1 The (co)homology of groups

1.1 The definition of group homology

Definition I. (Group homology). Let G be a (discrete) group. Then the *homology of* G is defined to be

$$H_*(G) := \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}).$$

where $\mathbb{Z}$ is the $\mathbb{Z}[G]$-module with trivial G-action. More generally, if M is a $\mathbb{Z}[G]$-module, then we define the *homology of G with coefficients in M* to be

$$H_*(G; M) := \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Observe that $H_*(G)$ is equal to $H_*(G; \mathbb{Z})$.

Using the results on the Tor functor, we can therefore compute $H_*(G; M)$ in the following ways.

(1) Take a projective (or, more generally, flat) resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ by right $\mathbb{Z}[G]$-modules,

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

Delete the term $P_{-1} = \mathbb{Z}$ and apply the functor $- \otimes_{\mathbb{Z}[G]} M$. Then $H_*(G; M)$ is the homology of the complex

$$\cdots \longrightarrow P_{n+1} \otimes_{\mathbb{Z}[G]} M \longrightarrow P_n \otimes_{\mathbb{Z}[G]} M \longrightarrow P_{n-1} \otimes_{\mathbb{Z}[G]} M \longrightarrow \cdots \longrightarrow P_0 \otimes_{\mathbb{Z}[G]} M \longrightarrow 0.$$

(2) Take a projective (or, more generally, flat) resolution of M by left $\mathbb{Z}[G]$-modules,

$$\cdots \longrightarrow P_{n+1'} \longrightarrow P'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow 0$$

Delete the term $P'_{-1} = M$ and apply the functor $\mathbb{Z} \otimes_{\mathbb{Z}[G]} -$. Then $H_*(G; M)$ is the homology of the complex

$$\cdots \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_{n+1} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_n \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_{n-1} \longrightarrow \cdots \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_0 \longrightarrow 0$$

Proposition II. *Let R be a ring and B an $R[G]$-module. Then there are isomorphisms of abelian groups*

$$H_*(G; B) \cong \mathrm{Tor}_*^{R[G]}(R, B).$$

In particular, if V is a representation of G over $\mathbb{Q}$, then $H_*(G; V) \cong \mathrm{Tor}_*^{\mathbb{Q}[G]}(\mathbb{Q}, V)$.

Exercise 1. (Warm-up). What is $H_*(G; \mathbb{Z}[G])$? More generally, what is $H_*(G, M)$ when M is a flat $\mathbb{Z}[G]$-module?

Exercise 2. (Warm-up). What is the homology of the trivial group?

Exercise 3. In this exercise we prove Proposition II. Let R be a ring. Prove that

$$\mathrm{Tor}_*^{R[G]}(R, B) \cong \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, B)$$

for all right $R[G]$-modules B. Conclude that

$$H_*(G; B) \cong \mathrm{Tor}_*^{R[G]}(R, B).$$

Hint: Let $F_\bullet \to \mathbb{Z}$ be a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules. First verify that $R \otimes_{\mathbb{Z}} F_\bullet \to R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ is still exact, whether or not R is flat. It may be useful to note that $\mathbb{Z}[G]$ is flat as an abelian group.

1.2 The definition of group cohomology

Definition III. (Group cohomology). Let G be a (discrete) group. Then the *cohomology of G* is defined to be

$$H^*(G) := \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}).$$

More generally, if M is a $\mathbb{Z}[G]$-module, then we define the *cohomology of G with coefficients in M* to be

$$H^*(G; M) := \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Proposition IV. *Let R be a ring and A an $R[G]$-module. Then there are isomorphisms of abelian groups*

$$H^*(G; A) \cong \mathrm{Ext}^*_{R[G]}(R, A).$$

In particular, if V is a representation of G over $\mathbb{Q}$, then $H^*(G; V) \cong \mathrm{Ext}^*_{\mathbb{Q}[G]}(\mathbb{Q}, V)$.

Exercise 4. (Warm-up). What is the cohomology of the trivial group?

Exercise 5. Describe two procedures for computing $H^*(G; M)$ in the style of Section 1.1.

1.3 Group invariants and coinvariants

Definition V. (Invariants; Coinvariants). Let G be a group and M a left $\mathbb{Z}[G]$-module. The group of *invariants* of M, denoted M^G, is the submodule of M

$$M^G := \{m \in M \mid gm = m \text{ for all } g \in G\}.$$

The group of *coinvariants* of M, denoted M_G, is defined to be the quotient of M

$$M_G := M/\langle gm - m \mid g \in G, m \in M\rangle.$$

The group M^G is the largest submodule of M with trivial G action, and M_G is the largest quotient of M with trivial G action.

Proposition VI. *If M is a left $\mathbb{Z}[G]$-module and $\mathbb{Z}$ the trivial right $\mathbb{Z}[G]$-module, then there is a natural isomorphism*

$$\mathbb{Z} \otimes_{\mathbb{Z}[G]} M \cong M_G.$$

Proposition VII. *If A is a left $\mathbb{Z}[G]$-module and $\mathbb{Z}$ the trivial left $\mathbb{Z}[G]$-module, then there is a natural isomorphism*

$$Hom_{\mathbb{Z}[G]}(\mathbb{Z}, A) \cong A^G.$$

Thus, given a a flat resolution $P'_\bullet \to M$ of M, we can identify $H_*(G; M)$ with the homology of the complex

$$\cdots \longrightarrow (P'_{n+1})_G \longrightarrow (P'_n)_G \longrightarrow (P'_{n-1})_G \longrightarrow \cdots \longrightarrow (P'_0)_G \longrightarrow 0.$$

In particular, if we can construct a flat resolution of M with $(P'_k)_G = 0$ for some n, we can deduce that $H_k(G; M) = 0$.

Analogously, given an injective coresolution $A \to I_\bullet$ of a left $\mathbb{Z}[G]$-module A, then we can identify $H^*(G; A)$ with the homology of the complex

$$0 \longrightarrow (I_0)^G \longrightarrow (I_1)^G \longrightarrow \cdots \longrightarrow (I_{n-1})^G \longrightarrow (I_n)^G \longrightarrow (I_{n+1})^G \longrightarrow \cdots.$$

Exercise 6. (Warm-up). Formulate the universal properties of the $\mathbb{Z}[G]$-module maps $M \twoheadrightarrow M_G$ and $M^G \hookrightarrow M$.

Exercise 7. (Warm-up). Verify the natural isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M \cong M_G$.

Exercise 8. (Warm-up). Verify the natural isomorphism $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \cong A^G$.

Exercise 9. (Warm-up). Formulate and prove the analogue of Definition V and Proposition VI for right $\mathbb{Z}[G]$-modules M and the functor $- \otimes_{\mathbb{Z}[G]} \mathbb{Z}$.

Exercise 10. Compute $H_*(G)$ directly from a free resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$ ...

(a) ... when G is a finite cyclic group.

(b) ... for $G = \mathbb{Z}$.

Hint: Word of caution: For $G = \mathbb{Z}$, identify the group ring $\mathbb{Z}[\mathbb{Z}]$ with $\mathbb{Z}[t, t^{-1}]$. This notation helps avoid confusion between coefficients (and addition of coefficients), and elements of the group G (and the group operation on G).

Exercise 11. Compute $H^*(G)$ directly from a free resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$ ...

(a) ... when G is a finite cyclic group.

(b) ... for $G = \mathbb{Z}$.

1.4 Functoriality

Let $\alpha : G \to G'$ be a group homomorphism. Then α induces maps on group (co)homology, as follows.

Let $P_\bullet \to \mathbb{Z}$ and $P'_\bullet \to \mathbb{Z}$ be resolutions of $\mathbb{Z}$ by projective $\mathbb{Z}[G]$- and $\mathbb{Z}[G']$-modules, respectively. The homomorphism $\alpha : G \to G'$ gives every $\mathbb{Z}[G']$-module an induced $\mathbb{Z}[G]$-module structure. Thus we can view $P'_\bullet \to \mathbb{Z}$ as an (exact but not necessarily projective) resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules, and view the identity map $\mathbb{Z} \xrightarrow{=} \mathbb{Z}$ as a map of $\mathbb{Z}[G]$-modules.

The fundamental theorem of homological algebra therefore allows us to extend $\mathbb{Z} \xrightarrow{=} \mathbb{Z}$ to a $\mathbb{Z}[G]$-equivariant chain map $P_\bullet \to P'_\bullet$, which is well-defined up to homotopy. When we apply the functors $\mathbb{Z} \otimes_{\mathbb{Z}[G]} -$ or $\mathrm{Hom}_{\mathbb{Z}[G]}(-, \mathbb{Z})$ and take homology, we obtain well-defined maps of (co)homology groups

$$\alpha_* : H_*(G) \to H_*(G') \qquad \text{or} \qquad \alpha^* : H^*(G') \to H^*(G).$$

Theorem VIII. *Group homology $H_*(-, \mathbb{Z})$ and cohomology $H^*(-, \mathbb{Z})$ are functors from groups to graded abelian groups.*

The following functoriality result follows from the formal properties of Tor and Ext.

Theorem IX. *Group homology $H_*(G, -)$ and cohomology $H^*(G, -)$ are functors from $\mathbb{Z}[G]$-modules to graded abelian groups.*

We can prove the following proposition by a direct analysis of the induced map on the level of chains.

Proposition X. *Let G be a group and $g \in G$. Then the action of G on G via conjugation by g induces the trivial map on $H_*(G)$.*

Corollary XI. *Let G be a group and N a normal subgroup. Then the action of G on N by conjugation induces an action of the quotient G/N on $H_*(N)$.*

Exercise 12. (Warm-up).

(a) Verify that a group homomorphism $\alpha : G \to G'$ induces a map of rings $\mathbb{Z}[G] \to \mathbb{Z}[G']$. Explain how this induces a $\mathbb{Z}[G]$-module structure on any $\mathbb{Z}[G']$-module.

(b) Let M be a $\mathbb{Z}[G]$-module and M' a $\mathbb{Z}[G']$-module. Verify that a map of abelian groups $\tau : M \to M'$ is a map of $\mathbb{Z}[G]$-modules (with respect to the $\mathbb{Z}[G]$-module structure on M' induced by τ) precisely when it satisfies the condition

$$\tau(gm) = \alpha(g)\tau(m) \qquad \text{for all } m \in M \text{ and all } g \in G.$$

Exercise 13. (Warm-up). Let $\alpha : G \to G'$ be a group homomorphism. Explain the construction of the induced homomorphisms $\alpha_* : H_*(G) \to H_*(G')$ and $\alpha^* : H^*(G') \to H^*(G)$, and verify that these maps are well-defined.

Exercise 14. Prove Proposition X.

Exercise 15. Let m be a positive integer. Compute the maps induced on group (co)homology by the homomorphism of groups $\mathbb{Z} \xrightarrow{m} \mathbb{Z}$.

Exercise 16. (Bonus). Compute the maps induced on group (co)homology by the following homomorphisms.

(a) The surjection $\mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z}$.

(b) The inclusion $\mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$

(c) The inclusion $\mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathbb{Z}/9\mathbb{Z}$

(d) The surjection $\mathbb{Z}/9\mathbb{Z} \xrightarrow{\text{mod } 3} \mathbb{Z}/3\mathbb{Z}$.