1 Review: The Künneth formula and the cup product

1.1 The Künneth formula for homology

Theorem I. (The homology Künneth formula for PIDs). *Let R be a PID. Let X and Y be topological spaces. Then for each $p \geq 0$ there exists a short exact sequence, natural in X and Y,*

$$0 \longrightarrow \bigoplus_{i+j=p} H_i(X;R) \otimes_R H_j(Y;R) \longrightarrow H_p(X \times Y;R) \longrightarrow \bigoplus_{i+j=p-1} \mathrm{Tor}_1^R(H_i(X;R), H_j(Y;R)) \longrightarrow 0.$$

The sequence splits, so we obtain isomorphisms of R-modules

$$H_p(X \times Y;R) \quad \cong \quad \left(\bigoplus_{i+j=p} H_i(X;R) \otimes_R H_j(Y;R) \right) \oplus \left(\bigoplus_{i+j=p-1} \mathrm{Tor}_1^R(H_i(X;R), H_j(Y;R)) \right).$$

This splitting, however, is not natural in X and Y.

The Künneth formula holds in particular for the PID $R = \mathbb{Z}$. When $R = \mathbb{F}$ is a field, since the functors $\mathrm{Tor}_1^{\mathbb{F}}$ vanish identically, we obtain the following corollary.

Corollary II. (The Künneth formula for a field). *Let $\mathbb{F}$ be a field. Let X and Y be topological spaces. Then for each p there is a natural isomorphism of functors from* **Top** $\times$ **Top** *to $\mathbb{F}$-vector spaces,*

$$\bigoplus_{i+j=p} H_i(X;\mathbb{F}) \otimes_{\mathbb{F}} H_j(Y;\mathbb{F}) \xrightarrow{\cong} H_p(X \times Y;\mathbb{F}).$$

Exercise 1. (Warm-up). Let $T^3 \cong S^1 \times S^1 \times S^1$ be the 3-torus. Compute the homology of T^3…
(i) directly from a cell structure, and (ii) using the Künneth formula.

Exercise 2. (Warm-up). Reconcile the Künneth formula with the isomorphism $\pi_1(X \times Y, (x,y)) \cong \pi_1(X,x) \times \pi_1(Y,y)$ and (for X path-connected) the Hurewicz isomorphism $H_1(X) \cong \pi_1(X)^{ab}$.

Exercise 3. (Warm-up). State the special case of the Künneth theorem when $R = \mathbb{Z}$ and assuming the homology groups $H_*(X)$ are free abelian.

Exercise 4. (Warm-up). Given a space Z, let $\beta_p^Z := \mathrm{rank}\,(H_p(Z))$ denote its pth Betti number. Assuming Z has finite Betti numbers, let $p_Z(t)$ denote the associated generating function $p_Z(t) := \beta_0^Z + \beta_1^Z t + \beta_2^Z t^2 + \beta_3^Z t^3 + \cdots$.

- (a) Prove that, for any spaces X and Y with finite Betti numbers, $p_{X \times Y}(t) = p_X(t) p_Y(t)$.
- (b) Let $\mathbb{F}$ be a field. Can you generalize the above result to the values $\beta_p^{Z;\mathbb{F}} := \dim_{\mathbb{F}}(H_p(Z;\mathbb{F}))$?

Exercise 5. Does the Künneth formula hold if we replace homology with reduced homology?

Exercise 6. (Bonus). Let X and Y be CW complexes, and consider the induced cell structure on the product $X \times Y$.

- (a) Let e^i be an i-cell of X and e^j a j-cell of Y. Verify that the cellular boundary map ∂ satisfies
$$\partial(e^i \times e^j) = (\partial e^i) \times e^j + (-1)^i e_i \times (\partial e^j).$$
- (b) Verify the following is a chain map, and induces a homomorphism $H_i^{CW}(X) \otimes_{\mathbb{Z}} H_j^{CW}(Y) \to H_{i+j}^{CW}(X \times Y)$.
$$\begin{aligned} C_i^{CW}(X) \times C_j^{CW}(Y) &\longrightarrow C_{i+j}^{CW}(X \times Y) \\ (e^i, e^j) &\longmapsto e^i \times e^j \end{aligned}$$

Exercise 7. (Bonus). For a general commutative ring R, there exists a spectral sequence satisfying

$$E^2_{pq} = \bigoplus_{q_1+q_2=q} \mathrm{Tor}_p^R(H_{q_1}(X;R), H_{q_2}(Y;R)) \quad \Longrightarrow \quad H_{p+q}(X \times Y;R).$$

Deduce Theorem I, the Künneth formula for PIDs, from this spectral sequence.

Exercise 8. (Bonus). Compare the Künneth formula to a calculation of $H^*(X \times Y)$ using the Serre spectral sequence for the fibration given by projection $X \times Y \to X$ onto a factor.

1.2 The Künneth formula for cohomology and the cup product

Let X and Y be spaces, and R a ring. There is a map on cohomology, called the *cross product*,

$$H^p(X;R) \otimes H^q(Y;R) \xrightarrow{\times} H^{p+q}(X \times Y;R).$$

When one of X or Y is *fintie type*–that is, the singular chain complex is equivalent to one that is finitely generated in each degree (say, as with a level-wise finite CW complex)—then from the cross product map we obtain a Künneth theorem for cohomology.

Theorem III. (The cohomology Künneth formula). *Let X and Y be topological spaces such that Y has finite type. Then for each $p \geq 0$ there exists a short exact sequence, natural in X and Y,*

$$0 \longrightarrow \bigoplus_{i+j=p} H^i(X) \otimes_{\mathbb{Z}} H^j(Y) \xrightarrow{\times} H^p(X \times Y) \longrightarrow \bigoplus_{i+j=p+1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(X), H_j(Y)) \longrightarrow 0.$$

The sequence splits, so we obtain isomorphisms of abelian groups

$$H^p(X \times Y) \quad \cong \quad \left(\bigoplus_{i+j=p} H^i(X) \otimes_{\mathbb{Z}} H^j(Y) \right) \oplus \left(\bigoplus_{i+j=p+1} \mathrm{Tor}_1(H_i(X), H_j(Y)) \right).$$

This splitting, however, is not natural in X and Y.

One of the great strengths of cohomology is that (given the cross product) the diagonal map $X \to X \times X$ induces a graded ring structure on the cohomology groups with respect to a product called the cup product.

Definition IV. Let X be a space, and R a commutative ring. Let $\Delta : x \mapsto (x,x)$ denote the diagonal map $X \to X \times X$. Then the *cup product* operation on $H^*(X)$ is defined to be the map

$$H^p(X;R) \otimes H^q(X;R) \xrightarrow{\times} H^{p+q}(X \times X;R) \xrightarrow{\Delta^*} H^{p+q}(X;R).$$

It is defined as follows on the level of singular cochains. Let $\Delta^{p+q} = [v_0, v_1, \ldots, v_{p+q}]$ be the standard $(p+q)$-simplex.

$$\begin{aligned} &C^p(X;R) \otimes C^q(X;R) \longrightarrow C^{p+q}(X;R) \\ &\qquad \alpha^p \otimes \beta^q \longmapsto \Big((\alpha^p \cup \beta^q) : [\sigma : \Delta^{p+q} \to X] \longmapsto \alpha^p(\sigma|_{[v_0,\ldots,v_p]})\beta^q(\sigma|_{[v_p,\ldots,v_{p+q}]}) \Big). \end{aligned}$$

Theorem V. *The cup product is* graded commutative. *For any space X and classes $\alpha^p \in H^p(X)$ and $\beta^q \in H^q(Y)$ it satisfies*

$$\alpha^p \smile \beta^q = (-1)^{pq} \beta^q \smile \alpha^p.$$

Theorem VI. *Let R be a ring. Cohomology $H^*(-;R)$ defines a functor from topological spaces to graded rings.*

Example VII. Let X_α, $\alpha \in I$ be a collection of spaces. Then the following isomorphisms are in fact isomorphisms of rings, with respect to the usual coordinate-wise multiplication of products of rings.

$$H^*\left(\coprod_\alpha X_\alpha \right) \cong \prod_\alpha H^*(X_\alpha) \qquad \widetilde{H}^*\left(\bigvee_\alpha X_\alpha \right) \cong \prod_\alpha \widetilde{H}^*(X_\alpha)$$

Example VIII. The following are ring isomorphisms of cohomology groups.

- **(The n-torus).** Let T^n denote the n-torus $T^n \cong (S_1)^n$. Then $H^*(T^n)$ is the exterior algebra

$$H^*(T^n) \cong \bigwedge_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_n) \qquad \text{for } |\alpha_i| = 1.$$

- **(The surface Σ_g).** Let Σ_g denote a closed orientable genus-g surface with

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, a_2, b_2, \ldots a_g, b_g \mid [a_1,b_1][a_2,b_2]\ldots[a_g,b_g] \rangle.$$

  Let α_i, β_i denote the cochains in $H^1(\Sigma_g)$ associated to the duals of a_i, b_i, respectively, viewed as cycles. Then

$$\alpha_i \smile \alpha_j \;=\; \beta_i \smile \beta_j = 0 \text{ for all } i,j, \quad \text{and} \quad \alpha_i \smile \beta_j = \begin{cases} 0, & i \neq j \\ \text{generator of } H^2(\Sigma_g) \cong \mathbb{Z}, & i = j \end{cases}$$

- **(Real and complex projective spaces).**

$$H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1}) \quad \text{and} \quad H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha] \qquad \text{for } |\alpha| = 1$$

$$H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}) \quad \text{and} \quad H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha] \qquad \text{for } |\alpha| = 2$$

Exercise 9. (Warm-Up). Consider a disjoint union $\coprod_\alpha X_\alpha$ or a wedge sum $\bigvee_\alpha X_\alpha$ of spaces. Deduce from Example VII that the cup product of a class in $H^*(X_\alpha)$ and $H^*(X_\beta)$ is zero if $\alpha \neq \beta$. Explain how the cup product on cohomology can be used to detect that a space does not have the homotopy type of a given wedge sum.

Exercise 10. (Warm-Up). Reconcile the cohomology Künneth formula with the homology Künneth formula and the universal coefficients theorem.

Exercise 11. Use cohomology to prove that the spaces $S^1 \vee S^1 \vee S^2$ and the torus T are not homotopy equivalent.

Exercise 12. (a) Let $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ be a map such that $f_* : \pi_1(\mathbb{R}\mathrm{P}^m) \to \pi_1(\mathbb{R}\mathrm{P}^n)$ is nonzero. Prove that $m \leq n$.

(b) Suppose $m > n \geq 1$. Prove that there is no antipodal map $S^m \to S^n$, i.e., no map that takes pairs of antipodal points to pairs of antipodal points. *Hint:* Covering space theory.

(c) Prove the **Borsuk–Ulam Theorem**: For any continuous map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.

Exercise 13. Use the Lefschetz fixed-point theorem to show that every map $\mathbb{C}\mathrm{P}^{2n} \to \mathbb{C}\mathrm{P}^{2n}$ has a fixed point.

Exercise 14. (Bonus). Prove Example VII.

Exercise 15. (Bonus).

(a) Let R be a ring. Consider a pair (X, A). Construct a relative cup product operation

$$H^*(X, A; R) \otimes_R H^*(X, A; R) \xrightarrow{\smile} H^*(X, A; R).$$

(b) Let X be a nonempty space. Conclude in particular that there is a ring structure on $\widetilde{H}^*(X) \cong H^*(X, *)$. Does $\widetilde{H}^*(X)$ embed in $H^*(X)$ as a subring?

(c) Let A and B be subcomplexes of a CW complex X. Construct a more general operation on cellular homology

$$H^*_{CW}(X, A; R) \otimes_R H^*_{CW}(X, B; R) \xrightarrow{\smile} H^*_{CW}(X, A \cup B; R).$$

(d) Let A and B be open subsets of a space X. Construct a map

$$H^*(X, A; R) \otimes_R H^*(X, B; R) \xrightarrow{\smile} H^*(X, A \cup B; R).$$