Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X be a topological space, and let $x \in X$. Show that the constant sequence $a_n = x$ converges to x. Could it also converge to other points of X?

2. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$.
   (a) Which elements of X are limits of the constant sequence $x_n = d$? The constant sequence $x_n = a$? The constant sequence $x_n = b$?
   (b) Give an example of a sequence in X that does not converge.

3. Let $X = \{0, 1\}$. Find a topology on X for which the following sequence converges: $0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \cdots$ Can you find more than one topology?

4. Suppose that $(X, \mathcal{T})$ is a topological space, and that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to $a_\infty \in X$. Prove that any subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to $a_\infty \in X$.

5. Let X be a topological space with the indiscrete topology. Prove that any sequence of points in X converges to every point in X.

6. Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions.
   (a) Show that if f and g are both surjective, then $g \circ f$ is surjective.
   (b) Show that if f and g are both injective, then $g \circ f$ is injective.
   (c) Show by example that, if we only assume one of f and g is surjective, then $g \circ f$ need not be surjective.
   (d) Show by example that, if we only assume one of f and g is injective, then $g \circ f$ need not be injective.

7. Determine which of the following subsets of of $\mathbb{R}^2$ can be expressed as the Cartesian product of two subsets of $\mathbb{R}$.

   (a) $\{(x, y) \mid x \in \mathbb{Q}\}$ (c) $\{(x, y) \mid x > y\}$ (e) $\{(x, y) \mid x^2 + y^2 < 1\}$ (g) $\{(x, y) \mid x = 3\}$
   (b) $\{(x, y) \mid x, y \in \mathbb{Q}\}$ (d) $\{(x, y) \mid 0 < y \leq 1\}$ (f) $\{(x, y) \mid x^2 + y^2 = 1\}$ (h) $\{(x, y) \mid x + y = 3\}$

8. (a) For which values of r is the square $(-r, r) \times (-r, r) \subseteq \mathbb{R}^2$ contained in the unit ball $\{(x, y) \mid x^2 + y^2 < 1\}$?
   (b) For which values of r is the r-ball $\{(x, y) \mid x^2 + y^2 < r^2\}$ contained in the square $(-1, 1) \times (-1, 1) \subseteq \mathbb{R}^2$?

Worksheet problems

(Hand these questions in!)

- Worksheet #13 Problems 2.
- Worksheet #14 Problems 2.

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a basis for $\mathcal{T}$. For a subset $A \subseteq X$, prove that $\overline{A} = \{x \in X \mid \text{any basis element } B \in \mathcal{B} \text{ containing } x \text{ must contain a point of } A\}$.

2. For each of the following sequences: find the set of all limits, or determine that the sequence does not converge. No justification needed.

   - Let $X = \{a, b, c, d\}$ have the topology $\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}\}$.

     (i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$ (ii) $c,\ c,\ c,\ c,\ c,\ c,\ c,\ c,\ c, \ldots$

   - Let $\mathbb{R}$ have the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

     (iii) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ (iv) $(n)_{n\in\mathbb{N}}$ (v) $(-n)_{n\in\mathbb{N}}$

   - Let $\mathbb{R}$ have the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.

     (vi) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ (vii) $1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \cdots$

3. Let $f : X \to Y$ be a continuous map of topological spaces. If X is Hausdorff, does it follow that $f(X)$ (viewed as a subspace of Y with the subspace topology) must be Hausdorff? Give a proof, or find a counterexample.

4. (a) Suppose that $(X, \mathcal{T}_X)$ is a topological space, and that $(Y, \mathcal{T}_Y)$ is a **Hausdorff** topological space. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Suppose that $A \subseteq X$ is a subset such that
$$f(a) = g(a) \qquad \text{for all } a \in A.$$
Prove that
$$f(x) = g(x) \qquad \text{for all } x \in \overline{A}.$$
This says that the values of a continuous function on $\overline{A}$ are completely determined by its values on A.

   (b) Suppose you have a function of topological spaces $f : \mathbb{R} \to Y$, where $\mathbb{R}$ has the standard topology and Y is Hausdorff. In a sentence, explain why the previous problem implies that a continuous function f is completely determined by its values on $\mathbb{Q}$.

5. **Definition (path).** Let X be a topological space, and let $I = [0, 1] \subseteq \mathbb{R}$ be the unit interval with the standard topology. A *path* in X is a continuous function $\gamma : I \to X$. For points $x, y \in X$, we say that a path γ is a *path from x to y* if $\gamma(0) = x$ and $\gamma(1) = y$.

   Note that the path γ does not need to be injective. For example, the constant path $\gamma(t) = x$ is a path from x to x.

   For these problems—as always—you may assume any standard results about which functions between Euclidean spaces, with the standard topology, are continuous.

   (a) Given points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, construct a path from x to y.

(b) Let x, y be points in the space $\mathbb{R}$ with the cofinite topology. Construct (with justification) a path from x to y.

(c) Let $X = \{a, b, c, d\}$ be a topological space with the topology

$$\mathcal{T} = \Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\Big\}.$$

Construct (with justification) a path in X from a to d.

6. Let X be a topological space.

(a) Let $x, y \in X$ and suppose that there exists a path from x to y. Show that there exists a path from y to x.

(b) Let $x, y, z \in X$. Show that, if there exists a path from x to y, and a path from y to z, then there exists a path from x to z.
Hint: The Pasting Lemma from Homework #7 Assignment Problem 6.

(c) Let X be a topological space. We introduce some nonstandard notation that we will use in this question only: For $x, y \in X$, write $x \sim y$ if there exists a path from x to y. Deduce that the binary relation $\sim$ is:

(i) Reflexive: $x \sim x$ for all $x \in X$.

(ii) Symmetric: $x \sim y$ if and only if $y \sim x$ for all $x, y \in X$.

(iii) Transitive: For any $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

(d) A binary relation $\sim$ on a set X that is reflexive, symmetric, and transitive is called an *equivalence relation*. Prove that, given a set X with an equivalence relation $\sim$, the *equivalence classes*

$$[x] = \{y \in X \mid x \sim y\}$$

define a *partition* of X: this means that every element of X is contained in one and only one equivalence class. It implies that X is a union of its equivalence classes, and any two equivalence classes are either equal or disjoint.

In the case that X is a topological space and the relation $x \sim y$ is "existence of a path from x to y", the equivalence classes are called the *path components* of X.