1 The Product Metric

Definition 1.1. (Product metric). Let (X, d_X) and (Y, d_Y) be metric spaces. Then their Cartesian product $X \times Y$ has a metric space structure, defined by the metric

$$\begin{aligned} d_{X\times Y} :\ & (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R} \\ & d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}. \end{aligned}$$

We will call $d_{X\times Y}$ the *product metric* on $X \times Y$.

Example 1.2. Let $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ be Euclidean spaces with the Euclidean metric. What is the product metric on $X \times Y$?

In-class Exercises

1. Verify that $d_{X\times Y}$ does in fact define a metric on $X \times Y$.
2. **(Optional.)** Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that $(X \times Y, d_{X\times Y})$ is a bounded metric space if and only if both X and Y are bounded.
3. **(Optional.)** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subseteq X$ and $B \subseteq Y$ be subsets. Let $x_0 \in X$ and $y_0 \in Y$ be elements.
   (a) Suppose that x_0 is an accumulation point of A, and y_0 is an accumulation point of B. Prove or disprove: (x_0, y_0) is an accumulation point of $A \times B$.
   (b) Suppose that x_0 is an isolated point of A, and y_0 is an isolated point of B. Prove or disprove: (x_0, y_0) is an isolated point of $A \times B$.
   (c) Suppose that (x_0, y_0) is an accumulation point of $A \times B$. Prove or disprove: x_0 is an accumulation point of A, and y_0 is an accumulation point of B.
   (d) Suppose that (x_0, y_0) is an isolated point of $A \times B$. Prove or disprove: x_0 is an isolated point of A, and y_0 is an isolated point of B.
4. **(Optional.)** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subseteq X$ and $B \subseteq Y$ be subsets. Consider the subset $A \times B \subseteq X \times Y$. We now have two ways to define a metric on this subset:
   - View A and B as metric spaces, by restriction of the metric d_X and d_Y respectively. Take the product metric on $A \times B$.
   - Take the product metric on $X \times Y$. Restrict this metric to the subspace $A \times B \subseteq X \times Y$.

   Verify that these two metrics on $A \times B$ are in fact equal.