1 Bases for topological spaces

Definition 1.1. (Basis of a topology.) Let $(X, \mathcal{T})$ be a topological space. We say that a collection $\mathcal{B}$ of subsets of X is a *basis* for the topology $\mathcal{T}$ if

- $\mathcal{B} \subseteq \mathcal{T}$, that is, every basis element is open, and
- every element of $\mathcal{T}$ can be expressed as a union of elements of $\mathcal{B}$.

We say that the basis $\mathcal{B}$ *generates* the topology $\mathcal{T}$.

Remark 1.2. By convention, we say that the empty set $\varnothing$ is the union of an empty collection of open sets. So a basis $\mathcal{B}$ does not need to include $\varnothing$.

Example 1.3. Let X be a set.

1. Find a basis for the discrete topology on X.

2. Find a basis for the indiscrete topology on X.

Example 1.4. The topology of a metric space (X, d) is defined by its basis

$$\mathcal{B} = \{ \ B_r(x_0) \mid x_0 \in X, r \in \mathbb{R}, r > 0 \ \}.$$

In-class Exercises

1. **(The basis criteria).** Let $(X, \mathcal{T})$ be a topological space. Show that a collection $\mathcal{B}$ of subsets of X is a basis for $\mathcal{T}$ if and only if it satisfies the following two conditions:
   (i) Every basis element is an **open** set, that is, $\mathcal{B} \subseteq \mathcal{T}$.
   (ii) For every open set $U \in \mathcal{T}$, and every $x \in U$, there exists some element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

2. Let (X, d_X) and (Y, d_Y) be metric spaces, and give $X \times Y$ the product metric $d_{X \times Y}$. Conclude from our "basis criteria" that

$$\mathcal{B} = \{ \ U \times V \mid U \subseteq X \text{ is open}, V \subseteq Y \text{ is open} \ \}$$

forms a basis for the topology induced by $d_{X \times Y}$.

3. **(Optional).** Let X be a metric space. Show that the following collection of open subsets is a basis for X:
$$\left\{ B_{\frac{1}{n}}(x) \mid x \in X,\ n \in \mathbb{N} \right\}.$$

4. **(Optional).**
   (a) Consider the topology on $\mathbb{R}^n$ induced by the Euclidean metric. Prove that the following set is a basis for $\mathbb{R}^n$.
$$\mathcal{B} = \{\ B_\epsilon(x) \mid\ \epsilon > 0,\ \epsilon \text{ is } \textbf{rational};\ x \in \mathbb{R}^n, \text{ all coordinates } x_i \text{ of } x \text{ are } \textbf{rational.}\}$$
   (b) Show that $\mathbb{R}^n$ has uncountably many open sets, but that the basis $\mathcal{B}$ is countable.

5. **(Optional).** Let (X, d) be a metric space, and $\mathcal{B}$ a basis for the topology $\mathcal{T}_d$ induced by d. Let $S \subseteq X$ be a subset, and $s \in S$. Show that s is an interior point of S if and only if there is some element $B \in \mathcal{B}$ such that $s \in B$ and $B \subseteq S$.

6. **(Optional) (Bases of closed subsets).** Let X be a topological space. Let $\mathcal{C}$ be a collection of **closed** subsets. Prove that the following conditions are equivalent.
   (i) The set $\mathcal{B} = \{U \subseteq X \mid X \setminus U \subseteq C\}$ is a basis for the topology on X.
   (ii) Every closed subset of X can be expressed as an intersection of elements of $\mathcal{C}$.
   (iii) For each closed set A in X and each $x \notin A$ there exists an element of $\mathcal{C}$ containing A and not containing x.
   (iv) The collection $\mathcal{C}$ satisfies $\bigcap_{C \in \mathcal{C}} C = \varnothing$, and for each $C_1, C_2 \in \mathcal{C}$ the union $C_1 \cup C_2$ can be expressed as the intersection of elements of $\mathcal{C}$.

   A collection of closed subsets $\mathcal{C}$ satisfying these equivalent conditions is called a *basis for the closed sets* of X.

7. **(Optional). Definition (Subbases).** Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X whose union is equal to X. Then the *topology generated by the subbasis* $\mathcal{S}$ is the collection of all arbitrary unions of all finite intersections of elements in $\mathcal{S}$. *Remark*: In contrast to a basis, we are permitted to take finite intersections of sets in a subbasis.
   (a) Show that the set $\mathcal{T}$ generated by a subbasis $\mathcal{S}$ really is a topology, and is moreover the coarsest topology containing $\mathcal{S}$.
   (b) Verify that $\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, a) \mid a \in \mathbb{R}\}$ is a subbasis for the standard topology on $\mathbb{R}$.
   (c) Prove the following proposition.

      Proposition. Let $f : X \to Y$ be a function of topological spaces, and let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if $f^{-1}(U)$ is open for every subbasis element $U \subseteq \mathcal{S}$.