1 Accumulation points

Definition 1.1. (Accumulation points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if . . .

Note that an accumulation point x of a set S may or may not itself be an element of S.

Definition 1.2. (Isolated points of a set.) Let (X, d) be a metric space. An element s of a subset S of X that is not an accumulation point of S is called an *isolated point* of S. This means . . .

Example 1.3. Consider $\mathbb{R}$ with the Euclidean metric. What are the accumulation points of the following subsets?

(a) $S = \mathbb{R}$

(b) $S = \{0\}$

(c) $S = (0, 1)$

In-class Exercises

1. Prove that the following definition of accumulation point is equivalent to the one above. In other words, show that a point $x \in X$ is an accumulation point of a set $S \subseteq X$ if and only if it satisfies the following property.

   Alternative Definition (Accumulation points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every open subset U of X containing x also contains a point in S distinct from x.

2. Let (X, d) be a metric space and let $S \subseteq X$ be a **closed** subset. Let x be an accumulation point of S. Show that x is contained in S.

3. **(Optional)** Find, with proof, the set of accumulation points of the following subsets of $\mathbb{R}$ (with the Euclidean metric).

   (a) $\mathbb{N}$ (b) $\mathbb{Q}$ (c) $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$ (d) $\mathbb{R} \setminus \mathbb{N}$.

4. **(Optional)** Let (X, d) be a metric space and $S \subseteq X$ a subset. Let S' denote the set of accumulation points of S. For each of the following statements, either prove the statement, or find a counterexample.

   (a) $S \subseteq S'$
   (b) $S' \subseteq S$.
   (c) If S is nonempty, then S' is nonempty.
   (d) The union $S \cup S'$ is always closed.
   (e) The union $(X \setminus S) \cup S'$ is always closed.
   (f) Any point of X can be an accumulation point for at most one of S and its complement $X \setminus S$.
   (g) Every point of X must be an accumulation point for at least one of S or its complement $X \setminus S$.