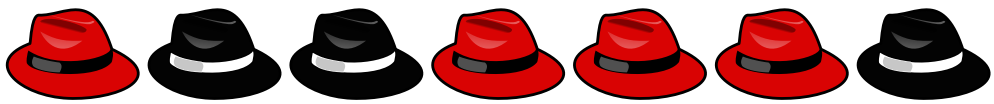
Hamming Codes & Coloured Hats

The problem

A mathematical villain has kidnapped Stanford's mascot, the colour cardinal. She will release the cardinal only on the condition that a group of $2^k - 1$ undergraduate students win at the following game.

- Each student is given a hat in one of two colours, red or black, each with probability $\frac{1}{2}$.

  Every student can see every other student's hat colour, but they cannot see their own.

- After a minute, each student must declare one of three things: *Red*, *Black*, or *Pass*. All students must announce their answers simultaneously.

- The students win if at least one student correctly identifies his or her hat colour, and no students incorrectly identify their hat colours.

- The students can convene and decide on a strategy in advance, but once the game begins they cannot communicate.

What strategy should the students adopt? What are their chances for rescuing their beloved mascot?

Already know the solution? Consider the following variations:

1. What strategy should the students adopt if the two hat colours are not distributed with equal likelihood? What if (say) red hats are given out with probability p?

2. What happens if there are N students for some N not of the form $2^k - 1$? First, verify that there are no perfect codes of length N and minimal distance 3. What is the optimal strategy in these cases, and how good are the chances of success?

3. How would the strategy change if, instead of two colours, there were three possible hat colours? M possible hat colours?

The strategy:

The students can optimize their chances by adopting the following strategy (in the case of $2^3 - 1 = 7$ students)

- Before the game begins, the students number themselves 1 through 7.
- When the hats are distributed, we will identify the hat pattern with a vector over the finite field $\mathbb{F}_2$. Let's say that red hats correspond to zeroes and black hats correspond to ones. Then, when the students line up in order, the hats represent a vector of length 7 with entries either zero or one.

  For example, the hats on the opposite page correspond to the vector $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

  Every student knows all but one entry of this vector.
- Each student writes down two vectors: the vector they would get if their hat were red, and the vector they would get if their hat were black. Using the $\mathbb{F}_2$ arithmetic rules, they multiply both vectors through the following matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- If both matrix-vector products are nonzero, then the student says "Pass". If the matrix-vector product is zero for the red-hat option, the student says "Black". If the matrix-vector product is zero for the black-hat option, the student says "Red".

  (It is impossible for both matrix-vector products to be zero. Why?)

What's going on? Why does this work? We'll find out today!