1 Homeomorphisms

When are two topological spaces “the same”?

Definition 1.1. (Homeomorphisms of topological spaces). Let X and Y be topological spaces. A map $f : X \to Y$ is a *homeomorphism* if

- f is continuous,
- f has an inverse function $f^{-1} : Y \to X$, and
- f^{-1} is continuous.

The topological space X is said to be *homeomorphic* to the topological space Y if there exists a homeomorphism $f : X \to Y$.

Example 1.2. Is it possible for a continuous function of topological space $f : X \to Y$ to have an inverse $f^{-1} : Y \to X$ that is not continuous?

Theorem 1.3. *If $f : X \to Y$ is a homeomorphism of topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, then f induces a bijection on their topologies,*

$$\begin{aligned} \mathcal{T}_X &\longrightarrow \mathcal{T}_Y \\ U &\longmapsto f(U) \\ f^{-1}(W) &\longleftarrow W. \end{aligned}$$

Corollary 1.4. *Let X be a set, and let d, D be two metrics on X. Then d and D are equivalent metrics if and only if the map*

$$\begin{aligned} (X, d) &\longrightarrow (X, D) \\ x &\longmapsto x \end{aligned}$$

is a homeomorphism.

In-class Exercises

1. This exercise shows that homeomorphism defines an *equivalence relation* on topological spaces.
   (a) Show that any topological space X is homeomorphic to itself.
   (b) Suppose $f : X \to Y$ a homeomorphism of topological spaces. Verify that $f^{-1} : Y \to X$ is also a homeomorphism. Conclude that X is homeomorphic to Y if and only if Y is homeomorphic to X. (We simply call the spaces “homeomorphic topological spaces”).
   (c) Show that, if X is homeomorphic to Y, and Y is homeomorphic to Z, then X is homeomorphic to Z.

2. Prove the following (very useful!) theorem.

   Theorem 1.5. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

3. Determine which of the following properties are preserved by homeomorphism. In other words, suppose X and Y are homeomorphic topological spaces. For each of the following properties P, prove or give a counterexample to the statement "X has property P if and only if Y has property P".
   (For some properties to be defined, you will need to assume that X and Y are metric spaces.)

   (i) discrete topology
   (ii) indiscrete topology
   (iii) T_1
   (iv) Hausdorff
   (v) normal
   (vi) number of connected components
   (vii) path-connected
   (viii) number of path components
   (ix) complete
   (x) sequentially compact
   (xi) compact
   (xii) bounded
   (xiii) metrizable

   Properties that are preserved by homeomorphisms are called *homeomorphism invariants*, *topological invariants*, or *topological properties* of a topological space.

4. Use the results of Problem 3 to explain why the following pairs of spaces are *not* homeomorphic.
   (a) $(0, 1)$ and $[0, 1]$ (with the Euclidean metric)
   (b) $\mathbb{R}$ with the Euclidean metric and $\mathbb{R}$ with the cofinite topology
   (c) $(0, 2)$ and $(0, 1] \cup (2, 3)$ (with the Euclidean metric)

5. **(Optional).** Let $f : X \to Y$ be a map of topological spaces. Prove the following statements.
   (a) The map f is a homeomorphism if and only if it is continuous, invertible, and open.
   (b) The map f is a homeomorphism if and only if it is continuous, invertible, and open.

6. **(Optional).** Let $f : X \to Y$ be a homeomorphism, and let $A \subseteq X$. Prove that f restricts to a homeomorphism $f|_A : A \to f(A)$ between the subspaces A and $f(A)$.

7. **(Optional).**
   (a) Prove that two spaces X and Y with the discrete topology are homeomorphic if and only if they have the same cardinality.
   (b) Prove that two spaces X and Y with the cofinite topology are homeomorphic if and only if they have the same cardinality.

8. **(Optional).** Let $X \times Y$ be the product of a space X and a nonempty space Y, endowed with the product topology. Fix $y_0 \in Y$. Prove that X is homeomorphic to the subspace $X \times \{y_0\} \subseteq X \times Y$.

9. **(Optional).** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $F : X \to Y$ a continuous function. Recall that the *graph* G of F is the set $G = \{(x, f(x)) \mid x \in X\}$ viewed as a subspace of $X \times Y$ with the product topology. Prove that G is homeomorphic to X.