Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space. Show that any subset $A = \{x\} \subseteq X$ of a single element is connected.

2. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

   Is X connected?

3. (a) Show that, for $a, b \in \mathbb{R}$, the subsets $\varnothing, \{a\}, (a, b), (a, b], [a, b), [a, b], (a, \infty), [a, \infty), (\infty, b), (\infty, b]$, and $\mathbb{R}$ of $\mathbb{R}$ are all intervals in the sense of Problem 5.
   (b) Show that every interval must have one of these forms.

4. Give an example of a subset A of $\mathbb{R}$ (with the standard topology) such that A is not connected, but $\overline{A}$ is connected. (Compare to Assignment Problem 2).

5. Let X be a disconnected topological space. Let $f : X \to Y$ be a continuous function from X to a topological space Y. Show by example that $f(X)$ may be disconnected, or may be connected. Contrast with Worksheet #17 Problem 3.

Worksheet problems

(Hand these questions in!)

- Worksheet #16 Problems 2.

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T}_X)$ be a topological space, and endow the product $X \times X$ with the product topology $\mathcal{T}_{X \times X}$. The set

$$\Delta = \{ \ (x, x) \mid x \in X \ \} \ \subseteq \ X \times X$$

   is called the *diagonal* of $X \times X$. Prove that X is Hausdorff if and only if the diagonal Δ is a closed subset of $X \times X$.

2. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$ be a connected subset. Let B be any subset such that $A \subseteq B \subseteq \overline{A}$. Prove that B is connected.
   Remark: This shows in particular that if A is connected, then so is $\overline{A}$.

3. **Definition (Connected components of a topological space).** Let $(X, \mathcal{T}_X)$ be a topological space. A subset $C \subseteq X$ is called a *connected component* of X if
   (i) C is connected;
   (ii) if C is contained in a connected subset A, then $C = A$.

That is, the connected components are the 'maximal' connected subsets of X.

(a) Show that any connected component of X is closed. (*Hint*: Problem 2).

(b) Let $x \in X$. Show that the following set is a connected component of X:

$$\bigcup_{\substack{A \text{ is a connected set,} \\ x \in A}} A.$$

(c) Show that the connected components form a partition of X. In other words, show that every point of X is contained in one, and only one, connected component. *Remark:* This implies that the binary relation "$x \sim y$ if $x, y \in X$ are contained in a connected subset of X" is an equivalence relation on a space X.

(d) Determine–with proof–the connected components of $\mathbb{Q}$ (with the Euclidean metric).

(e) Deduce from the example of $\mathbb{Q}$ that connected components need not be open.

(f) Suppose that X has the property that every point has a connected neighbourhood. Show that the connected components of X are open.

4. In this problem, we will prove the following result:

> **Theorem (Connectivity of product spaces).** Let X and Y be nonempty topological spaces. Then the product space $X \times Y$ (with the product topology) is connected if and only if both X and Y are connected.

Hint: See Worksheet #17, Lemma 1.9.

(a) Suppose that $X \times Y$ is nonempty and connected in the product topology $\mathcal{T}_{X \times Y}$. Prove that X and Y are connected.

(b) Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are nonempty, connected spaces, and suppose that $(a, b) \in X \times Y$. Prove that $\Big(X \times \{b\}\Big) \cup \Big(\{a\} \times Y\Big)$ is a connected subset of the product $X \times Y$ with the product topology $\mathcal{T}_{X \times Y}$.

(c) Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are nonempty, connected spaces. Prove that $X \times Y$ is connected in the product topology $\mathcal{T}_{X \times Y}$.

5. (a) Consider $\{0, 1\}$ as a topological space with the discrete topology. Show that a topological space $(X, \mathcal{T})$ is disconnected if and only if there is a continuous **surjective** function $X \to \{0, 1\}$.

(b) You may cite the Intermediate Value Theorem from real analysis without proof:

> **Intermediate Value Theorem.** If $f : [a, b] \to \mathbb{R}$ is continuous and d lies between $f(a)$ and $f(b)$ (i.e. either $f(a) \leq d \leq f(b)$ or $f(b) \leq d \leq f(a)$), then there exists $c \in [a, b]$ such that $f(c) = d$.

Define a subset $A \subseteq \mathbb{R}$ to be an *interval* if whenever $x, y \in A$ with $x < y$, and $x < z < y$ for some $z \in \mathbb{R}$, then $z \in A$. Prove that any interval of $\mathbb{R}$ is connected.

(c) Prove that any subset of $\mathbb{R}$ that is not an interval is disconnected.

These last two results together prove:

> **Theorem (Connected subsets of $\mathbb{R}$).** A subset of $\mathbb{R}$ is a connected if and only if it is an interval.