1 Review: foundations of cohomology

1.1 The cochain complexes and their cohomology

A *cochain complex* is a chain complex whose differentials increase the degree. All your Math 592 results for general chain complexes still apply!

Definition I. (The singular, cellular, and simplicial cochain complexes). Let X be a topological space, and let $\Big(C_*(X), \partial_*\Big)$ be its singular chain complex. We define, for $p \geq 0$, the *singular cochain groups* as the dual groups

$$C^p(X) := \mathrm{Hom}_{\mathbb{Z}}(C_p(X), \mathbb{Z}).$$

Elements of $C^p(X)$ are called *singular p-cochains*. These groups form the *singular cochain complex* $\Big(C^*(X), \delta^*\Big)$ with respect to the *coboundary maps*

$$\begin{aligned} \delta^p : \mathrm{Hom}_{\mathbb{Z}}(C_p(X), \mathbb{Z}) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_{p+1}(X), \mathbb{Z}) \\ (\alpha : C_p(X) \to \mathbb{Z}) &\longmapsto (\alpha \circ \partial_{p+1} : C_{p+1}(X) \to \mathbb{Z}) \end{aligned}$$

$$\begin{array}{ccc} C_p(X) & \xleftarrow{\partial_{p+1}} & C_{p+1}(X) \\ \alpha \downarrow & \swarrow_{\delta_p(\alpha) := \alpha \circ \partial_{p+1}} & \\ \mathbb{Z} & & \end{array}$$

defined by pre-composition with the boundary maps ∂_*. The resulting complex has the form

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_0(X), \mathbb{Z}) \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbb{Z}}(C_1(X), \mathbb{Z}) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathbb{Z}}(C_2(X), \mathbb{Z}) \xrightarrow{\delta^2} \mathrm{Hom}_{\mathbb{Z}}(C_3(X), \mathbb{Z}) \xrightarrow{\delta^3} \cdots$$

Elements of the kernel of δ^p are called *cocycles*, and elements of its image are called *coboundaries*.

When X is a CW complex, the *cellular cochain complex* $\Big(C^*_{CW}(X), \delta^*\Big)$ is defined analogously with respect to the cellular chain complex $\Big(C^{CW}_*(X), \partial_*\Big)$. When X is a Δ complex, the *simplicial cochain complex* $\Big(C^*_{\Delta}(X), \delta^*\Big)$ is defined analogously with respect to the simplicial chain complex $\Big(C^{\Delta}_*(X), \partial_*\Big)$.

In short, we construct the singular cochain complex of X by applying the contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to its singular chain complex.

Definition II. (Singular and cellular cohomology). Let X be a topological space. The *singular cohomology of X* is the homology of the singular cochain complex

$$H^p(X) := H_p\Big(C^*(X), \delta^*\Big),$$

that is, the group of cocycles modulo the coboundaries. The cellular cohomology of a CW complex and the simplicial cohomology of a Δ complex are defined analogously.

Definition III. (Reduced cohomology). The *reduced singular cohomology* $\widetilde{H}^*(X)$ of a topological space X is the cohomology of the cochain complex obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the *augmented* singular chain complex of X,

$$\cdots \longrightarrow C_4(X) \xrightarrow{\partial_4} C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Reduced cellular and simplicial cohomology are defined analogously.

Recall that a *pair* of spaces (X, A) is a topological space X and subspace $A \subseteq X$. A *map of pairs* $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ satisfying $f(A) \subseteq B$. A *homotopy* of maps of pairs $f_t : (X, A) \to (Y, B)$ is a homotopy of maps $f_t : X \to Y$ with $f_t(A) \subseteq B$ for all t.

Definition IV. (The cohomology of a pair). Let (X, A) be a pair of spaces. The *relative cohomology groups* $H_*(X, A)$ are defined by dualizing the relative chain complex $\Big(C_*(X, A), \partial_*\Big)$ with relative chain groups $C_p(X, A) := C_p(X)/C_p(A)$. Relative cellular and simplicial cohomology groups are defined analogously.

Exercise 9 verifies that $H^*(X) \cong H^*(X, \varnothing)$ and $\widetilde{H}^*(X) \cong H^*(X, *)$ for all spaces X.

Exercise 1. (Warm-Up). For each of the following chain complexes, apply the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ and compute the homology of the resulting cochain complex.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix}1\\2\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}-2 & 1\end{bmatrix}} \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix}2\\4\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}-2 & 1\end{bmatrix}} \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix}1\\2\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}-4 & 2\end{bmatrix}} \mathbb{Z} \longrightarrow 0$$

Exercise 2. (Warm-Up). Let (C_*, d_*) be a chain complex of abelian groups. Verify that the sequence obtained by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ to this chain complex is, in fact, a valid cochain complex.

Exercise 3. (Warm-Up). Verify that a cocycle is a cochain that vanishes on boundaries.

Exercise 4. (Warm-Up). View $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ as a contravariant endofunctor on the category of abelian groups.

(a) Prove that $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ is left-exact. Recall that this means that, if $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then the following sequence is exact.

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(B,\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$$

Equivalently, the kernel of the functorial image of a homomorphism is the functorial image of the cokernel.

(b) Prove that $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ fails to be exact, by showing that the functorial image of an injective homomorphism may fail to be surjective.

(c) Prove that the image of a *split* short exact sequence under $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ is again exact.

Exercise 5. (Warm-Up). Let X be a topological space.

(a) Explain how to interpret $H^0(X)$ as a group of functions $X \to \mathbb{Z}$ that are constant on path components.

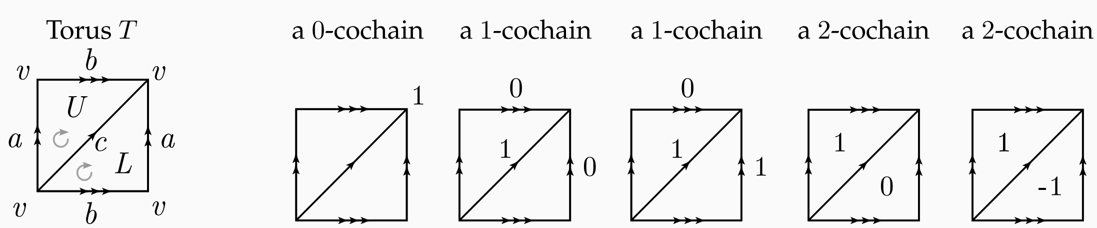
(b) Show that $H^0(X) \cong \prod_{\substack{\text{path components}\\ \text{of } X}} \mathbb{Z}$. Note that this is the product, and not the direct sum (coproduct).

(c) Explain how to interpret $\widetilde{H}^0(X)$ as a the quotient of the group of functions $X \to \mathbb{Z}$ that are constant on path components, modulo functions that are constant on X.

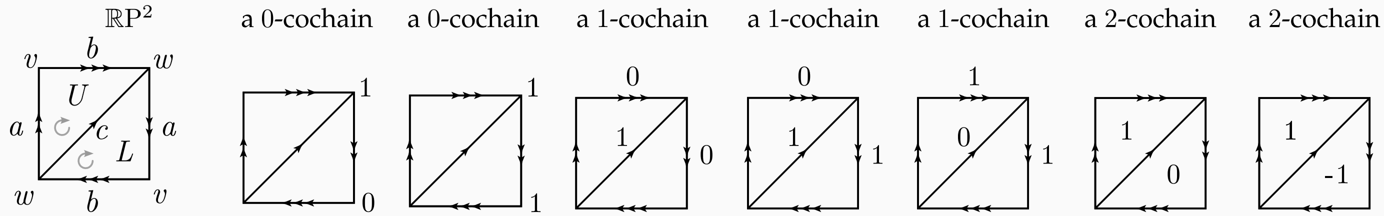
Exercise 6. (Warm-Up). Determine $\widetilde{H}^{-1}(\varnothing)$.

Exercise 7. (Warm-Up).

(a) Consider the Δ-complex structure on the torus T shown below. Each of the subsequent figures represents a simplicial p-cochain α on T. The p-simplices are labelled by their value under α. Compute the image of each cochain under the coboundary map. Verify that applying the coboundary map twice gives zero. Which cochains are cocycles? Coboundaries? Can you identify a principle for recognizing cocycles and coboundaries?



(b) Repeat this exercise for the Δ-complex structure on $\mathbb{R}\mathrm{P}^2$ and the cochains below.



Exercise 8. (Warm-Up). Compute, directly from a choice of CW complex structure, the cellular cohomology of the following spaces.

(a) The n-sphere S^n

(b) A closed genus-g surface Σ_g.

(c) Complex projective space $\mathbb{C}\mathrm{P}^n$.

(d) Real projective space $\mathbb{R}\mathrm{P}^n$.

1.2 The axioms for cohomology and other properties

The singular cohomology of a pair (X, A) satisfy axioms that are dual to the Eilenberg–Steenrod axioms for homology.

Theorem V. (Homotopy Axiom).

- *(Reduced) cohomology defines functors $H^*(-)$ and $\widetilde{H}^*(-)$ from topological spaces to abelian groups. Homotopic maps of spaces induce the same map on cohomology.*
- *Relative cohomology $H^*(-,-)$ defines a functor from pairs of topological spaces to abelian groups. Homotopic maps of pairs induce the same map on cohomology.*

Given a map f of spaces or of pairs, we write f^* for the map induced on cohomology.

Theorem VI. (Excision Axiom). *Let (X, A) be a pair and $U \subseteq A$ is a subspace such that the closure of U is contained in the interior of A. Then the inclusion map $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism on cohomology*

$$\iota^* : H^*(X, A) \xrightarrow{\cong} H^*(X \setminus U, A \setminus U).$$

Theorem VII. (Exactness Axiom). *Let (X, A) be a pair. There is a long exact sequence*

$$\cdots \longrightarrow H^{n-1}(A) \xrightarrow{\delta} H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i_*} H^n(A) \longrightarrow \cdots$$

where i^* is induced by the map $i : (A, \varnothing) \to (X, \varnothing)$ and j^* by the map $j : (X, \varnothing) \to (X, A)$. The long exact sequence of a pair is natural: given a map of pairs $f : (X, A) \to (Y, B)$, the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^{n-1}(A) & \xrightarrow{\delta} & H^n(X, A) & \xrightarrow{j^*} & H^n(X) & \xrightarrow{i^*} & H^n(A) & \longrightarrow \cdots \\
& f^* \uparrow & & f^* \uparrow & & f^* \uparrow & & f^* \uparrow & \\
\cdots \longrightarrow & H^{n-1}(B) & \xrightarrow{\delta} & H^n(Y, B) & \xrightarrow{j^*} & H^n(Y) & \xrightarrow{i^*} & H^n(B) & \longrightarrow \cdots
\end{array}$$

The long exact sequence in reduced cohomology is also natural.

$$\cdots \longrightarrow \widetilde{H}^{n-1}(A) \xrightarrow{\delta} H^n(X, A) \xrightarrow{j^*} \widetilde{H}^n(X) \xrightarrow{i_*} \widetilde{H}^n(A) \longrightarrow \cdots$$

Theorem VIII. (Additivity Axiom). *If $X = \coprod_\alpha X_\alpha$ is a decomposition into subspaces such that each X_α is a union of path components of X, then*

$$H^*(X) \cong \prod_\alpha H^*(X_\alpha).$$

Theorem IX. (Relative homology and quotients). *When the inclusion $A \to X$ is a cofibration, the quotient map $(X, A) \to (X/A, *)$ induces natural isomorphisms*

$$H^*(X, A) \cong H^*(X/A, *) \cong \widetilde{H}^*(X/A).$$

This holds in particular when (X, A) is a pair of CW complexes.

Theorem X. (The Mayer–Vietoris long exact sequence). *Let X be a space, and let $A, B \subseteq X$ be subspaces whose interiors cover X. Then there is a long exact sequence on cohomology groups*

$$0 \longrightarrow H^0(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{n-1}(A \cap B) \longrightarrow H^n(X) \longrightarrow H^n(A) \oplus H_n(B) \longrightarrow H^n(A \cap B) \longrightarrow \cdots$$

and a corresponding long exact sequence on reduced cohomology groups

$$0 \longrightarrow \widetilde{H}^{-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow \widetilde{H}^{n-1}(A \cap B) \longrightarrow \widetilde{H}^n(X) \longrightarrow \widetilde{H}^n(A) \oplus \widetilde{H}_n(B) \longrightarrow \widetilde{H}^n(A \cap B) \longrightarrow \cdots$$

Exercise 9. (Warm-Up). Let X be a topological space. Verify the following natural isomorphisms.

(a) $H^*(X) \cong H^*(X, \varnothing)$

(b) $\widetilde{H}^*(X) \cong H^*(X, *)$

Exercise 10. (Warm-Up). Compute, explicitly, the map induced on cohomology by the following maps of spaces.

(a) The inclusion $\iota : S^1 \to \Sigma_1$ of the meridian of a torus.

(b) The projection $\pi : \Sigma_1 \to S^1$ of the torus $\Sigma_1 \cong S^1 \times S^1$ onto its first factor.

(c) The degree-d covering map $p : S^1 \to S^1$.

(d) The quotient $q : \Sigma_1 \to S^2$ of the torus by the complement of an embedded open 2-disk.

Exercise 11. (Warm-Up). Let (X, A) be a pair of spaces. Verify that the following sequence is exact.

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_*(X)/C_*(A), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_*(X), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_*(A), \mathbb{Z}) \longrightarrow 0$$

Deduce the existence of the long exact sequence of a pair in cohomology.

Exercise 12. (Warm-Up). In this problem, we'll verify the special case of Mayer–Vietoris sequence where X is a CW complex and $A, B \subseteq X$ are CW subcomplexes whose union is X. By working with cellular cohomology, we avoid the technical subtleties concerning subdivision of singular simplices.

Verify that the sequence of cellular cochain complexes

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_*^{CW}(X), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_*^{CW}(A) \oplus C_*^{CW}(B), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_*^{CW}(A \cap B), \mathbb{Z}) \longrightarrow 0,$$

dual to the sequence defining the Mayer–Vietoris sequence, is exact.

Exercise 13. (Warm-Up). Inductively compute the singular cohomology groups of the n-sphere S^n in two different ways, using Mayer–Vietoris, and using the long exact sequence of a pair.

Exercise 14. (Warm-Up). Let X_α be a collection of CW complexes with distinguished basepoints. Show that

$$\widetilde{H}^*\left(\bigvee_\alpha X_\alpha\right) \cong \prod_\alpha \widetilde{H}^*\left(X_\alpha\right).$$

Exercise 15. (Warm-Up). Suppose that a subspace $A \subseteq X$ is a retract of X. Show that $H_n(X) \cong H_n(A) \oplus H_n(X, A)$.

Exercise 16. Let SX denote the suspension of X. What is the relationship between $H^*(X)$ and $H^*(SX)$?

Exercise 17. Let X be a space. Why don't we define $H^n(X)$ as the dual group $\mathrm{Hom}_{\mathbb{Z}}(H_n(X); \mathbb{Z})$?

Exercise 18. (Bonus) Let X be a space. What is the relationship between $H^1(X)$ and $[X, S^1]$, the homotopy classes of maps from X to the circle?