Final Exam

Math 490
16 December 2025
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Name: ____________________

Instructions: This exam has 5 questions for a total of 45 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	7	
2	21	
3	8	
4	4	
5	5	
Total:	45	

1. (7 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let $f : X \to Y$ be a function of sets. Then the binary relation
   $$a \sim b \quad \text{iff} \quad f(a) = f(b)$$
   satisfies all the axioms of an equivalence relation on X.

   (b) Let X be a metric space, and $(a_n)_{n \in \mathbb{N}}$ a Cauchy sequence in X. If this sequence has a convergent subsequence, then it converges.

   (c) Let $X = \mathbb{R}$. Then the collection of subsets $\mathcal{T} = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$ satisfies all the axioms of a topology on X.

   (d) Let X and Y be topological spaces. If X and Y are metrizable, then their product $X \times Y$ is metrizable.

   (e) There is no path from a to d in the topological space $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{d\}, \{a, d\}, \{a, c\}, \{a, c, d\}, X\Big\}$.

   (f) The topological space $\{a, b, c\}$ with the topology $\Big\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\Big\}$ does not have a sequence that converges to every element a, b, c.

   (g) The topological space $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a, b\}, \{c\}, \{a, b, c\}, \{a, b, d\}, X\Big\}$ does not have a 1-element dense subset.

2. (21 points) (a) Let $X = \mathbb{N}$ with the topology $\{A \mid X \setminus A \text{ is finite}\} \cup \{A \mid 1 \notin A\}$.

(i) Circle all properties that hold for X. No justification needed.

`Hausdorff` T_1 `connected` `path-connected` `compact`

(ii) For each of the following subsets S of X, compute the interior $\mathrm{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S.

$S = \{1, 2\}$

$\mathrm{Int}(S)$: ____________ $\overline{S}$: ____________ ∂S: ____________ S': ____________

$S = \{2, 4, 6, 8, \dots\}$

$\mathrm{Int}(S)$: ____________ $\overline{S}$: ____________ ∂S: ____________ S': ____________

(iii) Circle "`continuous`" or "`discontinuous`" to indicate which of the following functions $f : X \to X$ are continuous.

$$\begin{aligned} f : X &\longrightarrow X \\ n &\longmapsto n+1 \end{aligned}$$

`continuous` `discontinuous`

$$\begin{aligned} f : X &\longrightarrow X \\ n &\longmapsto \begin{cases} 1, & n \neq 2 \\ 2, & n = 2 \end{cases} \end{aligned}$$

`continuous` `discontinuous`

(iv) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \cdots$

$1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ \cdots$

(b) Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \Big\{ (a, \infty) \;\Big|\; a \in \mathbb{R} \Big\} \cup \Big\{ [a, \infty) \;\Big|\; a \in \mathbb{R} \Big\} \cup \Big\{ \varnothing, \mathbb{R} \Big\}$.

(i) Circle all properties that hold for X. No justification needed.

`metrizable` `Hausdorff` T_1 `connected` `compact`

(ii) For each of the following subsets S of X, compute the interior $\text{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S.

$S = \{1\}$

$\text{Int}(S)$: ________ $\overline{S}$: ________ ∂S: ________ S': ________

$S = \{1, 2, 3, 4, 5, \dots\}$

$\text{Int}(S)$: ________ $\overline{S}$: ________ ∂S: ________ S': ________

(iii) Circle "`continuous`" or "`discontinuous`" to indicate which of the following functions $f : X \to X$ are continuous.

$$\begin{aligned} f : X &\longrightarrow X \\ n &\longmapsto 2n+1 \end{aligned}$$

`continuous` `discontinuous`

$$\begin{aligned} f : X &\longrightarrow X \\ n &\longmapsto 1-n \end{aligned}$$

`continuous` `discontinuous`

(iv) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

$\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$\left(\frac{-1}{n}\right)_{n\in\mathbb{N}}$

3. (8 points) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set W. If $\mathcal{T} \subseteq \mathcal{T}'$, we say that the topology $\mathcal{T}$ is *coarser* than the topology $\mathcal{T}'$, and we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

   This means that any subset of W that is open in the coarser space $(W, \mathcal{T})$ is also open in $(W, \mathcal{T}')$. The finer topology $\mathcal{T}'$ has all the open subsets of $\mathcal{T}$, and may have additional open subsets.

   For each of the following: circle "`coarser`", "`finer`", neither, or both, according to which word truthfully completes the statement.

   (a) If a set $C \subseteq X$ is a closed subset of a topological space $(X, \mathcal{T})$, then C will also be closed with respect to any topology on X that is __________ than $\mathcal{T}$. `coarser` `finer`

   (b) If $(X, \mathcal{T})$ is Hausdorff, then X will also be Hausdorff with respect to any topology that is __________ than $\mathcal{T}$. `coarser` `finer`

   (c) If $(X, \mathcal{T})$ is compact, then X will also be compact with respect to any topology that is __________ than $\mathcal{T}$. `coarser` `finer`

   (d) If $(X, \mathcal{T})$ is connected, then X will also be connected with respect to any topology that is __________ than $\mathcal{T}$. `coarser` `finer`

   (e) If $\mathcal{T}$ is the indiscrete topology on a set X, then any other topology on X must be __________ than $\mathcal{T}$. `coarser` `finer`

   (f) If a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous, then f will still be continuous if the topology $\mathcal{T}_Y$ on the codomain Y is replaced by any __________ topology. `coarser` `finer`

   (g) If a sequence $(x_n)_{n \in \mathbb{N}}$ converges in a topological space $(X, \mathcal{T})$, it will also converge if the topology $\mathcal{T}$ is replaced by any __________ topology on X. `coarser` `finer`

   (h) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X, and $A \subseteq X$. Then the closure of A with respect to $\mathcal{T}_1$ will be contained in the closure of A with respect to $\mathcal{T}_2$ if $\mathcal{T}_1$ is __________ than $\mathcal{T}_2$. `coarser` `finer`

4. (4 points) Suppose that X is a nonempty topological space. Show that the following map is continuous with respect to the product topology on $X \times X$.

$$\begin{aligned} D : X \times X &\longrightarrow X \times X \\ (x, y) &\longmapsto (x, x) \end{aligned}$$

5. (5 points) Let X be a topological space. Suppose that X has the property that every point of X has a path-connected open neighbourhood. Prove that, if X is connected, then X is path-connected.

   Hint: First prove that the path components of X are open in X.