

1 Compactness and sequential compactness

Our definition of sequential compactness in metric spaces (Worksheet #8) generalizes to all topological spaces.

Definition 1.1. (Sequential compactness.) A topological space X is called *sequentially compact* if ...

Similarly, a subset $S \subseteq X$ is *sequentially compact* if every sequence of points in S has a subsequence that converges to a point in S .

For general topological spaces, a space can be sequentially compact but not compact, and a space can be compact but not sequentially compact. We will prove, however, that these two concepts are equivalent for metric spaces.

Theorem 1.2. (Compactness vs sequential compactness in metric spaces).

Let (X, d) be a metric space. Then X is compact if and only if X is sequentially compact.

Theorem 1.2 and Worksheet #8 Theorem 1.7 imply the following.

Theorem 1.3. (Compactness in metric spaces). *A compact subset of a metric space is closed and bounded.*

Theorem 1.2 and Worksheet #8 Theorem 1.8 allow us to fully classify the compact subsets of Euclidean space.

Theorem 1.4. (Compactness in $\mathbb{R}^n$). *A subspace $S \subseteq (\mathbb{R}^n, \text{Euclidean})$ is compact if and only if it is closed and bounded.*

In-class Exercises

1. (a) Let (X, d) be a metric space. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in X that contains no convergent subsequence. Prove that, for every $x \in X$, there is some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many points of the sequence.
 (b) Prove the first direction of Theorem 1.2: any compact metric space is sequentially compact.
2. (a) **Definition (Lebesgue number of an open cover).** Let (X, d) be a metric space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X . Then $\delta > 0$ is a *Lebesgue number* for $\mathcal{U}$ if, for every $x \in X$, there is some associated index $i_x \in I$ such that $B_\delta(x) \subseteq U_{i_x}$.
 Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of a metric space X . Suppose that $\mathcal{U}$ does not have a Lebesgue number. Explain why it is possible to construct a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $B_{\frac{1}{n}}(x_n)$ is not contained in U_i for any i .
 (b) Show that the sequence $(x_n)_{n \in \mathbb{N}}$ cannot have a convergent subsequence. Deduce that, if X is sequentially compact, then every open cover of X has a Lebesgue number.
 (c) **Definition (ϵ -nets of a metric space).** Fix $\epsilon > 0$. A subset A of a metric space X is called an ϵ -net if $\{B_\epsilon(a) \mid a \in A\}$ is an open cover of X .
 Fix $\epsilon > 0$. Prove that any sequentially compact metric space has a finite ϵ -net.
 (d) Prove the second direction of Theorem 1.2: a sequentially compact subset of a metric space is compact.

3. **(Optional).** Consider the following definitions.

Definition (Local compactness). A topological space X is *locally compact* if for all $x \in X$, there exists a compact subset of X that contains a neighbourhood of x .

Note that this implies that any compact space is locally compact.

- (a) Find an example of a space that is not locally compact.
 - (b) Suppose that X is a **Hausdorff** topological space. Show that X is locally compact if and only if every point $x \in X$ has a neighbourhood whose closure is compact.
 - (c) Suppose that X is a **Hausdorff** topological space. Show that X is locally compact if and only if for every point $x \in X$ and every neighbourhood U of x , the neighbourhood U contains a neighbourhood of x whose closure is compact.
 - (d) Do either of the previous two equivalences hold for all topological spaces?
 - (e) Find criteria for a subspace of a locally compact space to be locally compact.
4. **(Optional).** Given a Hausdorff and locally compact space X , our goal is to embed X into a compact Hausdorff space.

Definition (One-point compactification). Let X be a topological space, and let $\hat{X} = X \cup \{\infty\}$ be the set we obtain by adding a point ∞ which we call *the point at infinity*. We endow $\hat{X}$ with the following topology:

$$\mathcal{T} = \{U \subseteq \hat{X} \mid U \subseteq X \text{ is open, or } \hat{X} \setminus U \text{ is a closed, compact subset of } X\}.$$

The resulting topological space $\hat{X}$ is called the *Alexandroff extension* of X . If X is a noncompact, locally compact, Hausdorff space, then $\hat{X}$ is called the *one-point compactification* or the *Alexandroff compactification* of X .

Notice, in particular, that X is an open subset of $\hat{X}$.

- (a) Verify that $\mathcal{T}$ is in fact a topology.
- (b) Verify that $\hat{X}$ is compact.
- (c) Suppose that X is locally compact and Hausdorff. Verify that $\hat{X}$ is Hausdorff.
- (d) An embedding of topological spaces $f : X \rightarrow Y$ is an injective map that preserves the topology on X , in the following sense.

Definition (embedding). A map $f : X \rightarrow Y$ of topological spaces is an *embedding* if f is injective, and for all subsets $A \subseteq X$, the set $f(A)$ is open in the subspace $f(X) \subseteq Y$ if and only if A is open in X .

Show that the inclusion map $X \hookrightarrow \hat{X}$ is an embedding.

- (e) Show that, if X is not compact, then the closure of X (as a subset of $\hat{X}$) is $\hat{X}$. Show, in contrast, that if X is compact, then $X \cup \{\infty\}$ is a separation of $\hat{X}$.
- (f) Suppose that X is noncompact, locally compact and Hausdorff. Show that any compactification of X by one point must be homeomorphic to the Alexandroff compactification.
- (g) Suppose that a topological space X is a subspace of a compact Hausdorff space Y such that $Y \setminus X$ is a single point. Show that X must be Hausdorff and locally compact.