

1 Sequential Compactness

Definition 1.1. (Subsequences.) Let (X, d) be a metric space, and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X . Let

$$0 < n_1 < n_2 < \cdots < n_i < \cdots$$

be an infinite sequence of strictly increasing natural numbers. Then $(a_{n_i})_{i \in \mathbb{N}}$ is called a *subsequence* of $(a_n)_{n \in \mathbb{N}}$.

Proposition 1.2. *Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in a metric space (X, d) that converges to a point a_∞ . Show that any subsequence of $(a_n)_{n \in \mathbb{N}}$ also converges to a_∞ .*

Definition 1.3. (Sequentially compact metric spaces; sequentially compact subsets.) A metric space (X, d) is called *sequentially compact* if every sequence in X has a convergent subsequence. Similarly, a subset $S \subseteq X$ is *sequentially compact* if every sequence of points in S has a subsequence that converges to a point in S .

Is $\emptyset$ sequentially compact?

Example 1.4. Give examples of sequences $(a_n)_{n \in \mathbb{N}}$ of real numbers with the following properties:

- (a) The sequence $(a_n)_{n \in \mathbb{N}}$ has a subsequence that converges to 0, and a subsequence that converges to 1.
- (b) The sequence $(a_n)_{n \in \mathbb{N}}$ has no convergent subsequence.
- (c) The sequence $(a_n)_{n \in \mathbb{N}}$ is an unbounded sequence with a subsequence that converges to 0.

We will prove one important application of sequential compactness on the homework.

Theorem 1.5. (Extreme value theorem for metric spaces). *Let (X, d) be a metric space and C a nonempty, sequentially compact subset of X . Let $f : X \rightarrow \mathbb{R}$ be a continuous function. Then there is a point $c \in C$ so that*

$$f(c) = \sup_{x \in C} f(x).$$

In other words, the restriction $f|_C$ achieves its supremum.

Definition 1.6. Let (X, d) be a metric space. A subset $S \subseteq X$ is *bounded* if there is some $R > 0$ in $\mathbb{R}$ such that $d(x, y) < R$ for all $x, y \in S$.

In the exercises, we will prove the following properties of sequentially compact metric spaces.

Theorem 1.7. Let (X, d) be a metric space. If $S \subseteq X$ is a sequentially compact subset, then S is closed and bounded.

We will prove on the homework that, in Euclidean space, these conditions are in fact equivalent to sequential compactness.

Theorem 1.8. (Sequential compactness in $\mathbb{R}^n$). Consider the space $\mathbb{R}^n$ with the Euclidean metric. Let $S \subseteq \mathbb{R}^n$ be a subset. Then S is sequentially compact if and only if S is closed and bounded.

In-class Exercises

1. Let (X, d) be a metric space. Prove that every sequentially compact subset of X is closed.
2. Let (X, d) be a metric space. Prove that every sequentially compact subset of X is bounded.
3. Give an example (with a complete proof) of a metric space (X, d) and a subset S that is closed and bounded, but not sequentially compact.

4. **(Optional).** Let X be a metric space with the discrete metric $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$

Give necessary and sufficient conditions for a subset of X to be sequentially compact.

5. **(Optional).** Let (X, d) be a metric space. A subset $A \subseteq X$ is called *limit point compact* if every infinite subset of A has an accumulation point. Prove that a subset A of a metric space is limit point compact if and only if it is sequentially compact.
6. **(Optional).** Recall that the space $\mathcal{C}([a, b])$ of continuous functions $f : [a, b] \rightarrow \mathbb{R}$ is a metric space with metric

$$d_\infty(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|.$$

- (a) Show that $\mathcal{C}([a, b])$ is unbounded. Conclude that it is **not** sequentially compact.
 - (b) Is the subset $C = \{f(x) \in \mathcal{C}([0, 1]) \mid |f(x)| \leq 1 \text{ for all } x \in [0, 1]\}$ sequentially compact?
7. **(Optional).** Let X be a metric space, and let $a_\infty \in X$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X with the property that every subsequence of $(a_n)_{n \in \mathbb{N}}$ has a subsequence that converges to a_∞ . Prove that $(a_n)_{n \in \mathbb{N}}$ converges to a_∞ .
 8. **(Optional).** Consider the sequence of real numbers

$$0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \dots$$

What real numbers can be realized as the limit of a subsequence of this sequence?

9. **(Optional).** Is it possible to construct a sequence $(a_n)_{n \in \mathbb{N}}$ with subsequences converging to every real number $r \in \mathbb{R}$?