

Midterm Exam

Math 490
21 October 2025
Jenny Wilson

Name: _____

Instructions: This exam has 4 questions for a total of 35 points.

The exam is closed-book. No books, cell phones, calculators, or other devices are permitted.

Each student may bring in one double-sided standard-size (8.5 in $\times$ 11 in) sheet of notes, which they must prepare themselves. Notes may be handwritten or typed with font size at least 12.

Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	14	
2	8	
3	6	
4	7	
Total:	35	

1. (14 points) For each of the following: If the statement is true, write “**True**”. Otherwise, state a specific counterexample. **No further justification needed.**

If the statement is not true, you may receive partial credit for writing “**False**” without a counterexample.

- (a) Let X be a metric space. If $U \subseteq X$ is an open subset, then every point of U is an accumulation point of U .

- (b) Let X be a metric space and $S \subseteq X$. The set S' of accumulation points of S is closed in X .

- (c) Let X be a metric space and $S \subseteq X$. Then the set of isolated points of S is closed in X .

- (d) If a metric space X is bounded, then for any $\epsilon > 0$, the space X is the union of a finite number of open balls of radius ϵ .

- (e) Let X be a metric space. If, for some $\epsilon > 0$, the space X is the union of a finite number of open balls of radius ϵ , then X is bounded.

- (f) Let X be a metric space with at least two elements. Then there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in X that is **not** Cauchy.

- (g) Let X and Y be nonempty metric spaces with the **discrete** metrics. Then the product metric $d_{X \times Y}$ on $X \times Y$ is topologically equivalent to the discrete metric.
- (h) If A and B are dense subsets of a metric space X , then their intersection $A \cap B$ is dense in X .
- (i) Let $f : X \rightarrow Y$ be a continuous function of metric spaces. If x is an accumulation point of a set A , then $f(x)$ is an accumulation point of $f(A)$.
- (j) Let $f : X \rightarrow Y$ be a continuous **injective** function of metric spaces. If x is an accumulation point of a subset $A \subseteq X$, then $f(x)$ is an accumulation point of $f(A)$.
- (k) Let X be a topological space and $S \subseteq X$ a subspace. If $U \subseteq S$ is open in the subspace topology, then U is open in X .
- (l) Let X be a topological space, and $A \subseteq X$ a subset. Then $\overline{A} = A \cup A'$.

- (m) Let X be a topological space and $\mathcal{B}$ a basis for the topology on X . Let $A \subseteq X$ be a subset.

$$\text{Then } \text{Int}(A) = \bigcup_{\substack{B \in \mathcal{B} \\ \text{with } B \subseteq A}} B.$$

- (n) Let $f : X \rightarrow Y$ be a continuous function of metric spaces, and $A \subseteq Y$.

$$\text{Then } f^{-1}(\text{Int}(A)) \subseteq \text{Int}(f^{-1}(A)).$$

2. (a) (4 points) Let U_i , $i \in I$, be a (possibly infinite) collection of subsets of a metric space X . Consider the following statement.

If the subset U_i is _____ for all $i \in I$, then so is their union $\bigcup_{i \in I} U_i$.

Circle all properties that truthfully fill in the blank. **No justification needed.**

open closed sequentially compact non-sequentially-compact

complete non-complete bounded unbounded

(By “ U is complete” we mean U is a complete metric space with respect to the subspace metric)

- (b) (4 points) Let $U_1, \dots, U_n$ be a **finite** collection subsets of a metric space X . Consider the following statement.

If the subset U_i is _____ for all i , then so is their union $\bigcup_{i=1}^n U_i$.

Circle all properties that truthfully fill in the blank. **No justification needed.**

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complete non-complete bounded unbounded

3. (6 points) Let (X, d) be a metric space. Prove that, if X has two distinct accumulation points x_0 and y_0 , then X contains a subset that is neither open nor closed.

4. (7 points) Suppose that (X, d_X) and (Y, d_Y) are nonempty, complete metric spaces. Show that their product $X \times Y$, with the product metric $d_{X \times Y}$, is a complete metric space.