

# 1 Review: Poincaré Duality

## 1.1 $R$ -orientations and the fundamental class

**Definition I.** Let  $R$  be a commutative ring. Let  $M$  be an  $n$ -manifold. A *local  $R$ -orientation* of  $M$  at a point  $x$  is a choice of generator  $\mu_x$  of the free cyclic  $R$ -module

$$H_n(M, M \setminus \{x\}; R) \cong R.$$

An  $R$ -orientation of  $M$  is a function  $x \rightarrow \mu_x$  that assigns to each point  $x \in M$  a local  $R$ -orientation, subject to the following ‘local consistency’ condition.

Each  $x \in M$  has an open neighbourhood  $\mathbb{R}^n \subseteq M$  containing a finite radius ball  $B$  about  $x$  that satisfies the following. For every  $y \in B$ , the local orientations  $\mu_y$  are the images of a single generator  $\mu_B \in H_n(M, M \setminus B; R)$  under the maps  $H_n(M, M \setminus B; R) \rightarrow H_n(M, M \setminus \{y\}; R)$  induced by the inclusions  $(M \setminus B) \hookrightarrow (M \setminus \{y\})$ .

If an  $R$ -orientation exists, then  $M$  is called  *$R$ -orientable*.

If  $R$  is not specified, it is assumed to be  $\mathbb{Z}$ . In practice, the most important rings are  $R = \mathbb{Z}$  and  $R = \mathbb{Z}/2\mathbb{Z}$ .

Recall that a manifold is called *closed* if it is compact and has empty boundary. The (co)homology groups of closed manifolds are finitely generated.

**Theorem II.** Let  $M$  be a closed connected  $n$ -manifold. Let  $R$  be a commutative ring.

- If  $M$  is  $R$ -orientable, the natural map  $H_n(M; R) \rightarrow H_n(M, M \setminus \{x\}; R) \cong R$  is an isomorphism for all  $x \in M$ . A choice of generator for  $H_n(M; R)$ , often denoted  $[M]$ , is called a *fundamental class for  $M$  with coefficients in  $R$* .
- If  $M$  is not  $R$ -orientable, the natural map  $H_n(M; R) \rightarrow H_n(M, M \setminus \{x\}; R) \cong R$  is injective with image  $\{r \in R \mid 2r = 0\}$  for all  $x \in M$ .

In particular, if  $M$  is a closed connected  $n$ -manifold, then  $H_n(M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable} \\ 0, & \text{if } M \text{ is nonorientable.} \end{cases}$

**Theorem III.** An orientable manifold is  $R$ -orientable for all  $R$ . A nonorientable manifold is  $R$ -orientable if and only if  $-1 = 1$  in the ring  $R$ , that is,  $R$  is 2-torsion.

**Theorem IV.** Let  $M$  be a connected  $n$ -manifold. Let  $R$  be a commutative ring. Then

$$H_p(M; R) = 0 \quad \text{for all } p > n.$$

If  $M$  is non-compact, then in fact  $H_p(M; R)$  vanishes for all  $p \geq n$ .

**Exercise 1. (Warm-up).** Let  $M$  be an  $n$ -manifold. Verify that  $H_n(M, M \setminus \{x\}; R) \cong R$ .

**Exercise 2. (Warm-up).** Use local homology to show that the dimension of a connected manifold is well-defined.

**Exercise 3. (Warm-up).** Explain why **every** manifold is  $\mathbb{Z}/2\mathbb{Z}$ -orientable.

**Exercise 4. (Warm-up).** Use Theorem II to compute  $H_n(M; R)$  in the case that  $M$  is  $R$ -orientable, and in the case that  $M$  is not  $R$ -orientable, in each of the cases

(i)  $R = \mathbb{Z}$

(iii)  $R = \mathbb{Z}/3\mathbb{Z}$

(v)  $R = \mathbb{Z}/p\mathbb{Z}, p > 2$  prime

(ii)  $R = \mathbb{Z}/2\mathbb{Z}$

(iv)  $R = \mathbb{Z}/4\mathbb{Z}$

(vi)  $R = \mathbb{Z}/2^k\mathbb{Z}$

**Exercise 5. (Warm-up).** Use the universal coefficients theorem to prove a special case of Theorem III: for  $R = \mathbb{Z}/m\mathbb{Z}$ , an orientable manifold is  $R$ -orientable.

**Exercise 6. (Warm-up).** Show that  $M \times N$  is orientable if and only if both  $M$  and  $N$  are orientable.

**Exercise 7. (Warm-up).** For each of the 2-sphere and the torus, choose  $\Delta$ -complex structure, and explicitly compute 2-chain representing a choice of fundamental class. In general, how can you construct a fundamental class for a triangulated compact manifold?

**Exercise 8.** Suppose that  $M$  is a closed connected  $n$ -manifold. Prove that the torsion subgroup of  $H_{n-1}(M)$  is trivial if  $M$  is orientable and  $\mathbb{Z}/2\mathbb{Z}$  if  $M$  is non-orientable.

**Exercise 9. (Bonus).** Prove Theorem III.

**Exercise 10. (Bonus).** Let  $M$  be an  $n$ -manifolds. Consider the set

$$\widetilde{M} = \{\mu_x \mid x \in M \text{ and } \mu_x \text{ a choice of generator for } H_n(M, M \setminus \{x\}; \mathbb{Z})\}.$$

- (a) Describe how to topologize  $\widetilde{M}$  so that the map  $\mu_x \mapsto x$  is a 2-sheeted covering space of  $M$ .
- (b) Verify that  $\widetilde{M}$  is orientable.
- (c) Suppose that  $M$  is connected. Show that  $M$  is orientable if and only if  $\widetilde{M}$  has two connected components.
- (d) Conclude as a corollary that the fundamental group of any non-orientable manifold contains an index-2 subgroup. In particular, simply connected manifolds are orientable.
- (e) What is the cover  $\widetilde{M} \rightarrow M$  in the case that  $M$  is a Möbius strip, a Klein bottle, or  $\mathbb{R}P^n$ ?
- (f) Can you construct an analogue of  $\widetilde{M}$  for an arbitrary commutative ring  $R$ ?

**Exercise 11. (Bonus).** Suppose that  $R$  is a ring that contains at least  $N > 2$  units. Could a manifold have a connected  $N$ -sheeted orientation cover? Could  $M$  contain a loop that (when traversed) multiplies a choice of local orientation class by a unit  $u$  in  $R$  of multiplicative order greater than 2?

## 1.2 The cap product

Related to the cup product, there is a second bilinear map, which gives the homology groups  $H_*(X; R)$  the structure of a module over the  $R$ -algebra  $H^*(X; R)$ .

**Definition V.** Let  $R$  be a commutative ring, and  $X$  any space. The *cap product*  $\frown$  is an  $R$ -bilinear pairing  $H_{p+q}(X; R) \otimes_R H^q(X; R) \rightarrow H_p(X; R)$ . It is induced by the following  $R$ -bilinear map on level of singular chains and cochains as follows. Let  $\Delta^{p+q} = [v_0, v_1, \dots, v_{p+q}]$  be the standard  $(p+q)$ -simplex. Let  $\sigma : \Delta^{p+q} \rightarrow X$  be a singular  $(p+q)$ -simplex, and  $\alpha$  a  $q$ -cochain.

$$C_{p+q}(X; R) \otimes C^q(X; R) \xrightarrow{\frown} C_p(X; R)$$

$$\sigma \otimes \alpha \mapsto \sigma \frown \alpha = \alpha \left( \sigma|_{[v_0, \dots, v_q]} \right) \sigma|_{[v_q, \dots, v_{q+p}]} : [v_q, \dots, v_{q+p}] \rightarrow X$$

For maps  $f : X \rightarrow Y$  of spaces, the cap product satisfies the naturality condition

$$f_*(\sigma) \frown \alpha = f_*(\sigma \frown f^*(\alpha)).$$

The cap product is related to the cup product by the formulas

$$(\sigma \frown \alpha) \frown \beta = \sigma \frown (\alpha \smile \beta) \qquad \beta(\sigma \frown \alpha) = (\alpha \smile \beta)(\sigma).$$

**Definition VI.** Let  $X$  be a space and  $R$  a commutative ring. The operation

$$\langle -, - \rangle : C^p(X; R) \otimes_R C_p(X; R) \longrightarrow R$$

$$\alpha \otimes \sigma \mapsto \langle \alpha, \sigma \rangle := \alpha(\sigma)$$

induces a map on homology called the *evaluation pairing*,

$$\langle -, - \rangle : H^p(X; R) \otimes_R H_p(X; R) \longrightarrow R.$$

Thus the formula  $\beta(\sigma \frown \alpha) = (\alpha \smile \beta)(\sigma)$  above can be restated as the relationship

$$\langle \alpha \smile \beta, \sigma \rangle = \langle \beta, \sigma \frown \alpha \rangle.$$

Equivalently, there is a commutative diagram

$$\begin{array}{ccc} H^p(X; R) & \xrightarrow{\beta \mapsto \langle \beta, - \rangle} & \text{Hom}_R(H_p(X; R), R) \\ \downarrow \alpha \smile - & & \downarrow (-\smile \alpha)^\vee \\ H^{p+q}(X; R) & \longrightarrow & \text{Hom}_R(H_{p+q}(X; R), R) \end{array}$$

**Exercise 12. (Warm-Up).**

- (a) Verify that the cap product satisfies  $\partial(\sigma \frown \alpha) = ((\partial\sigma) \frown \alpha) \pm (\sigma \frown (\delta\alpha))$  on (co)chains.
- (b) Deduce that the cap product induces a well-defined map bilinear map on (co)homology groups.

**Exercise 13. (Warm-Up).** Let  $f : X \rightarrow Y$  be a continuous map. Verify (using the formula at the level of (co)chains) that the cap product satisfies  $f_*(\sigma) \frown \alpha = f_*(\sigma \frown f^*(\alpha))$ .

**Exercise 14. (Warm-Up).** Verify the formulas  $(\sigma \frown \alpha) \frown \beta = \sigma \frown (\alpha \smile \beta)$  and  $\beta(\sigma \frown \alpha) = (\alpha \smile \beta)(\sigma)$  at the level of (co)chains.

**Exercise 15. (Warm-Up).** Let  $X$  be a space and  $R$  a commutative ring. Verify that the cap product gives the homology groups  $H_*(X; R)$  the structure of a module over the ring  $H^*(X; R)$ .

**Exercise 16. (Warm-Up).** Verify that the evaluation pairing  $\langle -, - \rangle : C^p(X; R) \otimes_R C_p(X; R) \rightarrow R$  of induces a well-defined map on (co)homology groups, as stated in Definition VI.

**Exercise 17. (Warm-Up).** Let  $X$  be a space. What is a cochain representing the multiplicative identity 1 (with respect to cup product) of the ring  $H^*(X)$ ? For  $\tau \in C_0(X)$  a 0-chain, what is  $\langle 1, \tau \rangle$ ?

**Exercise 18. (Warm-Up).** Suppose that  $X$  is a space and  $R$  a ring such that the maps  $H^p(X; R) \rightarrow \text{Hom}_R(H_p(X; R), R)$  are isomorphisms, for example (by the universal coefficients theorem),  $R$  could be a field, or  $R = \mathbb{Z}$  and  $X$  a space with free abelian homology groups. Show that the cup and cap products determine one another.

**Exercise 19. (Warm-Up).** Explicitly compute the cap product in simplicial (co)homology for a selection of explicit classes for your choice of  $\Delta$ -complex structure on the 2-sphere, torus, and  $\mathbb{R}P^2$ .

**Exercise 20. (Bonus).** Construct relative forms of the cap product: for a subspace  $A \subseteq X$ ,

$$H_{p+q}(X, A; R) \otimes H^q(X; R) \xrightarrow{\frown} H_p(X, A; R)$$

$$H_{p+q}(X, A; R) \otimes H^q(X, A; R) \xrightarrow{\frown} H_p(X; R)$$

For open subspaces  $A, B \subseteq X$ ,

$$H_{p+q}(X, A \cup B; R) \otimes H^q(X, A; R) \xrightarrow{\frown} H_p(X, B; R)$$

### 1.3 Poincaré duality

In the 1890s and 1900s, Poincaré proved Poincaré duality for a triangulated manifold using the notion of a ‘dual triangulation’. The modern formulation, using the cap product, was not developed until the concept of cohomology was formalized in the 1930s.

**Theorem VII. (Poincaré duality).** Let  $M$  be a closed  $R$ -orientable  $n$ -manifold with a choice of fundamental class  $[M] \in H_n(M; R)$ . Then the map  $D$  defined by cap product with  $[M]$  is an isomorphism for all  $p$ ,

$$\begin{aligned} D : H^p(M; R) &\longrightarrow H_{n-p}(M; R) \\ \alpha &\longmapsto [M] \frown \alpha \end{aligned}$$

The homology class  $D(\alpha)$  is called the Poincaré dual of the cohomology class  $\alpha$ .

Recall that an  $R$ -bilinear form  $A \times B \rightarrow R$  is called *nonsingular* if the associated maps  $A \rightarrow \text{Hom}_R(B, R)$  and  $B \rightarrow \text{Hom}_R(A, R)$  are isomorphisms. Poincaré duality has the following consequence.

**Corollary VIII.** *Let  $M$  be a closed  $R$ -orientable  $n$ -manifold with a choice of fundamental class  $[M] \in H_n(M; R)$ . For  $R = \mathbb{Z}$  the cup product induces a nonsingular pairing*

$$\begin{aligned} H^p(M)/\text{torsion} \otimes_{\mathbb{Z}} H^{n-p}(M)/\text{torsion} &\longrightarrow \mathbb{Z} \\ \alpha \otimes \beta &\longmapsto \langle \alpha \smile \beta, [M] \rangle \end{aligned}$$

and, for  $R = \mathbb{F}$  a field, a nonsingular pairing

$$\begin{aligned} H^p(M; \mathbb{F}) \otimes_{\mathbb{F}} H^{n-p}(M; \mathbb{F}) &\longrightarrow \mathbb{F} \\ \alpha \otimes \beta &\longmapsto \langle \alpha \smile \beta, [M] \rangle. \end{aligned}$$

There are variations of the Poincaré Duality theorem that apply to noncompact manifolds, and to manifolds with boundary.

**Theorem IX.** *Let  $M$  be a (not necessarily compact) orientable  $n$ -manifold. Then there are isomorphisms*

$$H_c^p(M) \xrightarrow{\cong} H_{n-p}(M)$$

where  $H_c^p(M)$  denotes the cohomology of  $M$  with compact support, and

$$H^p(M) \xrightarrow{\cong} H_{n-p}^{BM}(M)$$

where  $H_{n-p}^{BM}(M)$  denotes the Borel–Moore homology of  $M$ .

**Theorem X. (Poincaré–Lefschetz Duality).** *Let  $M$  be a compact orientable  $n$ -manifold with boundary  $\partial M$ . Then there are isomorphisms*

$$H_c^p(M, \partial M) \xrightarrow{\cong} H_{n-p}(M) \quad \text{and} \quad H_p(M, \partial M) \xrightarrow{\cong} H^{n-p}(M).$$

**Exercise 21. (Warm-Up).** Describe the implications of Poincaré duality for symmetries of the Betti numbers of a closed orientable manifold.

**Exercise 22. (Warm-Up).** Compute the isomorphism  $D$  explicitly in the simplicial homology of a sphere and torus, and in the mod-2 homology of  $\mathbb{R}P^2$ .

**Exercise 23. (Warm-Up).** Suppose that  $A$  and  $B$  are finite dimensional vector spaces over a field  $R$ , or finite-rank free abelian groups over  $R = \mathbb{Z}$ . Show that an  $R$ -bilinear form  $\omega : A \times B \rightarrow R$  is nonsingular iff  $\omega(a, b) = 0$  for all  $b$  implies that  $a = 0$ .

**Exercise 24.** Show that an odd-dimensional closed manifold (orientable or non-orientable) has Euler characteristic zero.

**Exercise 25.** Prove Corollary VIII.

**Exercise 26.** Deduce from Corollary VIII that, for any  $\alpha$  that is an element of a basis for the free subgroup of  $H^p(M)$ , there exists a basis element  $\beta$  for the free part of  $H^{n-p}(M)$  such that  $\alpha \smile \beta$  generates  $H^n(M) \cong \mathbb{Z}$ . Formulate and prove the analogous result for cohomology with coefficients in a field  $\mathbb{F}$ .

**Exercise 27.** Use Exercise 26 to prove the following ring isomorphisms stated on Worksheet #3.

- **(Real and complex projective spaces).**

$$\begin{aligned} H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1}) \quad \text{and} \quad H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha] \quad \text{for } |\alpha| = 1 \\ H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}) \quad \text{and} \quad H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha] \quad \text{for } |\alpha| = 2 \end{aligned}$$

**Exercise 28. (Bonus).** Verify the following ring isomorphisms stated on Worksheet #3.

- **(The  $n$ -torus).** Let  $T^n$  denote the  $n$ -torus  $T^n \cong (S^1)^n$ . Then  $H^*(T^n)$  is the exterior algebra

$$H^*(T^n) \cong \bigwedge_{\mathbb{Z}} (\alpha_1, \dots, \alpha_n) \quad \text{for } |\alpha_i| = 1.$$

## 1.4 Cup product and geometric intersections

The following theorem is a useful tool for computing the cohomological ring structure of smooth manifolds.

**Theorem XI.** *Let  $M$  be a smooth, closed, orientable  $n$ -manifold. Let  $A$  and  $B$  be a codimension- $p$  submanifold and a codimension- $q$  submanifold, respectively, that intersect transversely, so their intersection  $A \cap B$  is a codimension- $(p+q)$  submanifold. Let  $[A]$  denote the image in  $H_*(M)$  of (a choice of) fundamental class of  $A$  under the inclusion  $A \hookrightarrow M$ . Let  $D([A])$  denote its Poincaré dual. Then*

$$D([A]) \smile D([B]) = D([A \cap B]) \in H^{p+q}(X).$$

We note that, implicit in this theorem: a choice of fundamental classes of  $A$ ,  $B$ , and  $M$ , and the order of the factors  $D([A]) \smile D([B])$ , induces a choice of fundamental classes for the components of the intersection  $A \cap B$ .

In the case that  $A$  and  $B$  have complementary dimensions in  $M$ , then  $[A \cap B]$  is represented by a 0-chain that is a linear combination of the points in the intersection  $A \cap B$ . Each point has coefficient  $\pm 1$ , as calculated in the definition of the *algebraic intersection number* of  $A$  and  $B$ .

### Exercise 29. (Warm-Up).

- (a) Let  $M$  be a smooth connected  $n$ -manifold. Let  $A$  be a submanifold, and consider the functional  $\sigma \mapsto \langle \sigma, D([A]) \rangle$ . Show that, if  $B$  is a submanifold of complementary dimension to  $A$  that intersects transversely, then

$$\langle [B], D([A]) \rangle = \text{algebraic intersection number of } B \text{ and } A.$$

*Hint:* Exercise 17.

- (b) Let  $\Sigma_g$  denote a closed orientable genus- $g$  surface with

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \dots [a_g, b_g] \rangle.$$

By abuse of notation, view the elements  $a_i, b_i$  as cycles (or elements of  $H_1(\Sigma_g)$ ). Describe the relationship between the linear duals of the cycles  $a_i, b_i$ , and their Poincaré duals.

**Exercise 30.** Use Theorem XI to verify the following ring isomorphisms stated on Worksheet #3.

- **(The surface  $\Sigma_g$ ).** Let  $\Sigma_g$  denote a closed orientable genus- $g$  surface with

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \dots [a_g, b_g] \rangle.$$

Let  $\alpha_i, \beta_i \in H^1(\Sigma_g)$  denote the classes of the cochains that are the linear duals of  $a_i, b_i$ , respectively, viewed as cycles. Then

$$\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = 0 \text{ for all } i, j, \quad \text{and} \quad \alpha_i \smile \beta_j = \begin{cases} 0, & i \neq j \\ \text{generator of } H^2(\Sigma_g) \cong \mathbb{Z}, & i = j \end{cases}$$