1 Topological spaces

Definition 1.1. (Topology; Topological space.) Let X be a set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X that satisfies the following properties:

(T1) The sets $\varnothing$ and X are elements of $\mathcal{T}$.

(T2) If $\{U_i\}_{i\in I}$ is any collection of elements of $\mathcal{T}$, then $\displaystyle\bigcup_{i\in I} U_i$ is in $\mathcal{T}$.

(T3) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

A set X endowed with a topology $\mathcal{T}$ is called a *topological space* and denoted by $(X, \mathcal{T})$, or simply denoted by X when $\mathcal{T}$ is clear from context. The elements of $\mathcal{T}$ are called the *open subsets* of the topological space X.

Definition 1.2. (Closed subsets of a topological space.) Let $(X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is called *closed* if its complement is open, that is, if $X \setminus C$ is an element of $\mathcal{T}$.

Example / Definition (The discrete and indiscrete topologies.) Let X be a set.

The topology $\mathcal{T} = \{\varnothing, X\}$ is the *indiscrete topology* on X.

The collection $\mathcal{T}$ of all subsets of X is the *discrete topology* on X.

We have already proved the following result:

Theorem 1.3. (Metrics induce a topology.) *Let (X, d) be a metric space. Then the collection $\mathcal{T}_d$ of all open sets in X forms a topology on X.*

The topology $\mathcal{T}_d$ is called the *topology induced by the metric d*. We will see that not every topology on a set X necessarily arises from a metric structure on X.

Definition 1.4. (Metrizable topologies.) A topology $\mathcal{T}$ on a set X is said to be *metrizable* if there exists a metric d on X such that $\mathcal{T}$ is the set $\mathcal{T}_d$ of open sets for the metric space (X, d).

In-class Exercises

1. Let X be a set.
   (a) Show that the discrete topology is metrizable.
   (b) Suppose that X contains at least 2 elements. Show that the indiscrete topology is not metrizable.

2. **(A useful criterion for openness).** Let $(X, \mathcal{T})$ be a topological space. Show that $V \in \mathcal{T}$ (that is, V is open) if and only if for every $x \in V$ there is some set $U_x \subseteq X$ containing x such that $U_x \in \mathcal{T}$ and $U_x \subseteq V$.

3. (a) Verify that X and $\varnothing$ are closed.
   (b) Suppose that B and C are closed subsets of X. Verify that $B \cup C$ is closed.
   (c) Suppose that $\{C_i\}_{i\in I}$ is a collection of closed sets in X. Verify that $\displaystyle\bigcap_{i\in I} C_i$ is closed.

4. **(Optional).**
   (a) Let $X = \{a\}$. Explain why the only topology on X is $\{\varnothing, X\}$.
   (b) Let $X = \{a, b\}$. Find all possible topologies on X.
   (c) Let $X = \{a, b, c\}$. Find all possible topologies on X.
   Hint: Look at the picture on our Canvas homepage.

5. **(Optional).** Verify that the following sets are topologies on $\mathbb{R}$.
   - The topology induced by the Euclidean metric
   - $\mathcal{T} = \{\mathbb{R}, \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, (0,1), \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, \{0,1\}, \{0\}, \{1\}, \varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, \mathbb{R} \setminus A \text{ is finite}\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 1 \in A\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{A \mid A \text{ is a union of intervals of the form } [a, b) \text{ for } a, b \in \mathbb{R}\ \} \cup \{\varnothing\}$

6. **(Optional).** Consider the following definitions.

   Definition (Coarser topology; finer topology). Let X be a set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then the topology $\mathcal{T}_1$ is said to be *coarser* than $\mathcal{T}_2$, and the topology $\mathcal{T}_2$ is said to be *finer* than the topology $\mathcal{T}_1$.

   (a) Let X be a set. Show that the indiscrete topology on X is coarser than any other topology on X.
   (b) Let X be a set. Show that the discrete topology on X is finer than any other topology on X.
   (c) Consider all the topologies on $\mathbb{R}$ in Problem 5. For each pair of topologies, determine either that one is coarser than the other, or show that they are not comparable.

7. **(Optional).** Let X be a set, and let $\mathcal{P}$ be its power set, that is, the collection of all subsets of X. A function $\mathbf{i} : \mathcal{P} \to \mathcal{P}$ is called a *Kuratowski interior operator* if it satisfies the following properties.
   - $\mathbf{i}(X) = X$.
   - For all subsets $A \subseteq X$, $\mathbf{i}(A) \subseteq A$.
   - For all subsets $A \subseteq X$, $\mathbf{i}(\mathbf{i}(A)) = \mathbf{i}(A)$ (*Idempotence*).
   - For all subsets $A, B \subseteq X$, $\mathbf{i}(A \cap B) = \mathbf{i}(A) \cap \mathbf{i}(B)$.

   (a) Given a Kuratowski interior operator $\mathbf{i}$ on X, show that the fixed points of $\mathbf{i}$ (that is, the subsets U of X satisfying $\mathbf{i}(U) = U$) form a topology on X.
   (b) Let X be a topological space, and for each $A \subseteq X$ let $\mathbf{i}(A)$ be the union of all open subsets of X that are contained in A. Prove that $\mathbf{i}$ is a Kuratowski interior operator.