

1 Separation Axioms

Definition 1.1. (T_1 -space). We say a topological space X has the T_1 property, or call X a T_1 -space, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists some neighbourhood U_x of x that does not contain y , and there exists some neighbourhood U_y of y that does not contain x .

Remark: The notation T originates from the German *trennungsaxiom*.

Proposition 1.2. A topological space X is a T_1 -space if and only if, for every point $x \in X$, the singleton set $\{x\}$ is closed.

Mathematicians often refer to this property by the slogan “points are closed”.

Proof:

Definition 1.3. (Hausdorff property; T_2 -space). We say a topological space X has the T_2 property, or call X a Hausdorff space, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists **disjoint** open neighbourhoods U_x of x and U_y of y .

Example 1.4. For each of the following topologies on $\mathbb{R}$, state whether the topology has the T_1 property, the T_1 and T_2 properties, or neither.

- (i) The topology induced by the Euclidean metric
- (ii) $\mathcal{T} = \{\mathbb{R}, \emptyset\}$
- (iii) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
- (iv) $\mathcal{T} = \{\mathbb{R}, (0, 1), \emptyset\}$
- (v) $\mathcal{T} = \{\mathbb{R}, \{0, 1\}, \{0\}, \{1\}, \emptyset\}$
- (vi) $\mathcal{T} = \{A \mid \mathbb{R} \setminus A \text{ is finite}\} \cup \{\emptyset\}$
- (vii) $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$
- (viii) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \in A\} \cup \{\emptyset\}$
- (ix) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \notin A\} \cup \{\mathbb{R}\}$
- (x) $\mathcal{T} = \{A \mid A \text{ is a union of intervals of the form } [a, b) \text{ for } a, b \in \mathbb{R}\} \cup \{\emptyset\}$

In-class Exercises

1. Show that every space satisfying the T_2 property must satisfy the T_1 property. Conclude in particular that, in a Hausdorff space, singleton sets $\{x\}$ are closed.

2. Prove the following result.

Theorem 1.5. *Metric spaces are Hausdorff.*

In particular, if a topological space is not Hausdorff, it is not metrizable.

3. Suppose that a topological space X is Hausdorff. Prove that, if a sequence of points $(x_n)_{n \in \mathbb{N}}$ in X converges, then its limit is unique.

4. **(Optional).** The following definitions give some of the additional separation axioms.

Definition 1.6. (T_0 -space, topological distinguishability). Let X be a topological space. We say that two distinct points x, y in X are *topologically indistinguishable* if they have precisely the same set of open neighbourhoods. Otherwise x and y are *topologically distinguishable*. We say that a space X is a T_0 -space if every pair of distinct points are topologically distinguishable.

Definition 1.7. ($T_{2\frac{1}{2}}$ -space, Urysohn). We say a topological space X has the $T_{2\frac{1}{2}}$ property, or call X a *Urysohn space*, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exist open neighbourhoods U_x of x and U_y of y whose closures are disjoint.

Definition 1.8. (Regular space). A topological space X is *regular* if, for every point $x \in X$ and every closed subset C not containing x , there exist disjoint open subsets U_x and U_C containing x and C , respectively.

Definition 1.9. (T_3 -space). A topological space X is defined to be a T_3 -space if it is a regular T_0 -space.

Definition 1.10. (Normal space). A topological space X is *normal* if, for any pair of disjoint closed subsets C and D , there exist disjoint open sets U_C and U_D containing C and D , respectively.

Definition 1.11. (T_4 -space). A topological space X is defined to be a T_4 -space if it is a normal Hausdorff space.

- (a) Prove that $T_4 \implies T_3 \implies T_{2\frac{1}{2}} \implies T_2 \implies T_1 \implies T_0$.
 - (b) Prove that metric spaces are regular and normal.
 - (c) Find an example of a topological space that is regular but not T_1 .
 - (d) Find an example of a topological space that is normal but not regular.
5. Show that, if a topological space is T_0 , T_1 , T_2 , or $T_{2\frac{1}{2}}$, then so is any finer topology on the same set. Is the same true of T_3 and T_4 ?
 6. **(Optional).** Let X be the set $(\mathbb{R} \setminus \{0\}) \cup \{0_\spadesuit, 0_\heartsuit\}$. The *line with two origins* is the space X with the topology generated by the following basis of subsets: all subsets of $\mathbb{R} \setminus \{0\}$ that are open in the Euclidean topology, and subsets of the form $(a, 0) \cup \{0_\spadesuit\} \cup (0, b)$ and $(a, 0) \cup \{0_\heartsuit\} \cup (0, b)$ for all $a < 0 < b \in \mathbb{R}$.
 - (a) Show that the line with two origins is T_1 .
 - (b) Show that it is not Hausdorff, but it is *locally Hausdorff* in the sense that every point has a neighbourhood with Hausdorff subspace topology.