

Configuration space summer school:

Topics for expository talks

Homological stability 1: Overview and compactly supported cohomology (talk 1)

The speaker will begin stating Segal's homological stability theorem for unordered configuration spaces of non-compact manifolds. This involves constructing a stabilization map for configuration spaces. Technical details involving manifold topology need not be emphasized when constructing the stabilization map. Then the speaker will review compactly supported cohomology. In particular, the speaker will describe how compactly supported cohomology is functorial with respect to proper maps and with respect to open embeddings. Then the speaker will describe the long exact sequence in compactly supported cohomology associated to a decomposition of a space into open and closed subspaces. The speaker should describe Poincaré duality using compactly supported cohomology and use this to reformulate homological stability in terms of compactly supported cohomology. In particular, the speaker will describe how the stabilization map extends to an open embedding. Time permitting, the speaker will describe improved stable ranges with certain coefficients and secondary homological stability.

Primary references: Segal [Seg79, Appendix A].

Secondary references: Knudsen [Knu18], Kupers–Miller [KM15], Himes [Him24].

Background talk: Fiber bundles, fibrations, quasi-fibrations, and homology fibrations (talk 2)

The speaker will define fiber bundles and give basic examples such as the tangent bundle and Möbius bundle. Then the speaker will define fibrations and give standard examples such as the path-loop fibration. The speaker will state (with some explanation but likely not full proofs) that fiber bundles are fibrations and fibrations are quasi-fibrations. The speaker will define homotopy fibers and use this to define quasi-fibrations. Then the speaker will give examples of fibrations that are not fiber bundles (triangle mapping to an interval) and quasi-fibrations that are not fiber bundles (an L shaped region mapping to an interval). Then the speaker will define homology fibrations. Assuming the existence of acyclic spaces, the speaker should give an example of a homology fibration using the mapping cone construction. The speaker will remind people that fibrations have associated long exact sequences for homotopy groups and Serre spectral sequence in homology. The speaker should point out that quasi fibration also have a long exact sequences for homotopy groups and a Serre spectral sequence in homology while homology fibrations only have a Serre spectral sequence.

Primary references: Cohen [Coh], Hatcher [Hat02], McDuff [McD75]

Secondary references: McDuff–Segal [MS76], Miller–Palmer [MP15], Segal [Seg79]

Braid groups: Definitions and $K(\pi, 1)$ property (talk 3)

The speaker will define braid groups of a manifold as the fundamental group of the configuration space of unordered points in that manifold and define pure braid groups as the fundamental group of the ordered configuration space. The speaker will describe how in the case of $\mathbb{R}^2$, elements of the braid group can be thought of as braids and elements of the pure braid group can be thought of as braids where each strand starts and ends at the same place. The speaker will talk about how the map forgetting the order gives a map from ordered to unordered configuration spaces and that this is a fiber bundle map. The speaker will also describe how forgetting a point gives a map between ordered configuration spaces on $k + 1$ and k points and how this map is a fiber bundle. Using that surfaces (except for S^2 and $\mathbb{R}P^2$) are $K(\pi, 1)$'s, the long exact sequence of homotopy groups, and induction, the speaker will explain how to show ordered and unordered configuration spaces are $K(\pi, 1)$'s. Time permitting, the speaker will explain connections between braid groups and mapping class groups.

Primary references: Fadell–Neuwirth [FN62, Corollary 2.2]

Secondary references: Knudsen [Knu18], Hatcher [Hat02], Williams [Wil20]

Homological stability 2: Arnol'd–Segal filtration (talk 4)

The goal of this talk is to reduce homological stability for unordered configuration spaces to stability for compactly supported cohomology for symmetric products (Segal [Seg79, Proof of A.1 assuming A.2]). The key tool is the Arnol'd–Segal filtration of symmetric products. The speaker should introduce symmetric products and the relevant filtration. The speaker should describe the long exact sequences in compactly supported cohomology associated to this filtration. The speaker should be careful to clearly explain the logic behind the double inductive argument (e.g. what are the base cases and which statements imply which other statements).

Primary references: Segal [Seg79, Proof of A.1 assuming A.2].

Secondary references: Kupers–Miller [KM15], Himes [Him24].

Scanning 1: Definitions and theorems (talk 5)

The goal of this talk is to define the scanning map for configuration spaces and state that it is a homology equivalence in a stable range [McD75, Theorem 1.2]. The speaker should define relative configuration spaces and define the scanning map for relative configuration spaces as well. The speaker should explain when the scanning map for relative configuration spaces is a homotopy equivalence. Then the speaker should prove that the configuration space of points in a disk relative to the boundary is homotopy equivalent to a sphere and explain how this verifies McDuff's theorem in a special case. This will be the base case of an inductive argument describe in later talks. Time permitting, the speaker should use homological stability and scanning to compute $\pi_{n+1}(S^n)$. The argument is as follows:

$$H_1(\mathbb{R}P^{n-1}) \cong H_1(\text{Conf}_2(R^n)) \cong H_1(\Omega^n S^n) \cong \pi_1(\Omega^n S^n)^{ab} \cong \pi_{n+1}(S^n)^{ab} \cong \pi_{n+1}(S^n).$$

Primary references: McDuff [McD75]

Secondary references: Knudsen [Knu18], Bökigheimer [Böd87]

Homological stability 2: Stability for symmetric products (talk 6)

The goal of this talk is to finish the proof of homological stability for unordered configuration spaces by proving stability for compactly supported cohomology of symmetric products (Segal [Seg79, Proof of A.2]). Following Segal, the speaker should reduce stability for symmetric products of M to

stability for symmetric products of $\mathbb{R}^n$. The speaker should explain that symmetric products have no reason to be manifolds because the symmetric group S_k does not act freely on M^k . Thus, even though symmetric products of $\mathbb{R}^n$ are contractible, the spaces may have interesting compactly supported cohomology. The speaker should explain that the k -fold symmetric product of $\mathbb{R}^2 \cong \mathbb{C}$ can be viewed as the space of monoic degree k -polynomials by considering roots of polynomials. By considering coefficients of polynomials, the speaker should explain how to see that the space of monoic polynomials is homeomorphic to $\mathbb{C}^k$. Using this, the speaker should complete the proof of homological stability in dimension 2. Time permitting, the speaker should explain aspects of the calculation of the compactly supported cohomology of $\mathbb{R}^n$ for $n > 2$ and how this implies homological stability.

Primary references: Segal [Seg79, Proof of A.2].

Secondary references: Kupers–Miller [KM15], Himes [Him24].

Scanning 2: Proof assuming projection is a quasi-fibration/homology fibration (talk 7)

The goal of this talk is to prove [McD75, Theorem 1.2]. The speaker will assume [McD75, Proposition 3.1] and [McD75, Proposition 4.1] which state that certain projection maps are quasi-fibrations or homology-fibrations. The proof of [McD75, Theorem 1.2] is via handle induction. The speaker should review handle decompositions of manifolds. Assuming the base case of the disk relative to its boundary, the speaker should prove that the scanning map is a homology equivalence for handles using [McD75, Proposition 3.1]. Then, using [McD75, Proposition 3.1] and handle induction, the speaker should prove the scanning map for relative configuration spaces is a homotopy equivalence. Finally, using [McD75, Proposition 4.1] the speaker should prove [McD75, Theorem 1.2] for (not relative) configuration spaces.

Primary references: McDuff [McD75]

Secondary references: Bödigheimer [Böd87], Segal [Seg79]

Scanning 3: Proof that projection is a quasi-fibration/homology fibration (talk 8)

The speaker will state the Dold–Thom quasi-fibration criterion [McD75, Lemma 3.3] and McDuff’s version for homology fibrations [McD75, Lemma 5.1]. The speaker should explain the intuition behind [McD75, Lemma 3.3] and [McD75, Lemma 5.1] by saying that these lemmas assert that if you stitch together fibrations along homotopy/homology equivalences, then you obtain a quasi/homology fibration. The speaker will explain the proof of [McD75, Proposition 3.1] using [McD75, Lemma 3.3] and the proof of [McD75, Proposition 4.1] using [McD75, Lemma 5.1]. The speaker should attempt to clearly explain why you need to work with relative configuration spaces to obtain a quasi-fibration and why you need to send the number of points to infinity to obtain a homology-fibration when you are not working relative to a subspace.

Primary references: McDuff [McD75]

Secondary references: Bödigheimer [Böd87]

References

- [Böd87] C.-F. Bödigheimer, *Stable splittings of mapping spaces*, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 174–187. MR 922926 (89c:55011) [2](#), [3](#)
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- [FN62] Edward Fadell and Lee Neuwirth, *Configuration spaces*, Mathematica Scandinavica **10** (1962), 111–118. [2](#)
- [Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001) [1](#), [2](#)

- [Him24] Zachary Himes, *Secondary homological stability for unordered configuration spaces*, Trans. Amer. Math. Soc. **377** (2024), no. 5, 3173–3241. MR 4744778 [1](#), [2](#), [3](#)
- [KM15] Alexander Kupers and Jeremy Miller, *Improved homological stability for configuration spaces after inverting 2*, Homology Homotopy Appl. **17** (2015), no. 1, 255–266. MR 3344444 [1](#), [2](#), [3](#)
- [Knu18] Ben Knudsen, *Configuration spaces in algebraic topology*, arXiv preprint arXiv:1803.11165 (2018). [1](#), [2](#)
- [McD75] Dusa McDuff, *Configuration spaces of positive and negative particles*, Topology **14** (1975), 91–107. MR 0358766 (50 #11225) [1](#), [2](#), [3](#)
- [MP15] Jeremy Miller and Martin Palmer, *A twisted homology fibration criterion and the twisted group-completion theorem*, Q. J. Math. **66** (2015), no. 1, 265–284. MR 3356291 [1](#)
- [MS76] D. McDuff and G. Segal, *Homology fibrations and the “group-completion” theorem*, Invent. Math. **31** (1975/76), no. 3, 279–284. MR 0402733 (53 #6547) [1](#)
- [Seg79] Graeme Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), no. 1-2, 39–72. MR 533892 (81c:55013) [1](#), [2](#), [3](#)
- [Wil20] Lucas Williams, *Configuration spaces for the working undergraduate*, Rose-Hulman Undergrad. Math. J. **21** (2020), no. 1, Art. 8, 27. MR 4205427 [2](#)