1 Connected topological spaces

Definition 1.1. (Disconnected spaces; connected spaces). A topological space $(X, \mathcal{T})$ is *disconnected* if there exist disjoint nonempty open subsets A and B in X such that $X = A \cup B$. We call the sets A and B a *separation* of X. If no separation of X exists, then X is called *connected*.

A subset S of X is said to be *disconnected* (respectively, connected) if it is disconnected (respectively, connected) when viewed with the subspace topology $(S, \mathcal{T}_S)$. Being disconnected means ...

Lemma 1.2. *A topological space X is disconnected if and only if there exists a subset $A \subseteq X$, with $\varnothing \subsetneq A \subsetneq X$, that is both closed and open.*

Example 1.3. Is this empty set connected?

Example 1.4. Determine which of the following topological spaces are connected.
(a) $X = \mathbb{Q}$, Euclidean

(b) $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, Euclidean

(c) $X = \mathbb{R}$, $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$

(d) $X = \{a, b, c, d\}$, $\mathcal{T} = \Big\{\varnothing, \{a, b\}, \{c\}, \{a, b, c\}, \{d\}, \{a, b, d\}, \{c, d\}, \{a, b, c, d\}\Big\}$

Example 1.5. (a) Can a connected space X have a disconnected subset $S \subseteq X$?

(b) Can a disconnected space X have a connected subset $S \subseteq X$?

You will prove the following important results on the homework.

Theorem 1.6. *A subset I of $(\mathbb{R}, Euclidean)$ is connected if and only if it has the following property: given any $x, y \in I$, if $z \in \mathbb{R}$ satisfies $x < z < y$, then $z \in I$.*

Such subsets I are called *intervals*.

Theorem 1.7. (Generalized Intermediate Value Theorem). *Let X be a* **connected** *topological space, and $f : X \to (\mathbb{R}, Euclidean)$ be a continuous function. If $x, y \in X$ and c lies between $f(x)$ and $f(y)$, then there exists $z \in X$ such that $f(z) = c$.*

In-class Exercises

1. Prove the following (often useful) lemma:

   Lemma 1.8. *Let X be a topological space, and let A, B be a separation of X. Let $S \subseteq X$. If S is connected, then $S \subseteq A$ or $S \subseteq B$.*

2. (a) Prove the following. *Hint:* Lemma 1.8.
   Lemma 1.9. *Let X be a topological space, and let A_i, $i \in I$, be a collection of* **connected** *subsets of X. Suppose that $\bigcap_{i\in I} A_i \neq \varnothing$. Then the union $\bigcup_{i\in I} A_i$ is connected.*

   (b) Show by example that the union $\bigcup_{i\in I} A_i$ may be disconnected if $\bigcap_{i\in I} A_i = \varnothing$.

3. Prove the following.

   Theorem 1.10. *Let $f : X \to Y$ be a continuous function of topological spaces. If X is connected, then so is $f(X)$.*

4. **(Optional).** Which of the following subsets of ($\mathbb{R}$, Euclidean) are connected?

$$\{x \in \mathbb{R} \mid d(x, 1) < 1 \text{ or } d(x, -1) < 1\}$$

$$\{x \in \mathbb{R} \mid d(x, 1) \leq 1 \text{ or } d(x, -1) < 1\}$$

$$\{x \in \mathbb{R} \mid d(x, 1) \leq 1 \text{ or } d(x, -1) \leq 1\}$$

5. **(Optional).** Consider the set $X = \{a, b, c, d\}$. For which of the following topologies $\mathcal{T}$ is the topological space $(X, \mathcal{T})$ connected?

   (a) $\mathcal{T} = \Big\{\varnothing, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\Big\}$

   (b) $\mathcal{T} = \Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{a, b, d\}, \{a, b, c, d\}\Big\}$

6. **(Optional).** Consider the following topologies on $\mathbb{R}$. Which of these topological spaces are connected?

   (a) indiscrete topology
   (b) discrete topology
   (c) Euclidean topology
   (d) cofinite topology
   (e) $\mathcal{T} = \{\mathbb{R}, (0, 1), \varnothing\}$
   (f) $\mathcal{T} = \{\mathbb{R}, \{0, 1\}, \{0\}, \{1\}, \varnothing\}$
   (g) $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   (h) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$
   (i) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$
   (j) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 1 \in A\} \cup \{\varnothing\}$

7. **(Optional).** Let $(X, \mathcal{T})$ be a topological space. Let $\{\ A_n \mid n \in \mathbb{Z}\ \}$ be a family of connected subspaces of X such that $A_n \cap A_{n+1} \neq \varnothing$ for every n. Prove $\bigcup_{n\in\mathbb{Z}} A_n$ is connected.

8. **(Optional).** Let $(X, \mathcal{T})$ be a topological space. Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of connected subspaces in X such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$. Is $\bigcap_{n\in\mathbb{N}} A_n$ is necessarily connected?

9. **(Optional).** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Suppose $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are both closed or both open, then A and B are connected.