1 Review: The Tor and Ext functors

1.1 Projective R-modules

Definition I. (Projective modules). Let R be a ring and P an R-module. Then P is called *projective* if it satisfies the equivalent conditions in the following theorem.

Theorem II. (Equivalent definitions of projective modules). *Let R be a ring and P an R-module.*

(i) For every surjective R-module homomorphism $d : N \to M$ and every R-module homomorphism $f : P \to M$, there is a (not necessarily unique) homomorphism $g : P \to N$ such that $d \circ h = f$.

$$\begin{array}{ccccc} P & & & & \\ {\scriptstyle \exists g}\,\vdots & \searrow^{f} & & & \\ N & \xrightarrow[d]{} & M & \longrightarrow & 0 \end{array}$$

(ii) Any short exact sequence of R-modules of the following form is split

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0.$$

(iii) P is a direct summand of a free R-module

(iv) The covariant functor $M \mapsto Hom_R(P, M)$ from the category of R-modules to the category of abelian groups is an exact functor.

Definition III. (Resolutions). Let R be a ring. Given an R-module M, an [adjective] resolution of M is an exact sequence of R-modules

$$\longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each term P_i is an [adjective] R-module. In particular, a projective resolution of M is a resolution as above by projective R-modules P_i.

Theorem IV. *Every R-module M admits a projective resolution.*

Theorem V. (The Fundamental Theorem of Homological Algebra). *Let R be a ring. Let (C, δ) and (C', δ') be chain complexes of R-modules and let r be an integer. Let $f_i : C_i \to C'_i$ be a family of maps for $0 \leq i \leq r$ making the following diagram commute.*

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{r+1}} & C_r & \xrightarrow{\partial_r} & C_{r-1} & \xrightarrow{\partial_{r-1}} & \ldots \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \longrightarrow 0 \\
 & & & & \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_{r-1}} & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & \\
\ldots & \longrightarrow & C'_{r+1} & \xrightarrow{\delta_{r+1}} & C'_r & \xrightarrow{\delta_r} & C'_{r-1} & \xrightarrow{\delta_{r-1}} & \ldots \xrightarrow{\delta_2} & C'_1 & \xrightarrow{\delta_1} & C'_0 & \longrightarrow 0
\end{array}$$

Assume

- *C_i is projective for all $i > r$,*
- *$H_i(C') = 0$ for all $i \geq r$.*

Then the maps f_i extend to a chain map $f : (C, \delta) \to (C', \delta')$,

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \longrightarrow 0 \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & \\
\ldots & \longrightarrow & C'_{n+1} & \xrightarrow{\delta_{n+1}} & C'_n & \xrightarrow{\delta_n} & C'_{n-1} & \xrightarrow{\delta_{n-1}} & \ldots \xrightarrow{\delta_2} & C'_1 & \xrightarrow{\delta_1} & C'_0 & \longrightarrow 0
\end{array}$$

and this chain map is unique up to homotopy. In fact, any two extensions are homotopic by a chain homotopy h such that $h_i = 0$ for all $i \leq r$.

Theorem VI. *Let R be a ring. Let $P_\bullet \to M$ and $P'_\bullet \to M$ be projective resolutions of an R-module M. Then there exists a chain map f as in the following commutative diagram,*

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & M & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} & & & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle id_M} & & \\
\cdots & \longrightarrow & P'_{n+1} & \xrightarrow{\delta_{n+1}} & P'_n & \xrightarrow{\delta_n} & P'_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots & \xrightarrow{\delta_1} & P'_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0
\end{array}$$

This chain map is unique up to homotopy, and is a homotopy equivalence.

Theorem VII. *Let M, N be modules over a ring R, and let $f_0 : M \to N$ be an R-module map. Then for any projective resolutions $P_\bullet \to M$ and $P'_\bullet \to N$ of M and N, there is a chain map (unique up to homotopy) extending the map f_0.*

Theorem VIII. (Simultaneous resolutions). *Let R be a ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of R-modules. Let $P_\bullet \to L$ and $P'_\bullet \to N$ be projective resolutions. Then there is a projective resolution of M by the projective modules $P_n \oplus P'_n$ making the following diagram commute.

$$\begin{array}{ccccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_{n+1} \oplus P'_{n+1} & \longrightarrow & P_n \oplus P'_n & \longrightarrow & P_{n-1} \oplus P'_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 \oplus P'_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P'_{n+1} & \longrightarrow & P'_n & \longrightarrow & P'_{n-1} & \longrightarrow & \cdots & \longrightarrow & P'_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

The columns of this diagram are split exact.

Exercise 1. (Warm-up). Verify that free modules are projective.

Exercise 2. (Warm-up). Is $\mathbb{Z}/2\mathbb{Z}$ a projective $\mathbb{Z}/4\mathbb{Z}$-module? A projective $\mathbb{Z}/6\mathbb{Z}$-module?

Exercise 3. (Warm-up). Give an example of a module that is projective but not free.

Exercise 4. (Warm-up). Prove Theorem II.

Exercise 5. (Warm-up). Prove Theorem IV.

Exercise 6. (Warm-up).

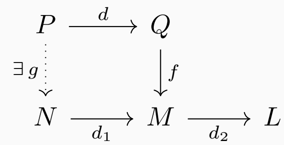
(a) Suppose the following diagram

$$\begin{array}{ccccc}
P & \xrightarrow{d} & Q & & \\
 & & \big\downarrow{\scriptstyle f} & & \\
N & \xrightarrow[d_1]{} & M & \xrightarrow[d_2]{} & L
\end{array}$$

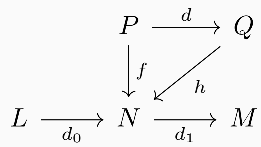
satisfies

- The bottom row is exact
- $d_2 \circ f \circ d = 0$
- P is projective

Show that there exists an R-module map g making the diagram commute:



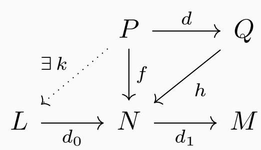
(b) Suppose the following (not necessarily commutative) diagram



satisfies

- The bottom row is exact
- $d_1 \circ h \circ d = d_1 \circ f$
- P is projective

Show that it is possible to find a map k as below such that $d_0 \circ k + h \circ d = f$.



(c) Prove Theorem V.

(d) Deduce Theorem VI.

(e) Deduce Theorem VII.

Exercise 7. (Bonus). Prove Theorem VIII.

Exercise 8. (Bonus). Is the tensor product of projective R-modules projective? What if R is commutative?

1.2 The Tor functor and flat modules

Definition IX. (The Tor functor). Let R be a ring. Let D be a right R-module and let B be a left R-module. Let

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0$$

be a projective resolution of B by left R-modules. Then we define the groups $\mathrm{Tor}_*^R(D, B)$ to be the homology of the chain complex

$$\cdots \longrightarrow D \otimes_R P_{n+1} \xrightarrow{1 \otimes d_{n+1}} D \otimes_R P_n \xrightarrow{1 \otimes d_n} D \otimes_R P_{n-1} \xrightarrow{1 \otimes d_{n-1}} \cdots \xrightarrow{1 \otimes d_1} D \otimes_R P_0 \longrightarrow 0$$

Theorem VI implies the following result, which states that this construction is well-defined.

Proposition X. *The groups* $\mathrm{Tor}_*^R(D, B)$ *do not depend on the choice of projective resolution of* B.

Theorem VII implies that a map of R-modules induces a homomorphism of the associated Tor groups.

Proposition XI. *For a fixed left* R*-module* D*, the assignment* $B \mapsto \mathrm{Tor}_*^R(D, B)$ *is a functor from right* R*-modules to abelian groups.*

The Tor functors measure, in a sense, the failure of the tensor product $\otimes_R$ to be exact, as we see in the following theorem.

Theorem XII. (The Tor long exact sequences). *Let R be a ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of left R-modules. Then there is a long exact sequence of abelian groups

$$\cdots \to \mathrm{Tor}^R_{i+1}(D,N) \to \mathrm{Tor}^R_i(D,L) \to \mathrm{Tor}^R_i(D,M) \to \mathrm{Tor}^R_i(D,N) \to \mathrm{Tor}^R_{i-1}(D,L) \to \cdots$$

$$\cdots \longrightarrow \mathrm{Tor}^R_1(D,N) \longrightarrow D \otimes_R L \longrightarrow D \otimes_R M \longrightarrow D \otimes_R N \longrightarrow 0$$

Theorem XII is the long exact sequence obtained from the short exact sequence of chain complexes we can construct by applying the functor $D \otimes_R -$ to the complex in Theorem VIII.

Definition XIII. (Flat modules). Let R be a ring. A left R-module D is *flat* if the functor $B \mapsto D \otimes_R B$ is an exact functor. Similarly, a right R-module D is *flat* if the functor $B \mapsto B \otimes_R D$ is exact.

Proposition XIV. *Let D be a right module over a ring R. The following are equivalent.*

(i) *D is flat.*

(ii) $\mathrm{Tor}^R_1(D,B) = 0$ *for every left R-module B.*

(iii) $\mathrm{Tor}^R_i(D,B) = 0$ *for every left R-module B and all $i \geq 1$.*

Proposition XV. *A module A over a PID is flat if and only if it is torsion-free.*

Exercise 9. (Warm-up). Verify that the sequence $(D \otimes P_\bullet, 1 \otimes d_\bullet)$ is indeed a chain complex, that is, verify that

$$(1 \otimes d_{n-1}) \circ (1 \otimes d_n) = 0$$

Exercise 10. (Warm-up). Verify that there is a natural isomorphism $\mathrm{Tor}^R_0(D,B) \cong D \otimes_R B$.

Exercise 11. (Warm-up).

(a) Show that a direct summand of a flat module is flat.

(b) Show that projective modules are flat.

Exercise 12. (Warm-up). Show that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, but $\mathbb{Q}/\mathbb{Z}$ is not.

Exercise 13. (Warm-up). Suppose that A and B are flat R-modules. Show that $A \otimes_R B$ is flat.

Exercise 14. Let R be a PID. Explain why $\mathrm{Tor}^R_n(B,D) = 0$ for all $n \geq 2$.

Exercise 15. (a) Verify that the following is a projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}/m\mathbb{Z}$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

(b) Let D be a $\mathbb{Z}$-module. Verify that

$$\mathrm{Tor}^{\mathbb{Z}}_0(D, \mathbb{Z}/m\mathbb{Z}) \cong D \otimes \mathbb{Z}/m\mathbb{Z} \cong D/mD$$
$$\mathrm{Tor}^{\mathbb{Z}}_1(D, \mathbb{Z}/m\mathbb{Z}) \cong \{d \in D \mid md = 0\}, \text{ the subgroup annihilated by } m,$$
$$\mathrm{Tor}^{\mathbb{Z}}_n(D, \mathbb{Z}/m\mathbb{Z}) = 0 \text{ for all } n \geq 2.$$

(c) Let D be a $\mathbb{Z}/m\mathbb{Z}$-module (and, in particular, a $\mathbb{Z}$-module). What is $\mathrm{Tor}^{\mathbb{Z}/m\mathbb{Z}}_*(D, \mathbb{Z}/m\mathbb{Z})$? Conclude that Tor groups depend on the ring R.

Exercise 16. Prove Proposition XIV.

Exercise 17. (Bonus). Verify the details of Theorem XII.

Exercise 18. (Bonus). Prove Proposition XV.

Computing Tor

The groups $\mathrm{Tor}_*^R(D,B)$ can in fact be calculated using projective resolutions of either D or B. The following result will be easier to prove once we complete our unit on spectral sequences.

Theorem XVI. (The Tor via resolutions of either variable). *Let R be a ring. Let D be a right R-module and let B be a left R-module. Let*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} D \longrightarrow 0$$

be a projective resolution of D by right R-modules. Then $\mathrm{Tor}_*^R(D,B)$ is equal to the homology of the chain complex

$$\cdots \longrightarrow P_{n+1}\otimes_R B \xrightarrow{d_{n+1}\otimes 1} P_n\otimes_R B \xrightarrow{d_n\otimes 1} P_{n-1}\otimes_R B \xrightarrow{d_{n-1}\otimes 1} \cdots \xrightarrow{d_1\otimes 1} P_0\otimes_R B \longrightarrow 0$$

We obtain a second long exact sequence on Tor groups, analogous to Theorem XII.

Theorem XVII. (The Tor long exact sequences). *Let R be a ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of right R-modules. Then there is a long exact sequence of abelian groups

$$\cdots \to \mathrm{Tor}_{i+1}^R(N,B) \to \mathrm{Tor}_i^R(L,B) \to \mathrm{Tor}_i^R(M,B) \to \mathrm{Tor}_i^R(N,B) \to \mathrm{Tor}_{i-1}^R(L,B) \to \cdots$$

$$\cdots \longrightarrow \mathrm{Tor}_1^R(N,B) \longrightarrow L\otimes_R B \longrightarrow M\otimes_R B \longrightarrow N\otimes_R B \longrightarrow 0$$

In fact, to compute $\mathrm{Tor}_*^R(D,B)$, it suffices to take *flat* resolutions $P_\bullet \to B$ or $Q_\bullet \to D$.

Theorem XVIII. (Tor via flat resolutions). *Let R be a ring. Let D be a right R-module and let B be a left R-module. Then the description of $\mathrm{Tor}_*^R(D,B)$ given in Definition IX (respectively, Theorem XVI) holds even if we assume the resolution of B (respectively, D) is merely flat and not necessarily projective.*

Theorem XIX. (Change of rings for Tor). *Let $\phi: R\to S$ be a ring homomorphism (preserving unit), so every S-module may be viewed as an R-module. Then*

- *Let B be a right S-module and C a left R-module. If S is flat as a R-module, then there are natural isomorphisms*

$$\mathrm{Tor}_p^R(B,C) \cong \mathrm{Tor}_p^S(B, S\otimes_R C) \qquad \text{for all } p\in\mathbb{Z}.$$

- *Let B be a right R-module and C a left S-module. If S is flat as a R-module, then there are natural isomorphisms*

$$\mathrm{Tor}_p^R(B,C) \cong \mathrm{Tor}_p^S(B\otimes_R S, C) \qquad \text{for all } p\in\mathbb{Z}.$$

Exercise 19. (Warm-up). State the analogue of Proposition XIV for left R-mdoules B.

Exercise 20. (Warm-up). Let G be a group.

(a) Show that $\mathbb{Q}[G]$ is a flat $\mathbb{Z}[G]$-module. *Hint:* First verify $\mathbb{Q}[G]\cong \mathbb{Z}[G]\otimes_{\mathbb{Z}}\mathbb{Q}$.

(b) Let M a left $\mathbb{Z}[G]$-module, and V a right $\mathbb{Q}[G]$-module. Use Theorem XIX to prove that

$$\mathrm{Tor}_*^{\mathbb{Z}[G]}(M,V) \cong \mathrm{Tor}_*^{\mathbb{Q}[G]}(M\otimes_{\mathbb{Z}}\mathbb{Q}, V)$$

Exercise 21. (Bonus).

(a) Let R be a ring. Let D be a right R-module and let B be a left R-module. Let $P_\bullet \to B$ and $Q_\bullet \to D$ be projective resolutions. Use the spectral sequence associated to the double complex $P_\bullet \otimes_R Q_\bullet$ to prove Theorem XVI.

(b) Would your proof work if we had only assumed that $P_\bullet \to B$ and $Q_\bullet \to D$ are flat resolutions? Deduce Theorem XVIII.

Exercise 22. (Bonus). Prove Theorem XIX. *Hint:* Consider $S \otimes_R P_\bullet$, where $P_\bullet$ is a free resolution of C by R-modules.

1.3 The Ext functor and injective modules

Definition XX. (The Ext functor). Let A and D be modules over a ring R. Let

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

be a projective resolution of A. Then we define the groups $\operatorname{Ext}^*_R(A, D)$ to be the homology of the cochain complex

$$0 \longrightarrow \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1^*} \operatorname{Hom}_R(P_1, D) \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} \operatorname{Hom}_R(P_{n-1}, D) \xrightarrow{d_n^*} \operatorname{Hom}_R(P_n, D) \xrightarrow{d_{n+1}^*} \cdots$$

Proposition XXI. *The groups $\operatorname{Ext}^*_R(A, D)$ do not depend on the choice of projective resolution of A.*

Proposition XXII. *For fixed D, the assignment $A \mapsto \operatorname{Ext}^*_R(A, D)$ defines a functor from (left) R-modules to abelian groups.*

As with Tor, the Ext functor in a sense measures the failure of the Hom functor to be exact. It determines the following two long exact sequences.

Theorem XXIII. (The Ext long exact sequences). *Let R be a ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of R-modules. Then there is a long exact sequence

$$0 \longrightarrow Hom_R(N, D) \to Hom_R(M, D) \to Hom_R(L, D) \to \operatorname{Ext}^1_R(N, D) \longrightarrow \cdots$$

$$\cdots \to \operatorname{Ext}^{i-1}_R(L, D) \longrightarrow \operatorname{Ext}^i_R(N, D) \longrightarrow \operatorname{Ext}^i_R(M, D) \to \operatorname{Ext}^i_R(L, D) \to \operatorname{Ext}^{i+1}_R(N, D) \to \cdots$$

and a long exact sequence

$$0 \longrightarrow Hom_R(D, L) \longrightarrow Hom_R(D, M) \to Hom_R(D, N) \to \operatorname{Ext}^1_R(D, L) \longrightarrow \cdots$$

$$\cdots \to \operatorname{Ext}^{i-1}_R(D, N) \longrightarrow \operatorname{Ext}^i_R(D, L) \longrightarrow \operatorname{Ext}^i_R(D, M) \to \operatorname{Ext}^i_R(D, N) \to \operatorname{Ext}^{i+1}_R(D, L) \to \cdots$$

Definition XXIV. (Injective modules). Let R be a ring and I an R-module. Then D is called *injective* if it satisfies the equivalent conditions in the following theorem.

Theorem XXV. (Equivalent definitions of injective modules). *Let D be a left module over a ring R. Show that the following are equivalent.*

(i) *The functor $B \mapsto Hom_R(B, D)$ is exact.*

(ii) *For any R-modules M, N, any short exact sequence of the following form is split*

$$0 \longrightarrow D \longrightarrow M \longrightarrow N \longrightarrow 0.$$

(iii) *If D is a submodule of an R-module M, then D has a direct complement in M, that is, there is some $L \subseteq M$ so that M is the internal direct sum $M = D \oplus L$.*

(iv) *If $f : L \to M$ is an injective map of R-modules, and $g : L \to D$ is any map of R-modules, then there exists a (not necessarily unique) extension of g to M making the following diagram commute.*

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M \\ & & & \searrow^{g} & \downarrow{\scriptstyle \exists h} \\ & & & & D \end{array}$$

(v) $\operatorname{Ext}^1_R(M, D) = 0$ *for every right R-module M.*

(vi) $\operatorname{Ext}^i_R(M, D) = 0$ *for every right R-module M and all $i \geq 1$.*

Proposition XXVI. *An module A over a PID is injective if and only if it is divisible, that is, $nA = A$ for all $n \in \mathbb{Z}_{>0}$.*

Exercise 23. (Warm-up). Verify that the sequence $(\mathrm{Hom}_R(P_\bullet, D), d_\bullet^*)$ is indeed a chain complex.

Exercise 24. (Warm-up). Verify that there is a natural isomorphism $\mathrm{Ext}^0_R(A, D) \cong \mathrm{Hom}_R(A, D)$.

Exercise 25. (Warm-up). Prove Theorem XXV.

Exercise 26. (Warm-up). Verify that every module over a field is projective, injective, and flat.

Exercise 27. Determine whether the following are free, projective, injective, and/or flat as $\mathbb{Z}$-modules.
(i) $\mathbb{Z}$ (ii) $\mathbb{Z}^2$ (iii) $\mathbb{Z}/m\mathbb{Z}$ (iv) $\mathbb{Q}$ (v) $\mathbb{Q}/\mathbb{Z}$ (vi) $\mathbb{Q}^2$ (vii) $\mathbb{Q} \oplus \mathbb{Z}$ (viii) $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}$

Exercise 28. Let D be a $\mathbb{Z}$-module. Verify that

$$\begin{aligned}
&\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong \{d \in D \mid md = 0\}, \text{ the subgroup annihilated by } m,\\
&\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD\\
&\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0 \text{ for all } n \geq 2.
\end{aligned}$$

Exercise 29. (Bonus). Prove Proposition XXVI.

Exercise 30. (Bonus). Prove that every R-module M has an injective coresolution

$$0 \longrightarrow M \xrightarrow{\ \epsilon\ } I_0 \xrightarrow{\ d_1\ } I_1 \xrightarrow{\ d_2\ } I_2 \longrightarrow \cdots$$

Computing Ext

As with Tor, we can compute the groups $\mathrm{Ext}^*_R(A, D)$ using a resolution of either variable.

Theorem XXVII. (Ext via injective resolutions). *Let A and D be modules over a ring R. Let*

$$0 \longrightarrow D \xrightarrow{\ \epsilon\ } I_0 \xrightarrow{\ d_1\ } I_1 \xrightarrow{\ d_2\ } I_2 \longrightarrow \cdots$$

be an injective coresolution of D. Then the groups $\mathrm{Ext}^*_R(A, D)$ are equal to the homology of the cochain complex

$$0 \longrightarrow Hom_R(A, I_0) \xrightarrow{d_1^*} Hom_R(A, I_1) \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} Hom_R(A, I_{n-1}) \xrightarrow{d_n^*} Hom_R(A, I_n) \xrightarrow{d_{n+1}^*} \cdots$$

Theorem XXVIII. (Change of rings for Ext). *Let $\phi : R \to S$ be a ring homomorphism (preserving unit), so every S-module may be viewed as an R-module.*

- *Let A be a left S-module and C a left R-module. If S is flat as an R-module, then there are natural isomorphisms*

$$\mathrm{Ext}^p_R(C, A) \cong \mathrm{Ext}^p_S(S \otimes_R C, A) \qquad \text{for all } p \in \mathbb{Z}.$$

- *Let A be a left R-module and C a left S-module. If S is projective as an R-module, then there are natural isomorphisms*

$$\mathrm{Ext}^p_R(C, A) \cong \mathrm{Ext}^p_S(C, Hom_R(S, A)) \qquad \text{for all } p \in \mathbb{Z}.$$

Exercise 31. (Warm-up). Let G be a group, M a right $\mathbb{Z}[G]$-module, and V a right $\mathbb{Q}[G]$-module. Prove that

$$\mathrm{Ext}^*_{\mathbb{Z}[G]}(M, V) \cong \mathrm{Ext}^*_{\mathbb{Q}[G]}(M \otimes_{\mathbb{Z}} \mathbb{Q}, V)$$

Exercise 32. (Bonus). Prove Theorem XXVII.

Exercise 33. (Bonus). Prove Theorem XXVIII. *Hint:* Consider $S \otimes_R P_\bullet$, where $P_\bullet$ is a free resolution of C by R-modules. Then, consider $\mathrm{Hom}_R(S, I_\bullet)$, where $A \to I_\bullet$ is an injective coresolution of A by R-modules.