Final Exam

Math 490
16 December 2025
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Name: ______________________________

Instructions: This exam has 5 questions for a total of 45 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	7	
2	21	
3	8	
4	4	
5	5	
Total:	45	

1. (7 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let $f : X \to Y$ be a function of sets. Then the binary relation

   $$a \sim b \quad \text{iff} \quad f(a) = f(b)$$

   satisfies all the axioms of an equivalence relation on X.

   True. *Hint:* We can check it is reflexive, symmetric, and transitive. It partitions the domain X into subsets called the *fibres* of f.

   (b) Let X be a metric space, and $(a_n)_{n\in\mathbb{N}}$ a Cauchy sequence in X. If this sequence has a convergent subsequence, then it converges.

   True. *Hint:* You proved this as a step in Homework #4 Problem 8.

   (c) Let $X = \mathbb{R}$. Then the collection of subsets $\mathcal{T} = \{[a,\infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$ satisfies all the axioms of a topology on X.

   False. The collection $\mathcal{T}$ is not closed under unions. For example, the union of elements $\bigcup_{a>0}[a,\infty) = (0,\infty)$ is not an element of $\mathcal{T}$.

   (d) Let X and Y be topological spaces. If X and Y are metrizable, then their product $X \times Y$ is metrizable.

   True. *Hint:* You proved on Homework #2 Problem 6 that if X and Y are metric spaces, then the product metric induces the product topology on $X \times Y$.

   (e) There is no path from a to d in the topological space $X = \{a,b,c,d\}$ with the topology $\Big\{\varnothing, \{a\}, \{d\}, \{a,d\}, \{a,c\}, \{a,c,d\}, X\Big\}$.

   False. For example, $\gamma(t) = \begin{cases} a, & t \in \left[0, \frac{1}{2}\right) \\ b, & t = \frac{1}{2}, \\ d & t \in \left(\frac{1}{2}, 1\right] \end{cases}$ is a path from a to d.

   (f) The topological space $\{a,b,c\}$ with the topology $\Big\{\varnothing, \{a\}, \{a,b\}, \{a,b,c\}\Big\}$ does not have a sequence that converges to every element a, b, c.

   False. For example, the sequence $a, a, a, a, a, \ldots$ converges to every element.

   (g) The topological space $X = \{a,b,c,d\}$ with the topology $\Big\{\varnothing, \{a,b\}, \{c\}, \{a,b,c\}, \{a,b,d\}, X\Big\}$ does not have a 1-element dense subset.

   True. *Hint:* A subset D is dense if and only if $\overline{D} = X$. In this space, the closure of every singleton set is a proper subset of X,

   $$\overline{\{a\}} = \{a,b,d\}, \quad \overline{\{b\}} = \{a,b,d\}, \quad \overline{\{c\}} = \{c\}, \quad \overline{\{d\}} = \{d\}.$$

2. (21 points) (a) Let $X = \mathbb{N}$ with the topology $\{A \mid X \setminus A \text{ is finite}\} \cup \{A \mid 1 \notin A\}$.

(i) Circle all properties that hold for X. No justification needed.

(Hausdorff) **(T_1)** connected path-connected **(compact)**

(ii) For each of the following subsets S of X, compute the interior $\text{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S.

$S = \{1, 2\}$

$\text{Int}(S)$: $\{2\}$ $\overline{S}$: $\{1, 2\}$ ∂S: $\{1\}$ S': $\varnothing$

$S = \{2, 4, 6, 8, \dots\}$

$\text{Int}(S)$: S $\overline{S}$: $S \cup \{1\}$ ∂S: $\{1\}$ S': $\{1\}$

(iii) Circle "continuous" or "discontinuous" to indicate which of the following functions $f : X \to X$ are continuous.

$$f : X \longrightarrow X, \quad n \longmapsto n + 1$$

continuous **(discontinuous)**

$$f : X \longrightarrow X, \quad n \longmapsto \begin{cases} 1, & n \neq 2 \\ 2, & n = 2 \end{cases}$$

(continuous) discontinuous

(iv) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "Does not converge". **No justification necessary.**

$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \cdots$ Limits: $\{1\}$

$1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ \cdots$ Does not converge

(b) Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \left\{(a, \infty) \;\middle|\; a \in \mathbb{R}\right\} \cup \left\{[a, \infty) \;\middle|\; a \in \mathbb{R}\right\} \cup \left\{\varnothing, \mathbb{R}\right\}$.

(i) Circle all properties that hold for X. No justification needed.

`metrizable` `Hausdorff` T_1 (`connected`) `compact`

(ii) For each of the following subsets S of X, compute the interior $\mathrm{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S.

$S = \{1\}$

$\mathrm{Int}(S)$: $\varnothing$ $\overline{S}$: $(-\infty, 1]$ ∂S: $(-\infty, 1]$ S': $(-\infty, 1)$

$S = \{1, 2, 3, 4, 5, \dots\}$

$\mathrm{Int}(S)$: $\varnothing$ $\overline{S}$: X ∂S: X S': X

(iii) Circle "`continuous`" or "`discontinuous`" to indicate which of the following functions $f : X \to X$ are continuous.

$$f : X \longrightarrow X, \quad n \longmapsto 2n + 1$$

(`continuous`) `discontinuous`

$$f : X \longrightarrow X, \quad n \longmapsto 1 - n$$

`continuous` (`discontinuous`)

(iv) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

$\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ Limits: $(-\infty, 0]$

$\left(\frac{-1}{n}\right)_{n \in \mathbb{N}}$ Limits: $(-\infty, 0)$

3. (8 points) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set W. If $\mathcal{T} \subseteq \mathcal{T}'$, we say that the topology $\mathcal{T}$ is *coarser* than the topology $\mathcal{T}'$, and we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

   This means that any subset of W that is open in the coarser space $(W, \mathcal{T})$ is also open in $(W, \mathcal{T}')$. The finer topology $\mathcal{T}'$ has all the open subsets of $\mathcal{T}$, and may have additional open subsets.

   For each of the following: circle "`coarser`", "`finer`", neither, or both, according to which word truthfully completes the statement.

   (a) If a set $C \subseteq X$ is a closed subset of a topological space $(X, \mathcal{T})$, then C will also be closed with respect to any topology on X that is ________ than $\mathcal{T}$. `coarser` **`finer`**

   (b) If $(X, \mathcal{T})$ is Hausdorff, then X will also be Hausdorff with respect to any topology that is ________ than $\mathcal{T}$. `coarser` **`finer`**

   (c) If $(X, \mathcal{T})$ is compact, then X will also be compact with respect to any topology that is ________ than $\mathcal{T}$. **`coarser`** `finer`

   (d) If $(X, \mathcal{T})$ is connected, then X will also be connected with respect to any topology that is ________ than $\mathcal{T}$. **`coarser`** `finer`

   (e) If $\mathcal{T}$ is the indiscrete topology on a set X, then any other topology on X must be ________ than $\mathcal{T}$. `coarser` **`finer`**

   (f) If a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous, then f will still be continuous if the topology $\mathcal{T}_Y$ on the codomain Y is replaced by any ________ topology. **`coarser`** `finer`

   (g) If a sequence $(x_n)_{n \in \mathbb{N}}$ converges in a topological space $(X, \mathcal{T})$, it will also converge if the topology $\mathcal{T}$ is replaced by any ________ topology on X. **`coarser`** `finer`

   (h) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X, and $A \subseteq X$. Then the closure of A with respect to $\mathcal{T}_1$ will be contained in the closure of A with respect to $\mathcal{T}_2$ if $\mathcal{T}_1$ is ________ than $\mathcal{T}_2$. `coarser` **`finer`**

4. (4 points) Suppose that X is a nonempty topological space. Show that the following map is continuous with respect to the product topology on $X \times X$.

$$\begin{aligned} D : X \times X &\longrightarrow X \times X \\ (x, y) &\longmapsto (x, x) \end{aligned}$$

Solution. By definition, the product topology is the topology generated by the basis of sets $\{U \times V \mid U, V \text{ are open in } X\}$.

Worksheet #13 Theorem 1.4 states that a map $f : Z \to Y$ of topological spaces is continuous if and only if the preimage of every element in a basis for Y is open in Z. Thus, to check that D is continuous, it suffices to show that $D^{-1}(U \times V)$ is open in $X \times X$ for any pair of open subsets U, V of X. But

$$\begin{aligned} D^{-1}(U \times V) &= \{(x, y) \mid D(x, y) \in U \times V\} \\ &= \{(x, y) \mid (x, x) \in U \times V\} \\ &= \{(x, y) \mid x \in U \text{ and } x \in V\} \\ &= (U \cap V) \times X \end{aligned}$$

But the intersection $(U \cap V)$ of two open subsets of X is open in X. The product of open subsets of X is open in $X \times X$ by definition of the product topology. Thus $D^{-1}(U \times V)$ is open in $X \times X$, and we conclude that D is continuous.

Alternate Solution (outline). Observe that the map D factors as the composite of a the projection map π_1 of $X \times X$ onto its first factor, and the "diagonal map" Δ.

$$\begin{array}{ccccc} X \times X & \xrightarrow{\pi_1} & X & \xrightarrow{\Delta} & X \times X \\ (x, y) & \longmapsto & x & \longmapsto & (x, x) \end{array}$$

It is enough to check that both π_1 and Δ are continuous, since the composite of continuous function is continuous (Worksheet #13 Problem 1(b)). The projection map is continuous by Worksheet #16 Theorem 1.3. To check that the map Δ is continuous, by Worksheet #16 Problem 2, it suffices to observe that its component functions—both equal to the identity function on X—are continuous.

5. (5 points) Let X be a topological space. Suppose that X has the property that every point of X has a path-connected open neighbourhood. Prove that, if X is connected, then X is path-connected.

Hint: First prove that the path components of X are open in X.

Solution. We first prove a lemma.

> **Lemma.** Suppose that every point of X has a path-connected neighbourhood. Then the path-components of X are open.
>
> *Proof of Lemma.* Let P be a path component of X. To prove that P is open, it suffices to show that an arbitrary element $x \in P$ is an interior point of P . By our definition of path component, P is the set of all points y in X such that there exists a path from x to y. By assumption, there exists a path-connected open neighbourhood of x. Since every point in that neighbourhood admits a path from x, the neighbourhood must be contained in P. But this implies that x is an interior point of P. We conclude that P is open in X.

We now use the lemma to prove that, if X is connected, then it is path-connected. We prove the contrapositive: if X has more than one path component, then it is disconnected.

Suppose that X had more than one path component. Let P be a path component of X. We claim that P and $X \setminus P$ are a separation of X. By assumption that X has more than one path component, $X \neq P$, so both P and its complement are nonempty. It suffices to check that both P and its complement $X \setminus P$ are open. But this follows from the lemma: since P is a path component, and its complement is a union of path components, both sets are open in X. Thus P and $X \setminus P$ are a separation of X, and X is disconnected.