1 Review: (co)homology with coefficients

1.1 The definition of (co)homology with coefficients

Definition I. (Homology with coefficients). Let X be a space, and let G be an abelian group. The *group* $C_p(X;G)$ *of singular* p*-chains on* X *with coefficients in* G is the abelian group of formal finite sums

$$C_p(X;G) = \left\{ \sum_i^{\text{finite}} g_i\sigma_i \;\middle|\; g_i \in G,\ \sigma_i \text{ a singular } p\text{-simplex of } X \right\}$$

The p-chains $C_p(X;G)$ have an addition defined pointwise by the group operation on G, giving an isomorphism of abelian groups

$$C_p(X;G) \cong \bigoplus_{\substack{\text{singular} \\ p\text{-simplices } \sigma}} G.$$

When $G=\mathbb{Z}$, the definition of $C_p(X;\mathbb{Z})$ corresponds to our original definition of the singular chain group $C_p(X)$.

The differential $\partial_p : C_p(X;G) \to C_{p-1}(X;G)$ is defined on a singular p-simplex by the same formula as when $G=\mathbb{Z}$, and extended G-linearly to $C_p(X;G)$. The resulting homology groups are denoted $H_*(X;G)$ and called *the homology of* X *with coefficients in* G.

We define the *relative homology* $H_*(X,A;G)$ *of the pair* (X,A) *with coefficients in* G as the homology of the chain complex $C_*(X;G)/C_*(A;G)$. We define the *augmented* chain complex

$$\longrightarrow C_2(X;G) \xrightarrow{\partial_2} C_1(X;G) \xrightarrow{\partial_1} C_0(X;G) \xrightarrow{\epsilon} G \longrightarrow 0$$
$$\sum_i g_i\sigma_i \longmapsto \sum_i g_i$$

and its homology is the *reduced homology of* X *with coefficients in* G, written $\widetilde{H}(X;G)$. Cellular and simplicial homology with coefficients in G, and their relative and reduced variants, are defined analogously.

When G is a ring, the chain complex $C_*(X;G)$ and its homology inherit a G-module structure.

Definition II. (Cohomology with coefficients). Let X be a space, and let G be an abelian group. The *group* $C^p(X;G)$ *of singular* p*-cochains on* X *with coefficients in* G is the group $\mathrm{Hom}_{\mathbb{Z}}(C_p(X),G)$. The *singular cochain complex on* X *with coefficients in* G is obtained by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,G)$ to the singular chain complex $C_*(X)$. The resulting cohomology groups are the *singular cohomology groups of* X *with coefficients in* G, and denoted $H^*(X;G)$.

Cellular and simplicial cohomology with coefficients, and their reduced and relative versions, are defined analogously. Again, when G is a ring, these groups have G-module structures.

Exercise 1. (a) **(Warm-up).** Compute, from the definition, the cellular homology of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. How does it compare to the homology of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}$?

(b) Determine for which $n \geq 1$ there exists a continuous map $\mathbb{RP}^n \to S^n$ that is not nullhomotopic.

Exercise 2. (Bonus).

(a) Let M, N be abelian groups and $f : M \to N$ a homomorphism. Show that f induces a natural transformation of homology functors $H_*(-;M) \to H_*(-;N)$ and cohomology functors $H^*(-;M) \to H^*(-;N)$. Note that both homology and cohomology are covariant with respect to their coefficient modules.

(b) Let $0 \to M \to N \to Q \to 0$ be a short exact sequence of abelian groups. Show that, for any pair (X,A), there is an induced long exact sequence of homology groups

$$\cdots \longrightarrow H_p(X,A;M) \longrightarrow H_p(X,A;N) \longrightarrow H_p(X,A;Q) \longrightarrow H_{p-1}(X,A;M) \longrightarrow \cdots$$

and a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^p(X,A;M) \longrightarrow H^p(X,A;N) \longrightarrow H^p(X,A;Q) \longrightarrow H^{p+1}(X,A;M) \longrightarrow \cdots$$

(c) Show that the above long exact sequences are natural.

(d) The connecting homomorphism in the long exact sequence is called the *Bockstein homomorphism*. Compute the Bockstein homomorphism for the following short exact sequences of coefficient modules for $X = \mathbb{R}\mathrm{P}^n$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \qquad 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

Exercise 3. (Bonus). Would our definition of cohomology with coefficients in G be equivalent if we had instead defined it using the following chain groups? Assume G has a ring structure where necessary.

$$\mathrm{Hom}_{\mathbb{Z}}(C_p(X;G);\mathbb{Z}) \qquad \mathrm{Hom}_{\mathbb{Z}}(C_p(X;G);G) \qquad \mathrm{Hom}_{G}(C_p(X;G);G)$$

1.2 Review: Properties of Tor and Ext

We will review the formal definitions of the Tor and Ext functors later in the course. For now, the following results are convenient for computation.

Proposition III. *The functor* $\mathrm{Tor}_1^{\mathbb{Z}}(-,-)$ *satisfies the following properties.*

- $\mathrm{Tor}_1^{\mathbb{Z}}(H,G) \cong \mathrm{Tor}_1^{\mathbb{Z}}(G,H)$
- $\mathrm{Tor}_1^{\mathbb{Z}}(\bigoplus_i H_i, G) \cong \bigoplus_i \mathrm{Tor}_1^{\mathbb{Z}}(H_i,G)$
- $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},G) \cong \ker(G \xrightarrow{n} G)$.
- $\mathrm{Tor}_1^{\mathbb{Z}}(H,G) \cong \mathrm{Tor}_1^{\mathbb{Z}}(T(H),G)$, *where* $T(H)$ *is the torsion subgroup of* H.

  In particular, $\mathrm{Tor}_1^{\mathbb{Z}}(H,G) \cong 0$ *if* H *or* G *are torsion-free.*

In general, if $0 \to P_1 \to P_0 \to H \to 0$ *is any presentation of* H *with* P_1, P_0 *free abelian groups, then*

$$\mathrm{Tor}_1^{\mathbb{Z}}(H,G) \cong \ker(P_1 \otimes G \to P_0 \otimes G).$$

Proposition IV. *The functor* $\mathrm{Ext}^1_{\mathbb{Z}}(-,-)$ *satisfies the following properties.*

- $\mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus_i H_i, G) \cong \prod_i \mathrm{Ext}^1_{\mathbb{Z}}(H_i,G)$

  In particular,
  $\mathrm{Ext}^1_{\mathbb{Z}}(H \oplus H', G) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H,G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H',G)$
- $\mathrm{Ext}^1_{\mathbb{Z}}(H,G) \cong 0$ *if* H *is free abelian*
- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},G) \cong G/nG$

In general, if $0 \to P_1 \to P_0 \to H \to 0$ *is any presentation of* H *with* P_1, P_0 *free abelian groups, then*

$$\mathrm{Ext}^1_{\mathbb{Z}}(H,G) \cong \mathit{Hom}_{\mathbb{Z}}(P_1,G)/\mathrm{im}(\mathit{Hom}_{\mathbb{Z}}(P_0,G)).$$

Exercise 4. (Warm-Up). Compute the following abelian groups.

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \qquad \mathrm{Tor}_1^{\mathbb{Z}}(H,G) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/12\mathbb{Z})$$

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \qquad \mathrm{Ext}^1_{\mathbb{Z}}(H,G) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/12\mathbb{Z})$$

Exercise 5. (Bonus). Prove Propositions III and IV.

1.3 The universal coefficients theorems

Let G be an abelian group, which we will view as a $\mathbb{Z}$-module. Since $C_*(X;G) \cong C_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G$, it is natural to ask whether the homology group $H_p(X;G)$ agrees with $H_p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G$. It turns out that the answer is "no" in general, and a result called the *Universal Coefficients Theorem* states that failure of this equality is measured by the group $\mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X);G)$.

Theorem V. (Universal coefficients theorem for homology). *Let C_* be a chain complex of* **free** *abelian groups. Let G be an abelian group. Then, for each p, there is a short exact sequence of abelian groups*

$$0 \longrightarrow H_p(C_*) \otimes_{\mathbb{Z}} G \longrightarrow H_p(C_* \otimes G) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(C_*), G) \longrightarrow 0$$

The short exact sequence is natural with respect to chain maps. This sequence splits, but the splitting is not natural with respect to chain maps.

The theorem applies in particular to the singular chain complexes $C_*(X)$ and $C_*(X, A)$ for any space or pair, giving the following short exact sequences.

$$0 \longrightarrow H_p(X) \otimes_{\mathbb{Z}} G \longrightarrow H_p(X; G) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X), G) \longrightarrow 0$$
$$0 \longrightarrow H_p(X, A) \otimes_{\mathbb{Z}} G \longrightarrow H_p(X, A; G) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X, A), G) \longrightarrow 0$$

Corollary VI. *Let G be an abelian group. Then for any space X or pair (X, A), there are, for all $p \geq 0$, isomorphisms of abelian groups (which are* **not** *natural in X or (X, A)),*

$$H_p(X; G) \cong \Big(H_p(X) \otimes_{\mathbb{Z}} G\Big) \oplus \Big(\mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X), G)\Big)$$

$$H_p(X, A; G) \cong \Big(H_p(X, A) \otimes_{\mathbb{Z}} G\Big) \oplus \Big(\mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X, A), G)\Big)$$

In particular, the homology groups $H_p(X; G)$ are determined as abelian groups by G and $H_*(X)$.

Corollary VII. *Let $G = \mathbb{Q}$. Then for any space X or pair (X, A), there are natural isomorphisms of $\mathbb{Q}$-vector spaces for all $p \geq 0$,*

$$H_p(X; \mathbb{Q}) \cong H_p(X) \otimes_{\mathbb{Z}} \mathbb{Q} \qquad\qquad H_p(X, A; \mathbb{Q}) \cong H_p(X, A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

There is a dual theorem that relates the cohomology groups $H^p(X; G)$ with the dual groups $\mathrm{Hom}_{\mathbb{Z}}(H_p(X), G)$.

Theorem VIII. (Universal coefficients theorem for cohomology). *Let C_* be a chain complex of* **free** *abelian groups. Let G be an abelian group. Then, for each p, there is a short exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(C_*), G) \longrightarrow H_p(Hom_{\mathbb{Z}}(C_*, G)) \longrightarrow Hom_{\mathbb{Z}}(H_p(C_*), G) \longrightarrow 0$$

The short exact sequence is natural with respect to chain maps. This sequence splits, but not naturally with respect to C_*.

In particular, given a space X or pair (X, A), we have the following short exact sequences.

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(X), G) \longrightarrow H^p(X; G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_p(X), G) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(X, A), G) \longrightarrow H^p(X, A; G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_p(X, A), G) \longrightarrow 0$$

The special case $G = \mathbb{Z}$ relates the homology and cohomology of a space or of a pair.

Corollary IX. *For any space X or pair (X, A), there are short exact sequences*

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(X), \mathbb{Z}) \longrightarrow H^p(X) \longrightarrow Hom_{\mathbb{Z}}(H_p(X), \mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(X, A), \mathbb{Z}) \longrightarrow H^p(X, A) \longrightarrow Hom_{\mathbb{Z}}(H_p(X, A), \mathbb{Z}) \longrightarrow 0$$

In particular, if $H_p(X)$ is a finitely generated abelian group $H_p(X) \cong \mathbb{Z}^{\beta_p} \oplus T_p$ with T_p torsion, then $H^p(X) \cong \mathbb{Z}^{\beta_p} \oplus T_{p-1}$.

Corollary X. *Let G be an abelian group. Then for any space X or pair (X, A), there are, for all $p \geq 0$, (non-natural) isomorphisms of abelian groups*

$$H^p(X; G) \cong \Big(Hom_{\mathbb{Z}}(H_p(X), G)\Big) \oplus \Big(\mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(X), G)\Big)$$

$$H^p(X, A; G) \cong \Big(Hom_{\mathbb{Z}}(H_p(X, A), G)\Big) \oplus \Big(\mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(X, A), G)\Big)$$

In particular, the cohomology groups $H^p(X; G)$ are determined as abelian groups by G and $H_*(X)$.

Exercise 6. (Warm-Up). Does Theorem V hold without the assumption that the chain groups of C_* are free abelian?

Exercise 7. (Warm-Up). Use the universal coefficient theorems to recompute the homology and cohomology of $\mathbb{R}\mathrm{P}^n$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Exercise 8. (a) **(Warm-Up).** Verify Corollary IX.

(b) **(Warm-Up).** Suppose a space X has finitely generated (co)homology groups. Prove $H^*(X)$ determines $H_*(X)$.

(c) **(Bonus).** Prove that the cohomology $H^*(X)$ of a space X need not determine its homology groups $H_*(X)$.

Exercise 9. (Warm-Up). Is it true that the Euler characteristic of X satisfies $\chi(X) = \sum_{i \geq 0} (-1)^i \mathrm{rank} H^i(X)$?

Exercise 10. (Warm-Up). Consider the following morphism of short exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{[0\ 1]} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow{\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]} & & \Big\downarrow{=} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{[0\ 1]} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Verify that the diagram commutes, but the middle vertical arrow does not respect the direct sum decomposition. Use this example to explain what it means that the short exact sequences in the universal coefficient theorems are natural, and that they are split, but that the splitting is not natural.

Exercise 11. Let G be an abelian group.

(a) **(Warm-Up).** Compute $H^*(S^n; G)$ using (i) directly from a CW complex structure on S^n, (ii) using the long exact sequence of a pair, (iii) using Mayer–Vietoris.

(b) Let $f : S^n \to S^n$ be a map of degree d of the n-sphere. Compute the induced maps $f_* : H_n(S^n; G) \to H_n(S^n; G)$ and $f^* : H^n(S^n; G) \to H^n(S^n; G)$.

Exercise 12. Let n and q be coprime integers. Recall that *lens space* $L(n, q)$ is a closed 3-manifold defined as follows. View S^3 as the unit sphere in $\mathbb{C}^2$. Then $L(n, q)$ is the quotient of S^3 by the $\mathbb{Z}/n\mathbb{Z}$ action generated by the map $(z_1, z_2) \mapsto (e^{2\pi i/n} z_1, e^{2\pi i q/n} z_2)$. It satisfies

$H_0(L(n, q); \mathbb{Z}) \cong \mathbb{Z}$ $\quad$ $H_1(L(n, q); \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ $\quad$ $H_2(L(n, q); \mathbb{Z}) \cong 0$ $\quad$ $H_3(L(n, q); \mathbb{Z}) \cong \mathbb{Z}$.

(a) **(Bonus).** Verify the calculation of the homology groups of $L(n, q)$.

(b) Compute the cohomology of $L(n, q)$.

(c) Compute $H_*(L(n, q); \mathbb{Z}/n\mathbb{Z})$ and $H^*(L(n, q); \mathbb{Z}/n\mathbb{Z})$. What about $\mathbb{Z}/q\mathbb{Z}$ coefficients? $\mathbb{Z}/p\mathbb{Z}$ for p prime, $p|n$?

Exercise 13. (Bonus). Consider a space X with finitely generated (as an abelian group) homology in each degree.

(a) Suppose we know $H_*(X; \mathbb{Z}/p^k\mathbb{Z})$ for each prime power p^k. Could we recover $H_*(X)$? If so, how?

(b) A certain finite CW complex X has the following nonzero homology groups $H_i(X; \mathbb{Z}/p^k\mathbb{Z})$. Find $H_*(X)$.

	$p^k = 2$	3	4	5	$p^k, p \geq 5$	8	$2^k, k \geq 2$	9	$3^k, k \geq 1$
$i = 0$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/p^k\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/2^k\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/3^k\mathbb{Z}$
1	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	0	0	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
2	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	0	0	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
3	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/4\mathbb{Z})^2$	$(\mathbb{Z}/5\mathbb{Z})^2$	$(\mathbb{Z}/p^k\mathbb{Z})^2$	$(\mathbb{Z}/8\mathbb{Z})^2$	$(\mathbb{Z}/2^k\mathbb{Z})^2$	$(\mathbb{Z}/9\mathbb{Z})^2$	$(\mathbb{Z}/3^k\mathbb{Z})^2$

(c) Given a space X, to what degree could we compute $H_*(X)$ if we only knew $H_*(X; \mathbb{Z}/p\mathbb{Z})$ for prime p? What about $H_*(X; \mathbb{F})$ for finite fields $\mathbb{F}$?

Exercise 14. (Bonus). Show that the isomorphisms of Corollary VI are not natural.
Hint: Consider the quotient map $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1 \cong S^2$.

Exercise 15. (Bonus). Prove Theorem V.
Hint: Let Z_p and B_p denote the subgroups of cycles and boundaries, respectively, in the chain group C_p. First consider the short exact sequences below. Which sequence splits? Which terms are flat as $\mathbb{Z}$-modules? Tensor the first short exact sequence with G and use it to construct a short exact sequence of chain complexes.

$$0 \longrightarrow Z_p \longrightarrow C_p \longrightarrow B_{p-1} \longrightarrow 0 \qquad\qquad 0 \longrightarrow B_p \longrightarrow Z_p \longrightarrow H_p(C_*) \longrightarrow 0$$