1 Subspaces of topological spaces

Definition 1.1. (Subspace topology.) Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be any subset. Then S inherits the structure of a topological space, defined by the topology

$$\mathcal{T}_S = \{U \cap S \mid U \in \mathcal{T}_X\}.$$

The topology $\mathcal{T}_S$ on S is called the *subspace topology*.

Example 1.2. Describe the subspace topology on the following subsets of $\mathbb{R}$, with the topology induced by the Euclidean metric (we call this the "standard topology").

(a) $S = \{0, 1, 2\}$

(b) $S = [0, 1)$

In-class Exercises

1. Verify that the subspace topology is, in fact, a topology.
2. Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be any subset. Let ι_S be the *inclusion map*

$$\iota_S : S \to X$$
$$\iota_S(s) = s$$

   Verify that the subspace topology on S is precisely the set $\{\iota_S^{-1}(U) \mid U \subseteq X \text{ is open}\}$.

   Remark: We haven't defined these terms, but we can summarize this result by the slogan "the subspace topology on S is the coarsest topology that makes the inclusion maps ι_S continuous".
3. Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be a subset endowed with the subspace topology $\mathcal{T}_S$. Show that a set $C \subseteq S$ is closed in S if and only if there is some set $D \subseteq X$ that is closed in X with $C = D \cap S$.
4. **(Optional).** Let $(X, \mathcal{T}_X)$ be a topological space, and let $Z \subseteq Y \subseteq X$ be subsets. Show that the subspace topology on Z as a subspace of X coincides with the subspace topology on Z as a subspace of Y (with the subspace topology as a subset of X). Conclude that there is no ambiguity in how to topologize the subset Z – to refer to its "subspace topology" we do not need to specify whether Y or X is the ambient space.

5. **(Optional).** Let (X, d) be a metric space, and let $\mathcal{T}_d^X$ be the topology induced by the metric. Let $S \subseteq X$ be a subset. We now have two methods of constructing a topology on S: we can restrict the metric from X to S, and take the topology $\mathcal{T}_d^S$ induced by the metric. We can also take the subspace topology $\mathcal{T}_S$ defined by $\mathcal{T}_d^X$. Show that these two topologies on S are equal, so there is no ambiguity in how to topologize a subset of a metric space.

6. **(Optional).** Let (X, d) be a metric space with the metric topology $\mathcal{T}_d$. Show that the subspace topology on any **finite** subset of X is the discrete topology.

7. **(Optional).** Consider $\mathbb{R}$ with the standard topology (that is, the topology induced by the Euclidean metric). For each of a the following statements, construct a nonempty subset S of $\mathbb{R}$ with that satisfies the description, or prove that none exists.
   (a) S is an infinite, closed subset of $\mathbb{R}$, and the subspace topology on S is discrete.
   (b) S is not a closed subset of $\mathbb{R}$, and the subspace topology on S is discrete.
   (c) S has the indiscrete topology.
   (d) The subspace topology on S consists of exactly 2 open subsets.
   (e) The subspace topology on S consists of exactly 3 open subsets.

8. **(Optional).** Let $X = \{a, b, c, d\}$. Consider the following topologies on X.
   - $\mathcal{T}_1 = \{\varnothing, \{b, c\}, \{a, d\}, \{a, b, c, d\}\}$
   - $\mathcal{T}_2 = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$
   - $\mathcal{T}_3 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$
   - $\mathcal{T}_4 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$
   - $\mathcal{T}_5 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, c, d\}\}$
   - $\mathcal{T}_6 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}\}$

   For each of these topologies . . . ,
   (a) Find the subspace topology on each of the subsets $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{a, b, c\}$, and $\{b, c, d\}$.
   (b) Identify all subspaces with the discrete topology.
   (c) Identify all subspaces with the indiscrete topology.

9. **(Optional).** Let $(X, \mathcal{T})$ be a topological space, and $S \subseteq X$ a subset endowed with the subspace topology.
   (a) Suppose X has the discrete topology. Must S have the discrete topology?
   (b) Suppose X has the indiscrete topology. Must S have the indiscrete topology?
   (c) Suppose X is metrizable. Is S metrizable?
   (d) A space has the T_1 *property* if every singleton subset $\{x\}$ is closed. If X is T_1, then must S be T_1?

   Remark: A property is called *hereditary* if, whenever a topological space has the property, all of its subspaces necessarily have the property.