

Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n -disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (n -sphere)
(we sometimes view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus- g surface
- $\mathbb{RP}^n$ real projective n -space
- $\mathbb{CP}^n$ complex projective n -space

Practice problems

1. Let X be a Hausdorff space. Prove that any cover $\tilde{X}$ of X must be Hausdorff.
Remark: The converse is not true!
2. Let $p : \tilde{X} \rightarrow X$ be a covering map with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\tilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.
3. Let $p : \tilde{X} \rightarrow X$ be a covering space. Prove that X has a basis of open subsets that are evenly covered by p .
4. For each of the following maps: prove that the map is a covering map, or prove that it is not a covering map.
 - (a) The defining quotient map $S^n \rightarrow \mathbb{RP}^n$.
 - (b) The defining quotient map $S^{2n+1} \rightarrow \mathbb{CP}^n$.
5. Let X, Y, Z be path-connected, locally path-connected, and semi-locally simply-connected spaces. Let p, q, r be continuous maps with $p = r \circ q$. Show that, if p and q are covering maps, then so is r .

$$\begin{array}{c}
 Z \\
 \downarrow q \\
 Y \\
 \downarrow r \\
 X
 \end{array}
 \quad
 \begin{array}{c}
 \curvearrowright p \\
 \curvearrowleft
 \end{array}$$

6. Let $p : \tilde{X} \rightarrow X$ be a covering space, and let $\{U_\alpha\}$ be an open cover of X with the property that each open subset U_α is evenly covered by p . Let $f : I \rightarrow X$ be a continuous path. Prove that there is a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each i , $f|_{[t_i, t_{i+1}]} \subseteq U_i$ for some $U_i \in \{U_\alpha\}$.
7.
 - (a) Suppose that X is a graph. Show that, if X is simply connected, then X is contractible.
 - (b) Let X be a graph. Conclude that the universal cover of X is contractible.
 - (c) Let X be a graph, and let L be a path-connected, locally path-connected space with a finite fundamental group. Prove that every continuous map from L to X is nullhomotopic.
8. Let $p : \tilde{Y} \rightarrow Y$ be a covering map, and let $f : X \rightarrow Y$ be any map.
 - (a) Define a *lift* of f to $\tilde{Y}$, and give sufficient conditions for a lift $\tilde{f}$ to exist.
 - (b) Explain how we constructed the lift $\tilde{f}$.
 - (c) In what sense is the lift unique? Show by example how, when X is not connected, the lift $\tilde{f}$ need not satisfy our uniqueness condition.
9. Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a covering map. Prove that the induced map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.

10. Let $n \geq 2$. Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a covering map, and let $f : (S^n, s_0) \rightarrow (X, x_0)$ be a based map. Prove that f lifts to a unique based map $\tilde{f} : (S^n, s_0) \rightarrow (\tilde{X}, \tilde{x}_0)$.

11. Let X be a path-connected, locally path-connected, semi-locally simply connected space.

- (a) Show that the universal cover $p : \tilde{X} \rightarrow X$ of X is universal in the following sense: Given any covering map $r : X' \rightarrow X$, there is covering map $q : \tilde{X} \rightarrow X'$ making the following diagram commute.

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow q & \searrow p & \\ X' & & \\ \downarrow r & \swarrow & \\ X & & \end{array}$$

- (b) Is the covering map $q : \tilde{X} \rightarrow X'$ unique?

12. (a) Let $p : \tilde{X} \rightarrow X$ be a covering map. Explain why, if $\tilde{X}$ is simply connected, then it must be the universal cover (or, more precisely, it must be isomorphic as a cover to the universal cover).

- (b) Read Tai-Danae Bradley's 'recipe' for constructing the universal cover of a wedge sum, and describe the procedure in your own words.

<https://www.math3ma.com/blog/a-recipe-for-the-universal-cover-of-x-y>

- (c) Give explicit descriptions (possibly by picture) of the universal covers of the following spaces.

(i) $S^1 \times S^1 \times S^1$

(iv) $\mathbb{RP}^n, n \geq 2$

(ii) $S^1 \vee S^1 \vee S^1$

(v) $\mathbb{CP}^n$

(iii) $S^2 \vee S^1$

13. Suppose that X is not semi-locally simply connected. Explain why X cannot have a universal cover.

14. (**Topology QR Exam, Jan 2021**). Let $X = \mathbb{RP}^3$ and let $Y = S^1 \vee S^1$.

- (a) Are all maps $f : X \rightarrow Y$ null-homotopic?

- (b) Are all maps $f : Y \rightarrow X$ null-homotopic?

For each of the above, give a proof if the answer is "yes" and give an example if the answer is "no".

15. (**Topology QR Exam, Aug 2020**). For each of the following cases, determine if there exists a covering space $f : X \rightarrow Y$. (If yes, then construct it; if not, then give a proof).

- (a) X is homotopy equivalent to $S^1 \times S^1$ and Y is homotopy equivalent to $S^1 \vee S^1$.

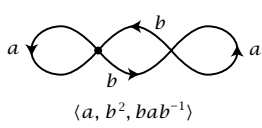
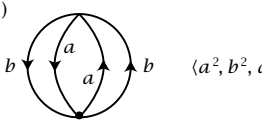
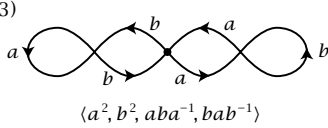
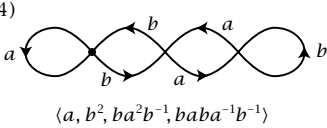
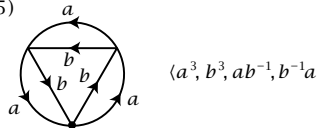
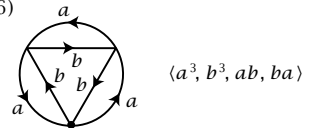
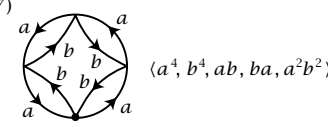
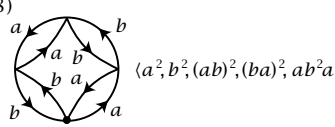
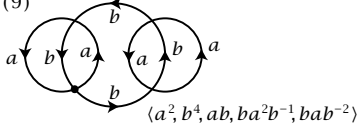
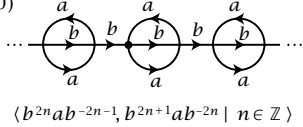
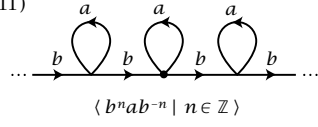
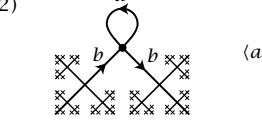
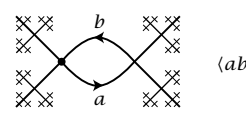
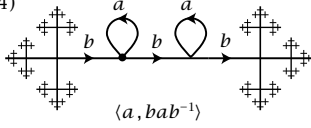
- (b) X is homotopy equivalent to $S^1 \vee S^1$ and Y is homotopy equivalent to $S^1 \times S^1$.

16. Let X and Y be path-connected, locally path-connected spaces with universal covers $p_X : \tilde{X} \rightarrow X$ and $p_Y : \tilde{Y} \rightarrow Y$. Show that a continuous map $f : X \rightarrow Y$ lifts to a continuous map $f' : \tilde{X} \rightarrow \tilde{Y}$ such that $p_Y \circ f' = f \circ p_X$. To what extent is f' unique?

17. Consider Hatcher's covers $\tilde{X}$ of $S^1 \vee S^1$ in the table below. Each cover has a distinguished basepoint $\tilde{x}_0$ (marked by a black dot) in the preimage of the single vertex x_0 in $S^1 \vee S^1$. We identify $\pi_1(S^1 \vee S^1, x_0)$ with the free group F_2 on letters a, b . For each cover, answer the following questions.

- (a) How many sheets is the cover?

- (b) What is $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ as a subgroup of $\pi_1(S^1 \vee S^1, x_0) = F_2$? Give a free generating set.

Some Covering Spaces of $S^1 \vee S^1$	
(1) 	(2) 
(3) 	(4) 
(5) 	(6) 
(7) 	(8) 
(9) 	(10) 
(11) 	(12) 
(13) 	(14) 

18. **True or counterexample.** For each of the following statements: if the statement is true, write “True”. If not, state a counterexample. No justification necessary.

Note: If the statement is false, you can receive partial credit for writing “False” without a counterexample.

- If $p : \tilde{X} \rightarrow X$ is a covering space map, then p cannot be nullhomotopic.
- If X is simply-connected, then X is semi-locally simply-connected.
- If X is connected, then X is semi-locally simply-connected.
- If X is locally simply-connected and semi-locally simply-connected, then X is simply-connected.
- Let $p : \tilde{Y} \rightarrow Y$ be a covering map, and let $f : X \rightarrow Y$ be a map such that a lift $\tilde{f} : X \rightarrow \tilde{Y}$ exists. Then the lift $\tilde{f}$ is the unique map lifting f .

19. Describe all (based) isomorphism classes of regular 3-sheeted covers of $S^1 \vee S^1$.

20. The *Euler number* of a finite graph X is the number of vertices of X minus the number of edges of X .

- (a) Suppose X is a finite connected graph with Euler number $\chi(X)$. What is the rank of the free group $\pi_1(X)$?
- (b) If X is a finite graph and $\tilde{X} \rightarrow X$ is an n -sheeted cover of X , what is the relationship between the Euler number of X and the Euler number of $\tilde{X}$?
21. (a) **(Centralizer)**. Let G be a group. Let S be a subgroup of G (or, more generally, a subset). The *centralizer* of S is defined to be

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

Prove that $C_G(S)$ is a subgroup of G .

- (b) Explain the difference between the normalizer $N_G(S)$ of S and the centralizer $C_G(S)$ of S .
- (c) Prove that $C_G(S)$ is contained in $N_G(S)$, and that it is a normal subgroup.
- (d) Under what conditions on S will we have containment $S \subseteq C_G(S)$?
- (e) We have another name for the subgroup $C_G(G)$. What is it?
22. (a) Consider the transposition (12) in the symmetric group S_4 . What is the normalizer of the subgroup $\langle(12)\rangle$ in S_4 ?
- (b) What is the normalizer of the subgroup $\langle a \rangle$ in the free group F_2 on a, b ?
23. **Definition (Abelian cover)**. Let X be path-connected, locally path-connected, and semi-locally simply-connected. A cover $p\tilde{X} \rightarrow X$ is *abelian* if it is a regular cover with an abelian deck group.
- (a) Prove that X has an abelian cover U that is universal in the sense that it is a cover of every other abelian cover of X .
- (b) Verify that the cover $U \rightarrow X$ is unique up to isomorphism of covers.
- (c) What is U when $X = S^1 \vee S^1$?
24. **(Topology QR Exam, May 2017)**. Let X be a connected CW-complex whose fundamental group is Σ_3 , the group of all permutations on 3 elements.
- (a) How many isomorphism classes of objects are there in the category $\text{Cov}_0(X)$ of connected covering spaces of X and continuous maps commuting with the covering map?
- (b) How many isomorphism classes of objects of $\text{Cov}_0(X)$ have degree 2?
- (c) How many isomorphism classes of objects of $\text{Cov}_0(X)$ are regular coverings?
25. **(Topology QR Exam, Jan 2018)**. Let X be a graph with one vertex and two edges. Does there exist a connected covering $f : Y \rightarrow X$ which is regular and a connected covering $g : Z \rightarrow Y$ which is regular such that $fg : Z \rightarrow X$ is not a regular covering? Prove your answer.
26. Let Σ_g be a genus- g surface, and let $n \leq 2g$. Prove or disprove: Σ_g has a regular covering space with deck group $\mathbb{Z}^{2g}$.
27. Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a path-connected covering map, and let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Prove that, if $[\gamma] \in \pi_1(X, x_0)$, then there is a point $\tilde{x}_1 \in p^{-1}(x_0)$ with $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = [\gamma]^{-1}H[\gamma]$.
28. Find free generating sets for the kernels of the following homomorphisms.
- (a)

$$\begin{aligned} h : F_2 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 1 \\ b &\longmapsto 0 \end{aligned}$$

(b)

$$h : F_2 \longrightarrow \mathbb{Z}$$

$$a \longmapsto 1$$

$$b \longmapsto 1$$

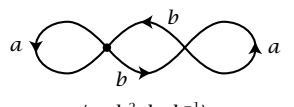
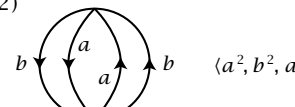
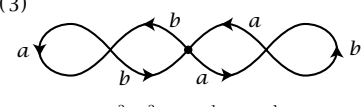
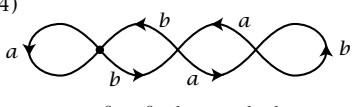
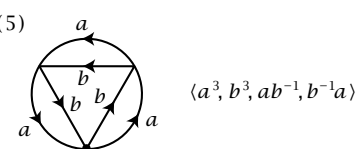
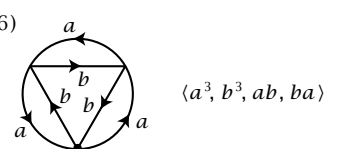
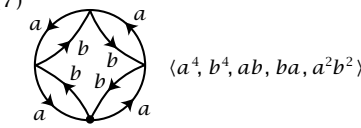
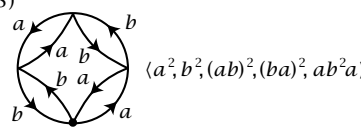
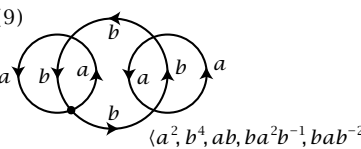
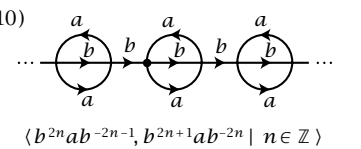
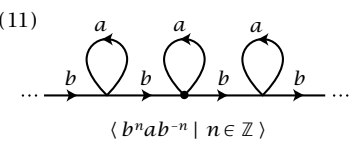
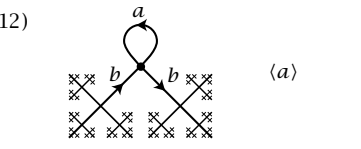
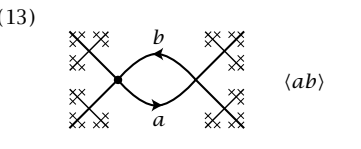
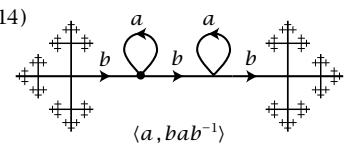
(c)

$$h : F_2 \longrightarrow \mathbb{Z}$$

$$a \longmapsto 1$$

$$b \longmapsto 2$$

29. (**Topology QR Exam, May 2020**). Describe a set of free generators of the subgroup of the free group on two generators a, b generated by b and all the conjugates of a^2 , b^2 , and $(ab)^3$. Is this a normal subgroup?
30. (**Topology QR Exam, Jan 2020**). Describe a set of free generators of the subgroup of the free group on two generators a, b generated by all conjugates of $aba^{-1}b^{-1}$.
31. Build an n -sheeted cover of $S^1 \vee S^1$, with vertices labelled $1, 2, \dots, n$, such that the action of a on the vertices is given by the permutation (in cycle notation):
- (a) (123)
 - (b) (12)(3)
 - (c) (1)(2)(3)
 - (d) (12)(34)
32. Let $p : \tilde{X} \rightarrow X$ be a connected, regular covering map, and let $x_0 \in X$. Let $G(\tilde{X})$ be the deck group. Build a bijection of sets between $G(\tilde{X})$ and the fibre $p^{-1}(x_0)$.
33. Suppose that $p : \tilde{X} \rightarrow X$ is a path-connected cover. Recall that we defined the cover to be *regular* if, for any $x \in X$, the group of Deck transformations of the cover acts transitively on the fibre $p^{-1}(x)$. Suppose that there is a single point $x_0 \in X$ such that the group of Deck transformations of the cover acts transitively on the fibre $p^{-1}(x_0)$. Prove that the cover is regular.
34. Let (X, x_0) be a based space, and let $p : \tilde{X} \rightarrow X$ be a cover.
- (a) Explain why the subgroup $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is independent of the choice of basepoint $\tilde{x}_0 \in p^{-1}(x_0)$ if and only if H is normal.
 - (b) Let $\gamma \in \pi_1(X, x_0)$. Show by example that the deck transformation of $\tilde{X}$ defined by γ may depend on the choice of basepoint $\tilde{x}_0$, even if H is normal.
- (Note that the subgroup H , its normalizer $N(H)$, and the action of an element of $N(H)$ all depend on our choice of basepoint $\tilde{x}_0$).
35. Consider Hatcher's covers $\tilde{X}$ of $S^1 \vee S^1$ in the table below.

Some Covering Spaces of $S^1 \vee S^1$	
(1)  $\langle a, b^2, bab^{-1} \rangle$	(2)  $\langle a^2, b^2, ab \rangle$
(3)  $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$	(4)  $\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$
(5)  $\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$	(6)  $\langle a^3, b^3, ab, ba \rangle$
(7)  $\langle a^4, b^4, ab, ba, a^2b^2 \rangle$	(8)  $\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$
(9)  $\langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$	(10)  $\langle b^{2n}ab^{-2n-1}, b^{2n+1}ab^{-2n} \mid n \in \mathbb{Z} \rangle$
(11)  $\langle b^nab^{-n} \mid n \in \mathbb{Z} \rangle$	(12)  $\langle a \rangle$
(13)  $\langle ab \rangle$	(14)  $\langle a, bab^{-1} \rangle$

Each cover has a distinguished basepoint $\tilde{x}_0$ (marked by a black dot) in the preimage of the single vertex x_0 in $S^1 \vee S^1$. We identify $\pi_1(S^1 \vee S^1, x_0)$ with the free group F_2 on letters a, b . For each cover, answer the following questions.

- How many sheets is the cover?
- What is $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ as a subgroup of $\pi_1(S^1 \vee S^1, x_0) = F_2$? Give a free generating set.
- Is the cover $\tilde{X}$ regular?
- Describe the group of deck transformations of $\tilde{X}$.
- Number the vertices of $\tilde{X}$, and compute the permutations on the vertices defined by the actions of a, b , and ab in $\pi_1(S^1 \vee S^1, x_0)$.
- In our proof identifying the deck group of $\tilde{X}$ with $N(H)/H$, we defined an action of $N(H) \subseteq F_2$ by deck transformations. For each of the covers, for the elements $\gamma = a$ and $\gamma = b$ in F_2 , determine whether γ is in $N(H)$, and, if so, describe the associated deck transformation of $\tilde{X}$.

36. Describe the universal cover of $S^1 \vee S^1$. Choose a cover from Hatcher's table, and explain/illustrate how it can be constructed as a quotient of the universal cover by a suitable choice of covering action.
37. Let $p : \tilde{X} \rightarrow X$ be a connected cover, and let $x_0 \in X$.
- (a) Let $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$. Prove that, if a deck transformation mapping $\tilde{x}_1$ to $\tilde{x}_2$ exists, then it is unique.
 - (b) Suppose a deck transformation τ mapping $\tilde{x}_1$ to $\tilde{x}_2$ exists. Explain how to determine where τ maps an arbitrary point $\tilde{x} \in \tilde{X}$. *Hint:* Recall our construction from our proof of the existence of lifts.
38. Given a group G and a normal subgroup N , show that there exists a normal covering space $\tilde{X} \rightarrow X$ with $\pi_1(X) \cong G$, $\pi_1(\tilde{X}) \cong N$, and deck transformation group $G(\tilde{X}) \cong G/N$.
39. Let $S^{2k-1} \subseteq \mathbb{C}^k \cong \mathbb{R}^{2k}$ be the unit sphere. Define an action of $\mathbb{Z}/m\mathbb{Z}$ on S^{2k-1} by rotation, generated by the map $v \mapsto e^{2\pi i/m}v$. Compute the fundamental group of its orbit space.
40. Let G_1 act on X_1 and G_2 act on X_2 by covering space actions.
- (a) Define an action of $G_1 \times G_2$ on $X_1 \times X_2$ by

$$(g_1, g_2) \cdot (x_1, x_2) = (g_1 \cdot x_1, g_2 \cdot x_2).$$

Prove this is a covering space action.

- (b) Prove that $(X_1 \times X_2)/(G_1 \times G_2)$ is homeomorphic to $X_1/G_1 \times X_2/G_2$.
41. Consider the following actions of a group G on a space X . Determine which actions are covering space actions.
- (a) $X = S^1$, $G = \mathbb{Z}/2\mathbb{Z}$, and the generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by 180° rotation.
 - (b) $X = S^2$, $G = \mathbb{Z}/2\mathbb{Z}$, and the generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by 180° rotation around the vertical axis.
 - (c) $X = \mathbb{R}^n$, $G = \mathbb{R}$, and $r \in \mathbb{R}$ acts by

$$r \cdot (x_1, x_2, x_3, \dots, x_n) = (x_1 + r, x_2, x_3, \dots, x_n).$$

- (d) $X = \mathbb{R}^n$, $G = \mathbb{Z}$, and $z \in \mathbb{Z}$ acts by

$$z \cdot (x_1, x_2, x_3, \dots, x_n) = (x_1 + z, x_2, x_3, \dots, x_n).$$

- (e) $X = Y^n$ for some space Y , $G = S_n$, and $\sigma \in S_n$ acts by

$$\sigma \cdot (y_1, y_2, \dots, y_n) = (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)}).$$

42. For each of the following actions: prove that they are covering actions, and identify the quotient (they are spaces we know by name!)
- (a) The group $\mathbb{Z}/2\mathbb{Z}$ acts on $S^1 \times I$. The generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $(x, t) \mapsto (-x, t)$.
 - (b) The group $\mathbb{Z}/2\mathbb{Z}$ acts on $S^1 \times I$. The generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $(x, t) \mapsto (-x, 1 - t)$.
43. Find a covering space action of $\mathbb{Z}/2\mathbb{Z}$ on the torus so that the quotient is a Klein bottle.
44. Let G be a group with a covering space action on a path-connected, locally path-connected space $\tilde{X}$.
- (a) What can you say about the relationship between G and $\pi_1(X/G)$? (Note: we did not assume X is simply connected).
 - (b) For a subgroup $H \subseteq G$, show that X/H is a cover of X/G .
 - (c) Show that the cover $X/H \rightarrow X/G$ is normal if and only if H is a normal subgroup of G .
 - (d) For subgroups H_1, H_2 of G , show that the covering space X/H_1 and X/H_2 of X/G are isomorphic if and only if H_1 and H_2 are conjugate subgroups of G .

45. Let $\tilde{X} \rightarrow X$ be a (not necessarily regular) cover, and let G be its group of deck transformations. Let $q: \tilde{X} \rightarrow \tilde{X}/G$ be the quotient map to the orbit space X/G . Show that there exists a map r making the following diagram commute.

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow q & \searrow p & \\ \tilde{X}/G & & \\ \downarrow r & \swarrow & \\ X & & \end{array}$$

46. Let $\{(C_*^i, d_*^i)\}_{i \in I}$ be a family of chain complexes.
- How should we define the complex $(\bigoplus_i C_*^i, d_*)$?
 - Prove that (for a suitable solution to part (a)),

$$H_n\left(\bigoplus_i C_*^i\right) = \bigoplus_i H_n(C_*^i).$$

47. (a) Let $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be a family of abelian groups. Construct a chain complex $\{(C_*, d_*)\}$ such that $H_n(C_*) = A_n$.
- (b) Let $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be a family of finitely generated abelian groups. Construct a chain complex $\{(C_*, d_*)\}$ such that C_n is free abelian for all n , and $H_n(C_*) = A_n$.
48. Consider the abelian groups A and subgroups B given below. Compute the isomorphism type of the quotient A/B (in the sense of the structure theorem for finitely generated abelian groups).
- $A = \mathbb{Z}^2$, B is the subgroup generated by $(2, 3)$.
 - $A = \mathbb{Z}^2$, B is the subgroup generated by $(2, 4)$.
 - $A = \mathbb{Z}^2$, B is the subgroup generated by $(1, 1)$ and $(1, -1)$.
 - $A = \mathbb{Z}^2$, B is the subgroup generated by $(2, 1)$ and $(7, 4)$.
 - $A = \mathbb{Z}^2$, B is the subgroup generated by $(2, 1)$ and $(1, 3)$.
49. Put the following matrices into Smith normal form. Choose some matrices of your own and compute their Smith normal forms.

$$\begin{bmatrix} 1 & 3 & -2 \\ 5 & 2 & -1 \\ 6 & -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ -6 & 6 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

50. For a matrix A , what is the relationship between the Smith normal form of A and its transpose A^T ?
51. Suppose A is a square matrix with entries in $\mathbb{Z}$.
- Suppose A is invertible over $\mathbb{Z}$. What are the possibilities for its Smith normal form?
 - Suppose A is invertible over $\mathbb{Q}$ (but not necessarily over $\mathbb{Z}$). What are the possibilities for its Smith normal form?

52. Consider the quotient of $\mathbb{Z}^3$ by the subgroup generated by $\begin{bmatrix} 7 \\ 6 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Compute this group (in the sense of the structure theorem for finitely generated abelian groups).
Hint: Smith normal form.

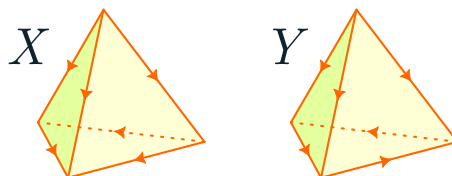
53. Compute (without computer assistance) the homology of the following chain complexes.

$$\begin{aligned}
 & \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 4 & -8 \\ -4 & 8 \\ 4 & -8 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} -4 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow \\
 & \longrightarrow \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 2 & 1 & 2 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ -1 & 1 & 2 \end{bmatrix}} \mathbb{Z}^3 \longrightarrow \\
 & \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 2 & 2 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & -3 \\ 1 & 1 & 1 \end{bmatrix}} \mathbb{Z}^3 \longrightarrow
 \end{aligned}$$

54. For each of the following spaces, choose a Δ -complex structure, and compute the simplicial homology.

- (a) a selection of finite graphs of your choosing
- (b) a cylinder
- (c) a Klein bottle
- (d) the wedge sum of a closed 2-disk and a circle
- (e) the disjoint union of a cylinder and a circle

55. A space X is formed by taking a quotient of a tetrahedron that identifies each of its 6 edges to a single edge, with the orientations shown. Only the 1-dimensional edges are glued; the interiors of the 4 triangular faces are distinct. A space Y is constructed in the same fashion. Compute $H_*(X)$ and $H_*(Y)$.



Note that for Y (unlike X) the edges of the bottom face are cyclically oriented.

56. Must the following maps be nullhomotopic? Give a proof or prove a counterexample.

- (a) $f : S^2 \rightarrow S^1 \times S^1$
- (b) $g : S^1 \times S^1 \rightarrow S^2$

57. **True or counterexample.** For each of the following statements: if the statement is true, write “True”. If not, state a counterexample. No justification necessary.

Note: If the statement is false, you can receive partial credit for writing “False” without a counterexample.

- (a) There is no 3-sheeted cover of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$.
- (b) There is no 4-sheeted cover of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$.
- (c) Every cover of a Mobius band is regular.
- (d) Every cover of $S^1 \vee S^1$ is regular.
- (e) Every covering map $p : \tilde{X} \rightarrow X$ is the quotient map to the orbit space of the action of the deck group $G(\tilde{X})$ on $\tilde{X}$.

- (f) Let $\tilde{X} \rightarrow X$ be a connected covering space, and let τ be a deck map $\tilde{X} \rightarrow \tilde{X}$. If τ fixes a point, then τ is the identity.
- (g) Let X be a path-connected, locally path-connected, semi-locally simply-connected based space. Let H be a subgroup of $\pi_1(X)$, and let Y be a space with $\pi_1(Y) \cong H$. Then there exists a covering map $Y \rightarrow X$.
- (h) Let $H_n(X)$ be the n th simplicial homology group of a Δ -complex X . Then the rank of $H_n(X)$ is at most the number of n -simplices in X .
- (i) Let X be a 2-dimensional Δ -complex X . Then $H_1(X)$ is free abelian.
- (j) Let X be a 2-dimensional Δ -complex X . Then $H_2(X)$ is free abelian.