1 A topological interpretation of group (co)homology

1.1 $K(G,1)$ spaces

Definition I. ($K(G,1)$ spaces). Fix a group G. A $K(G,1)$ space is a path-connected space with fundamental group isomorphic to G, and which has a contractible universal cover.

CW complex $K(G,1)$ spaces can equivalently be defined as spaces that are *aspherical*, in the following sense.

Theorem II. *A based CW complex space (X,x_0) has a contractible universal cover if and only if it is* aspherical *in the sense that $\pi_d(X,x_0)\cong 0$ for all $d\geq 2$. This means that for any $d\geq 2$ and any based map $f:(S^d,s_0)\to(X,x_0)$, the map f is nullhomotopic via a homotopy fixing s_0.*

The following theorem states that CW complex $K(G,1)$ spaces exist and have a unique homotopy type.

Theorem III. *Fix a group G. Then there exists a CW complex that is a $K(G,1)$ space, and this CW complex is unique up to homotopy equivalence.*

It follows that this homotopy type (and hence its homology and cohomology) are invariants of the group G. We will later see that the (co)homology of a $K(G,1)$ space agrees with the group (co)homology of G. Moreover, the following result shows that any group homomorphism $G\to H$ can be realized as the map induced on fundamental group by a unique (up to homotopy) map from a $K(G,1)$ to a $K(H,1)$.

Theorem IV. *Fix a group G. Let X be a connected CW complex with basepoint x_0 and let Y be a $K(G,1)$ space with basepoint y_0. Then every homomorphism $\pi_1(X,x_0)\to\pi_1(Y,y_0)$ is induced by a based map $(X,x_0)\to(Y,y_0)$ that is unique up to homotopy fixing x_0.*

Exercise 1. (Warm-up). Verify that the following are $K(G,1)$ spaces. What is G in each case?

$$S^1 \qquad \text{any graph} \qquad (S^1)^n \qquad \Sigma_g\ (g\geq 1) \qquad \mathbb{RP}^\infty$$

Exercise 2. (Warm-up). Describe a covering action of $G=\mathbb{Z}/m\mathbb{Z}$ on the infinite-dimensional sphere S^∞, and verify that the quotient is a $K(\mathbb{Z}/m\mathbb{Z},1)$ space. The quotient might be called an"infinite-dimensional lens space".

Exercise 3. (Warm-up). Verify that the product of a $K(G,1)$ space and a $K(H,1)$ space is a $K(G\times H,1)$ space.

Exercise 4.

(a) Let X be an aspherical CW complex. Show that X has a contractible universal cover. *Hint:* Whitehead's theorem.

(b) Let (X,x_0) be a based topological space with a contractible universal cover. Fix $d\geq 2$. Show that any based map $f:(S^d,s_0)\to(X,x_0)$ is nullhomotopic rel s_0.

Exercise 5. Use Theorem IV to deduce the 'uniqueness' statement in Theorem III: any two CW complex $K(G,1)$ spaces are homotopy equivalent. We will prove existence in Section 1.2.

Exercise 6. In this exercise we will prove Theorem IV. Note that we do not need to assume that Y is a CW complex.

(a) First reduce the result to the situation where X has a single vertex x_0.
Hint: Let T be a maximal tree in the 1-skeleton of X. Consider the quotient map $X\to X/T$.

(b) Let $f:\pi_1(X,x_0)\to\pi_1(Y,y_0)$ be a group homomorphism. Our goal is to construct a based map $F:(X,x_0)\to(Y,y_0)$. Explain how to define F on the 1-skeleton $X^{(1)}$ of X.

(c) Suppose we have map $f^{(k-1)}$ from the $(k-1)$-skeleton of a CW complex X to a space Y. Let e^k_α be a k-cell with attaching map $\psi^k_\alpha:S^{k-1}\to X$. Check that, to prove the map $f^{(k-1)}$ can be extended over the k-cell e^k_α, it suffices to show that $f^{(k-1)}\circ\psi^k_\alpha:S^{k-1}\to Y$ is nullhomotopic in Y.

(d) Explain why your map $F:X^{(1)}\to Y$ can be extended to a continuous map from the 2-skeleton of X.

(e) Argue that your map will extend over higher skeleta of X. *Hint:* Exercise 4.

(f) Suppose that $F_0,F_1:(X,x_0)\to(Y,y_0)$ are two based maps inducing the same map on fundamental group. Show that their restrictions to $X^{(1)}$ are homotopic.

(g) Explain how to extend this homotopy to a map $(X^{(1)} \times I) \cup (X \times \partial I) \to Y$, so in particular it is defined on the 2-skeleton of $X \times I$.

(h) Explain how to extend this homotopy over higher-dimensional cells of $X \times I$ to obtain a map from $X \times I$ to Y. Deduce that $F_0 \simeq F_1$.

1.2 A standard construction of a $K(G, 1)$ space

Fix a group G. In this section (following Hatcher Section 1.B) we will construct a contractible Δ-complex with a covering space action by G, so its quotient is a $K(G, 1)$ space.

Let EG be the Δ-complex whose n-simplices are in indexed by G^{n+1}. Its vertices are the elements of G, and an n-simplex is an ordered $(n+1)$-tuple $[g_0, g_1, \ldots, g_n]$. Note that the components $g_0, g_1, \ldots, g_n$ need not be distinct. The ith face of the n-simplex is glued to the $(n-1)$-simplex $[g_0, \ldots, g_{i-1}, \hat{g_i}, g_{i+1}, \ldots, g_n]$ with component g_i omitted.

The group G acts freely on EG. The element $g \in G$ maps the simplex $[g_0, g_1, \ldots, g_n]$ to $[gg_0, gg_1, \ldots, gg_n]$ via linear isomorphism. We let BG denote the orbit space EG/G of this action.

Proposition V. *The space EG constructed above is a contractible, and the quotient map $EG \to BG := EG/G$ is a covering space map. Thus the quotient BG is a $K(G, 1)$ space.*

Exercise 7. (Warm-up). What is the dimension of EG and BG?

Exercise 8. (Warm-up). Sketch a small-dimensional skeleton of EG and BG for $G = \mathbb{Z}/2\mathbb{Z}$. What about $G = \mathbb{Z}$?

Exercise 9. (Warm-up). Verify that the action of G on EG is continuous, is free, and is a covering space action.

Exercise 10. Verify that EG is contractible.
Hint: Define a homotopy h_t that slides a point x in the simplex $[g_1, \ldots, g_n]$ along the straight line connecting x to $[e]$ in the simplex $[e, g_1, \ldots, g_n]$. Check that h_t is continuous.

1.3 The bar construction

Consider the augmented simplicial chain complexes $\widetilde{C}_*(EG)$ and the chain complex $C_*(BG)$.

Proposition VI. *The augmented simplicial chain complex*

$$\cdots \longrightarrow C_2(EG) \longrightarrow C_1(EG) \longrightarrow C_0(EG) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. We can identify the G-coinvariants $C_p(EG)_G$ with the simplicial chain group $C_p(BG)$. Thus the homology $H_*(BG)$ of the chain complex

$$\cdots \longrightarrow C_2(BG) \longrightarrow C_1(BG) \longrightarrow C_0(BG) \longrightarrow 0$$

agrees with the group homology $H_*(G)$. Similarly the cohomology groups $H^*(BG)$ agree with the group cohomology $H^*(G)$.

Observe that, in every G-orbit of p-simplices of EG, there is a unique simplex with first component the identity $e \in G$. We can change our indexing convention to express this simplex in the form $[e, g_1, g_1g_2, g_1g_2g_3, \ldots]$. It is conventional to denote this simplex using the notation $[g_1|g_2|\ldots|g_p]$. With this "bar notation", the differential takes the form $d = \sum_{i=0}^{p}(-1)^i d_i$, where

$$\begin{aligned} d_0 &: [g_1|g_2|\ldots|g_p] \longmapsto [g_2|g_3|\ldots|g_p] \\ d_i &: [g_1|g_2|\ldots|g_p] \longmapsto [g_1|g_2|\ldots|g_ig_{i+1}|\ldots|g_p], \quad 1 \leq i \leq p-1 \\ d_p &: [g_1|g_2|\ldots|g_p] \longmapsto [g_1|g_2|\ldots|g_{p-1}] \end{aligned}$$

In this notation, the chain complex for BG is called the *bar construction*. (In the literature the term *bar construction* often refers to a more general concept, frequently formalized using the theory of *simplicial objects*.)

This model BG for a $K(G,1)$ is quite 'large' in general—for example, it is always infinite-dimensional—and it is not always convenient for explicit calculations. It does, however, have favourable formal properties, such as functoriality with respect to the group G.

Proposition VII. *The assignment* $G \mapsto (BG, [e])$ *defines a functor from the category of groups to the category of based topological spaces. A group homomorphism* $f : G \to H$ *defines a continuous based map*

$$\begin{aligned} Bf : BG &\longrightarrow BH \\ [g_1|g_2|\dots|g_p] &\longmapsto [f(g_1)|f(g_2)|\dots|f(g_p)] \end{aligned}$$

It satisfies $\pi_1(BG,[e]) \cong G$ *and* $\pi_1(Bf) = f$*, that is,* B *is a right inverse to the functor* π_1*.*

Corollary VIII. *The assignment* $G \mapsto C_*(BG)$ *defines a functor from the category of groups to the category of chain complexes.*

Corollary VIII gives an alternate proof that group homology $H_*(G)$ and cohomology $H^*(G)$ are functorial in G.

Exercise 11. Verify Proposition VI.

Exercise 12. Verify the formula for the differential of the simplex $[g_1|g_2|\dots|g_p]$ in bar notation.

Exercise 13. Verify that the map Bf of Proposition VII is continuous, and that it induces the map f on fundamental group.