1 Subgroups, (co)induced representations, and group (co)homology

1.1 Induced and coinduced representations

Let R be a commutative ring. We are primarily interested in the cases that $R = \mathbb{Z}$ or $R = \mathbb{Q}$.

Definition I. (Induction and co-induction). Let G be a group and $H \subseteq G$ a subgroup. If M is a $R[H]$-module, then we can construct from M an *induced* $R[G]$-module $\text{Ind}_H^G M$ by extension of scalars:

$$\text{Ind}_H^G M = R[G] \otimes_{R[H]} M.$$

and a *coinduced* $R[G]$-module by

$$\text{Coind}_H^G M = \text{Hom}_{R[H]}(R[G], M).$$

The following proposition gives a concrete description of an induced or coinduced representation.

Proposition II. *There are isomoprhisms of R-modules,*

$$\text{Ind}_H^G M \cong \bigoplus_{\sigma H \in G/H} M \qquad \text{and} \qquad \text{Coind}_H^G M \cong \prod_{\sigma H \in G/H} M.$$

In the exercises you will describe how to define the G-actions on the righthandside of these isomorphisms to promote them to isomorphisms of G-representations.

The following theorem gives, in practice, a useful way to identify a given representation as an induced representation.

Theorem III. (An orbit-stabilizer theorem). *Let H be a subgroup of a group G. Suppose that N is a $R[G]$-module such that*

- *As an R-module, $N \cong \bigoplus_{i \in I} M_i$*
- *For each $g \in G$ and $i \in I$, there is some j such that $gM_i = M_j$. In other words, G has a well-defined action by permutations on the set of summands $\{M_i\}_{i \in I}$.*
- *G permutes the summands transitively.*
- *$H \subseteq G$ is the stabilizer of M_{i_0} for some i_0*

Then $N \cong \text{Ind}_H^G M_{i_0}$.

Exercise 1. (Warm-up) Describe how $\mathbb{Z}[G]$ acts on $\text{Ind}_H^G M$ and $\text{Coind}_H^G M$.

Exercise 2. (Warm-up) Suppose $K \subseteq H \subseteq G$. Verify that $\text{Ind}_H^G\left(\text{Ind}_K^H M\right) \cong \text{Ind}_K^G M$.

Exercise 3. (Warm-up) Let $\mathbb{Z}$ denote the H-representation with trivial action. Verify that $\text{Ind}_H^G \mathbb{Z} \cong \mathbb{Z}[G/H]$, with the G-action induced by the action of G on its set of cosets G/H.

Exercise 4. (Warm-up) Let S_n denote the symmetric group on n letters. Describe the following representations. In all cases $\mathbb{Z}$ has trivial group action.

$$\text{Ind}_{S_{n-1}}^{S_n} \mathbb{Z} \qquad \text{Ind}_{S_{n-k} \times S_k}^{S_n} \mathbb{Z} \qquad \text{Ind}_{S_k}^{S_n} \mathbb{Z}$$

Exercise 5. (Warm-up) Let $\mathbb{Z}^n$ denote the canonical S_n-representation, where S_n acts by permuting a basis. Suppose $n \geq 3$. Use Theorem III to express $\bigwedge^3 \mathbb{Z}^n$ and $\text{Sym}^3 \mathbb{Z}^n$ as a (direct sum of) induced representations from proper subgroups of S_n.

Exercise 6.

(a) Describe $\mathbb{Z}[G]$ as a $\mathbb{Z}[H]$-module.

(b) Show that, as an abelian group, $\text{Ind}_H^G M \cong \bigoplus_{\sigma H \in G/H} M,$ and explain how G acts on the right-hand side.

(Some authors write the above decomposition as $\text{Ind}_H^G M \cong \bigoplus_{\sigma H \in G/H} \sigma M$ to be more suggestive of this G-action).

(c) Show that, as an abelian group, $\text{Coind}_H^G M \cong \prod_{\sigma H \in G/H} M$, and explain how G acts.

(d) Verify that if the index $[G : H]$ is finite, then there is an isomorphism of $\mathbb{Z}[G]$-modules

$$\text{Ind}_H^G M \cong \text{Coind}_H^G M$$

for any $\mathbb{Z}[H]$-module M. *Hint:* First show that the map

$$\phi : M \longrightarrow \text{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$$
$$\phi(m)(g) = \begin{cases} gm, & g \in H \\ 0, & g \notin H \end{cases}$$

extends to a $\mathbb{Z}[G]$-module map $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \to \text{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$.

Exercise 7. Prove Theorem III.

Exercise 8. (Bonus). Show that induction and coinduction are left and right adjoint functors, respectively, to restriction of scalars.

1.2 Shapiro's Lemma

Let H be a subgroup of a group G. The following result shows that the group (co)homology of H is a special case of the group (co)homology of G.

Theorem IV. (Shapiro's Lemma). *Let H be a subgroup of a group G and let M be an H-module. Then*

$$H_*(H, M) \cong H_*(G; \mathit{Ind}_H^G M) \quad \text{and} \quad H^*(H, M) \cong H^*(G; \mathit{Coind}_H^G M).$$

Exercise 9. Prove Shapiro's Lemma. *Hint:* First argue that if F is a free $\mathbb{Z}[G]$-module, then

$$F \otimes_{\mathbb{Z}[H]} M \cong F \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \cong F \otimes_{\mathbb{Z}[G]} (\text{Ind}_H^G M) \quad \text{and}$$

$$\text{Hom}_{\mathbb{Z}[H]}(F, M) \cong \text{Hom}_{\mathbb{Z}[G]}(F, \text{Coind}_H^G M).$$

Exercise 10. Let G be a group and X a CW complex $K(G, 1)$ space. Let H be a subgroup of G. Prove that

$$H_*(G; \mathbb{Z}[G/H]) \cong H_*(H) \cong H_*(\widetilde{X}),$$

where $\widetilde{X} \to X$ is the covering space of X corresponding to the subgroup H of its fundamental group G. Explain these isomorphisms using both an algebraic and a topological argument.

1.3 The rational cohomology of orbit spaces with finite stabilizers

We saw in on Worksheet #7 that, if Y is a contractible simplicial complex with a free simplicial action of a group G, then $H_*(G) \cong H_*(Y/G)$, in fact, $H_*(G; A) \cong H_*(X/G; A)$ for any abelian group A. In this section we will see that, in order to compute $H_*(G; \mathbb{Q})$, we may relax our assumptions on the simplicial G-complex Y. It will suffice to assume that G acts simplicially with finite stabilizers.

This result is key to relating, for example, the rational cohomology of a locally symmetric space to the rational cohomology of the associated arithmetic group, or the rational cohomology of moduli space to the rational cohomology of the mapping class group.

Theorem V. *Let G be a group, and let Y be a contractible simplicial complex on which G acts simplicially. Assume that the stabilizer subgroup G_σ is finite for every simplex σ of Y. Then*

$$H_*(G;\mathbb{Q}) \cong H_*(Y/G;\mathbb{Q}).$$

Exercise 11. (a) Prove the following lemma.

Lemma VI. *Let G be a group, and let Y be a simplicial complex with a simplicial action of G. For some $p \geq 0$, assume that the setwise stabilizer subgroup G_σ is finite for every p-simplex σ of Y. Then the $\mathbb{Q}[G]$-module $C_p(Y;\mathbb{Q})$ of simplicial p-chains is flat.*

Hint: Apply Theorem III to the $\mathbb{Q}[G]$-submodule generated by a simplex σ. You may assume the result that every rational representation of a finite group is projective, and hence a summand of a free module.

(b) Prove Theorem V.