lfloor \frac{1}{2} \right\rfloor = 2q.$$

Since $\mathrm{H}_c^{2q}(\mathbb{R}^1 \times C_1(M)) \cong \mathrm{H}_c^{2q}(C_2(M)) \cong \mathbb{Z}$, then

$$\mathrm{H}_c^i(\mathbb{R}^1 \times C_1(M)) \cong \mathrm{H}_c^i(C_2(M)) \quad \text{for all } i \geq 2q.$$

Let $n > 1$. Assume that for all $m < n$, F_m is a compactly supported cohomology equivalence above the dimension $(m+1)q - \lfloor \frac{m}{2} \rfloor$. Consider the open embedding induced by G_n,

$$\begin{aligned} G_{n,k} \colon \mathbb{R}^q \times A_{n,k}(M) &\to A_{n+1,k}(M) \\ (r, \xi = 2T + U) &\mapsto \iota(r) + \iota(\xi) = 2\iota(T) + (\iota(r) + \iota(U)) \end{aligned}$$

Claim. *For $k > 0$, $G_{n,k}$ is a compactly supported cohomology equivalence above the dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.*

We postpone the proof of the claim till the end and we proceed with the induction proof assuming the claim. (2.1) gives that

$$A_n(M) - A_{n,1}(M) = A_{n,0}(M) - A_{n,1}(M) = A_0(M) \times C_n(M) = C_n(M).$$

Consider the long exact sequences of compactly supported cohomology,

$$\begin{tikzcd}
\cdots & \mathrm{H}_c^i(\mathbb{R}^q \times A_{n,1}(M)) \arrow[l] \arrow[d, "(G_{n,1})_*"] & \mathrm{H}_c^i(\mathbb{R}^q \times A_n(M)) \arrow[l] \arrow[d, "(G_n)_*"] & \mathrm{H}_c^i(\mathbb{R}^q \times C_n(M)) \arrow[l] \arrow[d, "(F_n)_*"] & \mathrm{H}_c^{i-1}(\mathbb{R}^q \times A_{n,1}(M)) \arrow[l] \arrow[d, "(G_{n,1})_*"] & \cdots \arrow[l] \\
\cdots & \mathrm{H}_c^i(A_{n+1,1}(M)) \arrow[l] & \mathrm{H}_c^i(A_{n+1}(M)) \arrow[l] & \mathrm{H}_c^i(C_{n+1}(M)) \arrow[l] & \mathrm{H}_c^{i-1}(A_{n+1,1}(M)) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

By the claim, the maps $(G_{n,1})_*$ on the left and right are isomorphisms for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$.

By Proposition 1.4, the map $(G_n)_*$ is an isomorphism for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$.

It follows from the Five Lemma that

$$(F_n)_*\colon H_c^i(\mathbb{R}^q \times C_n(M)) \longrightarrow H_c^i(C_{n+1}(M))$$

is an isomorphism for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$. ☐

We now turn to prove the claim.

Proof of the claim. This a proof by downwards induction on k.

For $k > \lfloor \frac{n+1}{2} \rfloor$, $A_{n,k}(M) = A_{n+1,k}(M) = \emptyset$.

Let $k \leq \lfloor \frac{n+1}{2} \rfloor$. Assume that for all $m > k$, $G_{n,m}$ is a compactly supported cohomology equivalence above the dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$. We have by (2.1) that

$$\mathbb{R}^q \times A_{n,k}(M) - \mathbb{R}^q \times A_{n+1,k}(M) = A_k(M) \times C_{n-2k}(M).$$

Consider the long exact sequences of compactly supported cohomology,

$$\begin{tikzcd}
\cdots & \mathrm{H}_c^i(\mathbb{R}^q \times A_{n,k+1}(M)) \arrow[l] \arrow[d, "(G_{n,k+1})_*"] & \mathrm{H}_c^i(\mathbb{R}^q \times A_{n,k}(M)) \arrow[l] \arrow[d, "(G_{n,k})_*"] & \mathrm{H}_c^i(A_k(M) \times (\mathbb{R}^q \times C_{n-2k}(M))) \arrow[l] \arrow[d, "(F_{n-2k})_*"] & \mathrm{H}_c^{i-1}(\mathbb{R}^q \times A_{n,k+1}(M)) \arrow[l] \arrow[d, "(G_{n,k+1})_*"] & \cdots \arrow[l] \\
\cdots & \mathrm{H}_c^i(A_{n+1,k+1}(M)) \arrow[l] & \mathrm{H}_c^i(A_{n+1,k}(M)) \arrow[l] & \mathrm{H}_c^i(C_{n-2k+}(M)) \arrow[l] & \mathrm{H}_c^{i-1}(A_{n+1,k+1}(M)) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

By induction hypothesis on k, the maps $(G_{n,k+1})_*$ on the left and right are isomorphisms, for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$.

Since $k > 0$ and $n - 2k < n$, the map

$$\mathrm{H}_c^i(\mathbb{R}^q \times C_{n-2k}(M)) \to \mathrm{H}_c^i(C_{n-2k+1}(M))$$

is an isomorphism by the induction hypothesis on n for $i > (n-2k+1)q - \lfloor \frac{n-2k}{2} \rfloor$ and surjective for $i = (n-2k+1)q - \lfloor \frac{n-2k}{2} \rfloor$.

Since $\dim(A_k(M)) = kq$, this follows by Lemma 2.1 that

$$\mathrm{H}_c^i(A_k(M) \times (\mathbb{R}^q \times C_{n-2k}(M))) \to \mathrm{H}_c^i(A_k(M) \times C_{n-2k+1}(M))$$

is an isomorphism by the induction hypothesis on n for $i > (n-k+1)q - \lfloor \frac{n}{2} \rfloor + k$ and surjective for $i = (n-k+1)q - \lfloor \frac{n}{2} \rfloor + k$.

It follows from the Five Lemma that

$$(G_{n,k})_*\colon H_c^i(\mathbb{R}^q \times A_{n,k}(M)) \longrightarrow H_c^i(A_{n+1}(M))$$

is an isomorphism for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$. ☐