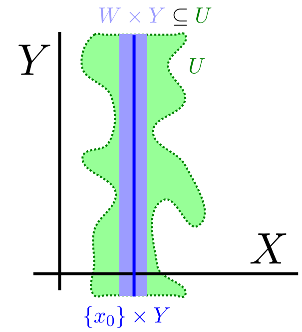
1 Products of compact spaces

The goal of this worksheet is to prove the following theorem.

Theorem 1.1. (Products of compact spaces). *Let X and Y be nonempty topological spaces. Then $X \times Y$ is compact with respect to the product topology if and only if both X and Y are compact.*

To prove this result, we will use the following lemma. Some other applications of this lemma appear in the bonus problems.

Lemma 1.2. (The tube lemma). *Let X and Y be nonempty topological spaces, with Y compact. Consider the product $X \times Y$ with the product topology. Suppose U is an open subset of $X \times Y$ containing $\{x_0\} \times Y$ for some $x_0 \in X$. Then there exists an open neighbourhood W of x_0 in X such that $W \times Y \subseteq U$.*



In-class Exercises

1. Prove the first direction of Theorem 1.1: Let X and Y be nonempty topological spaces. Suppose that their Cartesian product $X \times Y$ is compact. Prove that X and Y are compact.

2. (a) Suppose that Y is compact. Prove, for any $x_0 \in X$, that $\{x_0\} \times Y$ is a compact subspace of the product space $X \times Y$.

   (b) Prove Lemma 1.2.

3. In this problem, we will prove the second direction of Theorem 1.1. Let X and Y be nonempty, compact topological spaces. Let $\mathcal{U}$ be any open cover of $X \times Y$.

   (a) Show that, for any $x_0 \in X$, there exists an open neighbourhood W_{x_0} of x_0 so that $W_{x_0} \times Y$ is covered by finitely many elements of $\mathcal{U}$.

   (b) Prove that $X \times Y$ is covered by finitely many elements of $\mathcal{U}$.

4. **(Optional).** **Definition (Closed map).** A function $f : X \to Y$ of topological spaces is called *closed* if for all closed subsets $C \subseteq X$, the image $f(C)$ is closed in Y.

   (a) Let X and Y be topological spaces, and $X \times Y$ their product with the product topology. Let $\pi_X : X \times Y \to X$ be the projection map. Show by example that the map π_X may not be a closed map.

   (b) Now suppose that Y is compact. Show that $\pi_X : X \times Y \to X$ is a closed map.

   (c) Suppose that Y is compact and Hausdorff. Show that a function $f : X \to Y$ is continuous if and only if the graph G_f of f,

   $$G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y,$$

   viewed as a subset of $X \times Y$ with the product topology, is closed in $X \times Y$.

5. **(Optional).**

   Definition (Lindelöf). A topological space X is called *Lindelöf* if every open cover of X has a countable subcover.

   Suppose that X is a Lindelöf space and Y is a compact space. Prove that the product $X \times Y$, with the product topology, is Lindelöf.

6. **(Optional).** Prove the following generalization of the tube lemma.

   Theorem (Generalized tube lemma). Let A and B be subspaces of topological spaces X and Y, respectively. Let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U in X and V in Y such that $(A \times B) \subseteq (U \times V) \subseteq N$.