1 Spectral sequences

A *spectral sequence* is a computational tool that can be viewed as a generalization of the familiar long exact sequences from algebraic topology. For example,

- Just as there is a long exact sequence of a pair (X, A), there is a spectral sequence associated to a filtration of a space
$$\varnothing \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_N = X.$$
- Just as there is the Mayer-Vietoris long exact sequence, there is a spectral sequence associated to an open cover $\{\mathcal{U}_i\}$ of a space X.

As with the long exact sequences in algebraic topology, we can sometimes use spectral sequences to do (co)homology calculations simply using formal algebraic properties of the spectral sequences, without needing to understand its construction or determine the individual maps.

1.1 The structure of a (homology) spectral sequence

A spectral sequence is a "book" consisting of a sequence of *pages* (or *sheets*). These are typically denoted E^r (or E_r for a cohomology spectral sequence), for $r = 0, 1, 2, \ldots$. Each page is a bigraded abelian group with a differential structure. Concretely, each page has

- a 2D array of groups (or rings, or algebras) $E^r_{p,q}$, with $(p, q) \in \mathbb{Z}^2$,
- a map $d^r : E^r \to E^r$ satisfying $(d^r)^2 = 0$, called the *differential*.

The differentials d^r give E^r the structure of a chain complex. The page E^{r+1} is the homology of the complex (E^r, d^r), in the sense that
$$E^{r+1}_{p,q} = \frac{\text{kernel of } d^r \text{ at } E^r_{p,q}}{\text{image of } d^r \text{ in } E^r_{p,q}}.$$
In particular the group $E^{r+1}_{p,q}$ is always a subquotient of $E^r_{p,q}$.

Unfortunately, although the complex (E^r, d^r) determines the groups $E^{r+1}_{p,q}$, it does not determine the differential d^{r+1}.

For the spectral sequences we will consider,

- the group $E^r_{p,q}$ is only possibly nonzero for $p, q \geq 0$. Such a spectral sequence is called a *first quadrant spectral sequence*.
- The differentials satisfy
$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

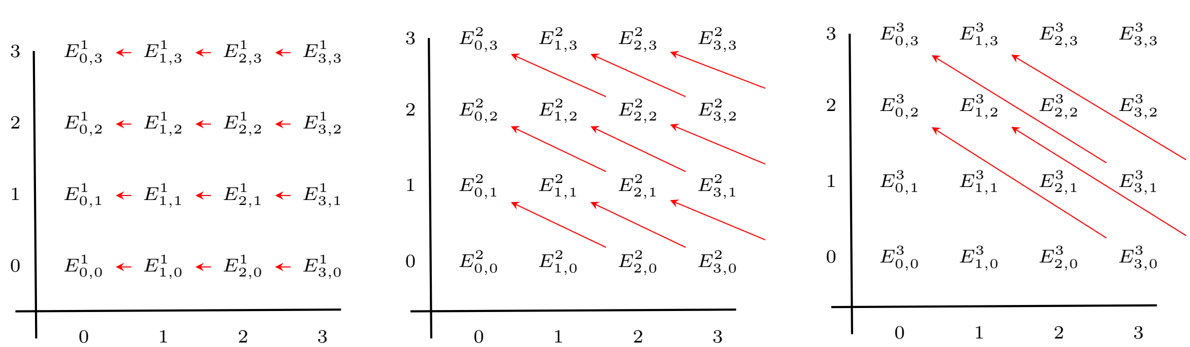


Figure 1: The pages E^1, E^2, and E^3.

Exercise 1. **(Warm-up)** Let $E^*_{*,*}$ be a first quadrant homology spectral sequence.

(a) Verify that the groups $E^r_{p,0}$ along the bottom $q = 0$ row are always subgroups of the groups $E^s_{p,0}$ on the previous pages $s < r$. The maps

$$E^r_{p,0} \hookrightarrow E^{r-1}_{p,0} \hookrightarrow \cdots \hookrightarrow E^3_{p,0} \hookrightarrow E^2_{p,0}$$

are called *edge maps*.

(b) Verify that the groups $E^r_{0,q}$ along the left $p = 0$ column are always quotients of the groups $E^s_{0,q}$ on the previous pages $s < r$. The maps

$$E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^{r-1}_{0,q} \twoheadrightarrow E^r_{0,q}$$

are also called *edge maps*.

(c) Show that, for fixed (p,q), there is equality $E^{r+1}_{p,q} = E^r_{p,q}$ if and only if both the incoming and outgoing differentials at $E^r_{p,q}$ are the zero map.

Abutment

Suppose for each (p,q), the groups $E^r_{p,q}$ eventually stabilize, in the sense that (for all r sufficiently large, depending on p and q),

$$E^r_{p,q} = E^{r+1}_{p,q} = E^{r+2}_{p,q} = \cdots$$

In this case, we write $E^\infty_{p,q}$ for the stable groups, and call the bigraded object $E^\infty = \{E^\infty_{p,q}\}_{p,q}$ the *limit* or *abutment* of the spectral sequence. This limit page does not have a differential.

Observe that, if we have a first quadrant spectral sequence, then at at any fixed point (p,q), for r sufficiently large, either the domain or the codomain of both differentials d^r to or from $E^r_{p,q}$ will be zero. Thus, the spectral sequence must stabilize at every point (p,q).

In general the sequence of groups $\{E^r_{p,q}\}_r$ stabilizes at a page r that depends on (p,q). If there is some r such that $E^r_{p,q} = E^\infty_{p,q}$ for all p and q, then we say that the spectral sequence *degenerates* on page E_r.

We say the spectral sequence *converges* to the graded A-module H_* if for each k there is some filtration of H_k

$$0 = F^{-1}_k \subseteq F^0_k \subseteq \cdots \subseteq F^k_k = H_k$$

such that the limiting groups $E^\infty_{p,q}$ are the associated graded terms

$$E^\infty_{p,q} = F^p_{p+q}/F^{p-1}_{p+q}.$$

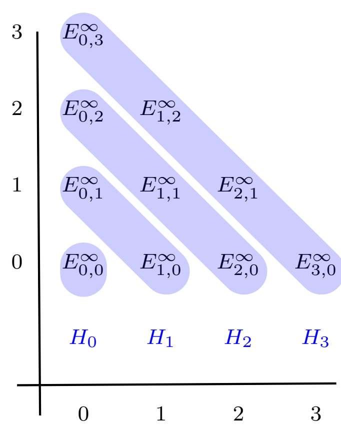


Figure 2: The limit of a spectral sequence.

Remark I. (Recovering H_*). In general, knowing the quotient groups $E^\infty_{p,q} = F^p_{p+q}/F^{p-1}_{p+q}$ is not enough to reconstruct the groups H_*; we can only determine these groups "up to extensions". Consider, for example, the simplest example: If $A = \mathbb{Z}$, and $E^\infty_{0,1} \cong E^\infty_{1,0} \cong \mathbb{Z}/2\mathbb{Z}$, then the theorem states that H_1 fits into a short exact sequence

$$0 \longrightarrow E^\infty_{0,1} \longrightarrow H_1 \longrightarrow E^\infty_{1,0} \longrightarrow 0$$

but the theorem does not distinguish between the two resultant possibilities, whether H_1 is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

Exercise 2. (Warm-up). Let $E^*_{*,*}$ be a first quadrant homology spectral sequence. For a given (p,q), on what page r must the group $E^*_{p,q}$ stabilize?

Exercise 3. (Warm-up). Let $E^*_{*,*}$ be a convergent spectral sequence of abelian groups. Suppose that, for some r, the page $E^r_{*,*}$ is supported on a single row. What is the graded group H_*? What if $E^r_{*,*}$ is supported on a single column?

Exercise 4. (Warm-up).

(a) Suppose that $A = \mathbb{F}$ is a field. Let $E^*_{*,*}$ be a spectral sequence that converges to a graded $\mathbb{F}$-vector space H_*. Show that, for each k,

$$H_k = \bigoplus_{p+q=k} E^\infty_{p,q}.$$

In this case there are no extension problems.

(b) Let $A = \mathbb{Z}$. Let $E^*_{*,*}$ be a spectral sequence that converges to a graded abelian group H_*. Suppose that, for some k, all the groups $E^\infty_{p,q}$ with $p+q=k$ are free abelian. Show that

$$H_k = \bigoplus_{p+q=k} E^\infty_{p,q}.$$

An example: the Leray–Serre spectral sequence (homology version)

Let A be an abelian group. Let

$$F \longrightarrow X \xrightarrow{\pi} B$$

be a fibration with a path-connected base space B. Then there is a *monodromy* action of $\pi_1(B)$ on the homology $H_*(F;A)$ of the fibre F. We use the notation $\mathscr{H}_*(F;A)$ when viewing this homology group as a $A[\pi_1(B)]$-module.

Theorem II. (The homology Leray–Serre spectral sequence). *Let A be an abelian group. Given a fibration*

$$F \longrightarrow X \xrightarrow{\pi} B$$

with a path-connected base space B, there is a spectral sequence $\{E^r_{p,q}, d_r\}$, called the Leray–Serre spectral sequence, with the following properties. The differentials satisfy

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

The homology $\mathscr{H}_q(F)$ is a $A[\pi_1(B)]$–module, and the E^2 page is the bigraded algebra of homology groups with twisted coefficients

$$E^2_{p,q} = H_p(B; \mathscr{H}_q(F;A)).$$

The spectral sequence converges to the homology groups

$$H_{p+q}(X;A)$$

in the sense that there is some filtration of $H_k(X;A)$

$$0 = F^{-1}_k \subseteq F^0_k \subseteq \cdots \subseteq F^k_k = H_k(X;A)$$

such that the limiting groups $E^\infty_{p,q}$ are the associated graded pieces

$$E^\infty_{p,q} = F^p_{p+q}/F^{p-1}_{p+q}.$$

We say that the spectral sequence *converges* to $H_{p+q}(X;A)$ and write

$$E^2_{p,q} \implies H_{p+q}(X;A).$$

Exercise 5. (Warm-up) Suppose the $\pi_1(B)$-action on $\mathscr{H}_q(F)$ is trivial—this is always the case if B is simply connected. Show that the E^2 page is simply the groups $E^2_{p,q} \cong H_p(B; H_q(F;A))$ with ordinary coefficients.

Exercise 6. (Warm-up). Let $\mathbb{F}$ be a field. Let F, B be path-connected topological spaces and consider a fibration $F \longrightarrow X \stackrel{\pi}{\longrightarrow} B$ with trivial monodromy.

(a) Use the Künneth formula to prove that, if all the differentials d^r are identically zero for $r \geq 2$, then

$$H_*(X) = H_*(F \times B).$$

(b) Explain the sense in which (with no assumptions on the differentials) the homology of $H_*(X)$ must be 'smaller' than the homology of $H_*(F \times B)$.

We can view the Serre spectral sequence as measuring how far the fibration π is from being the trivial fibration.

Exercise 7. (Warm-up). Let $F \longrightarrow X \stackrel{\pi}{\longrightarrow} B$ be a fibre bundle.

(a) Suppose that B is contractible. Use the Leray-Serre spectral sequence to verify that $H_*(X) \cong H_*(F)$. In fact, all fibre bundles over a contractible space are trivial bundles.

(b) Suppose alternatively that F is contractible. Show that $H_*(X) \cong H_*(B)$. In fact, if the spaces have the homotopy type of CW complexes, then in this case π must be a homotopy equivalence.

We can view the Serre spectral sequence as measuring how far the fibration π is from being the trivial fibration.

An instance of the Leray–Serre spectral sequence

As a toy example to illustrate the use of the Leray–Serre spectral sequence, we will address the following question. Consider a fibre bundle

$$S^3 \longrightarrow X \longrightarrow S^4$$

with total space X. What are the possibilities for the homology of X? The E_2 page of the Leray–Serre spectral sequence is shown in Figure 3. The only possibly-nonzero differential, a d_4 differential, is shown.

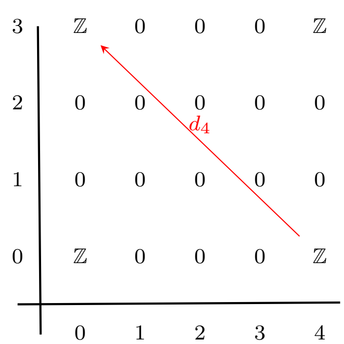


Figure 3: The E^2 page of the Leray–Serre spectral sequence for a fibration $S^3 \to X \to S^4$

Thus the spectral sequence collapses on the E_5 page, and the homology of X is given by

$$H_k(X) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/\mathrm{im}(d_4), & k = 3 \\ \ker(d_4), & k = 4 \\ \mathbb{Z}, & k = 7. \end{cases}$$

In the case this is the trivial bundle

$$S^3 \longrightarrow S^3 \times S^4 \longrightarrow S^4$$

the d_4 differential is zero. In the case of the generalized Hopf fibration

$$S^3 \longrightarrow S^7 \longrightarrow S^4$$

the nonzero d_4 differential must be an isomorphism, and kill the two copies of $\mathbb{Z}$ in homological degrees 3 and 4.

Exercise 8. As another toy application of the Leray–Serre spectral sequence, use the homology of S^1 and S^∞ and the fibration

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

to compute the homology of $\mathbb{C}P^\infty$. You may assume that $\mathbb{C}P^\infty$ is simply connected.

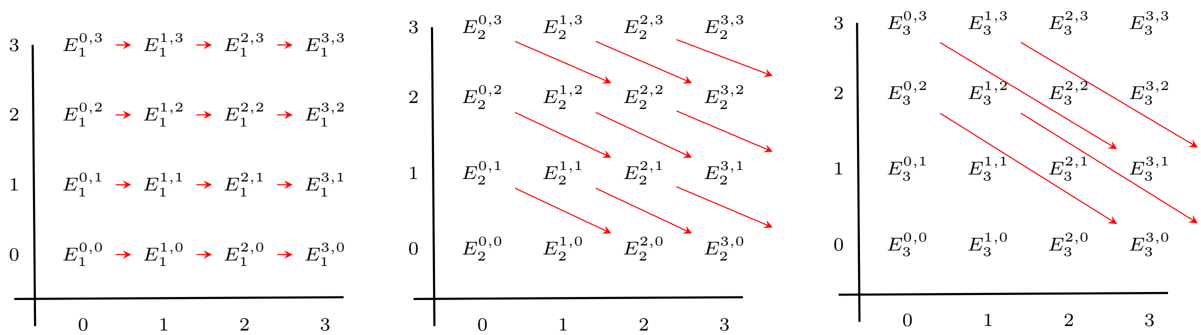
1.2 The structure of a (cohomology) spectral sequence

A (cohomology) *spectral sequence* is a sequence of bigraded abelian groups $E_r = \bigoplus_{p,q} E_r^{p,q}$, called *pages*, for $r = 0, 1, 2, \ldots$. Each page has a differential map $d^r : E_r \to E_r$ satisfying $d_r^2 = 0$, and the page E_{r+1} is the homology of the complex (E_r, d_r), in the sense that

$$E_{r+1}^{p,q} = \frac{\text{kernel of } d_r \text{ at } E_r^{p,q}}{\text{image of } d_r \text{ in } E_r^{p,q}}.$$

In particular $E_{r+1}^{p,q}$ is always a subquotient of $E_r^{p,q}$. In the examples we will study, the differentials satisfy

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$



The pages E_1, E_2, and E_3.

The Serre spectral sequence is an example of a *first quadrant spectral sequence*, that is, the groups $E_r^{p,q}$ can be nonzero only when p and q are nonnegative. This implies that, at any fixed point (p,q), for r sufficiently large, either the domain or the codomain of any differential d_r to or from $E_r^{p,q}$ will be zero. Hence, for r large we find (upon taking homology)

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots$$

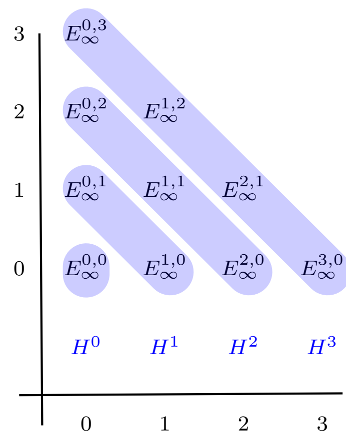
We call this stable group $E_\infty^{p,q}$, and call the bigraded abelian group $E_\infty^{*,*}$ the *limit* of the spectral sequence. In general the sequence of groups $\{E_r^{p,q}\}_r$ stabilizes at a page r that depends on (p,q). If there is some r such that $E_r^{p,q} = E_\infty^{p,q}$ for all p and q, then we say that the spectral sequence *collapses* on page E_r.

The spectral sequence *converges* to the graded A-modules H^* if, for each k, there is some filtration of H^k

$$0 \subseteq F_k^k \subseteq \cdots \subseteq F_0^k = H^k$$

such that the limiting groups $E_\infty^{p,q}$ are the associated graded pieces

$$E_\infty^{p,q} = F_p^{p+q} / F_{p+1}^{p+q}.$$



The limit of a spectral sequence.

Our example: the Serre spectral sequence (cohomology version)

Theorem III. ***(The cohomology Serre spectral sequence).*** *Let A be an abelian group. Given a fibration $F \to X \to B$ there is an associated (cohomology) spectral sequence $E_*^{p,q}$ with differentials*

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

as follows. The cohomology $\mathscr{H}^q(F)$ is a $A[\pi_1(B)]$–module, and the E_2 page is the bigraded algebra of cohomology groups with twisted coefficients

$$E_2^{p,q} = H^p(B; \mathscr{H}^q(F; A)).$$

The page E_r has a multiplication

$$E_r^{p,q} \times E_r^{s,t} \longrightarrow E_r^{p+s,q+t}$$

which is, on the E_2 page, $(-1)^{qs}$ times the cup product. The differentials d_r are derivations, satisfying

$$d_r(xy) = (d_r x)y + (-1)^{p+q} x(d_r y).$$

The spectral sequence converges to the cohomology groups

$$H^{p+q}(X)$$

in the sense that there is some filtration of $H^k(X)$

$$0 \subseteq F_k^k \subseteq \cdots \subseteq F_0^k = H^k(X)$$

such that the limiting groups $E_\infty^{p,q}$ are the associated graded pieces

$$E_\infty^{p,q} = F_p^{p+q} / F_{p+1}^{p+q}.$$

Exercise 9. (Warm-up). State and prove the analogue for Exercise 1 for a first-quadrant cohomology spectral sequence. What are the properties of the edge maps in this case?