Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show that $\mathcal{T}$ is a basis for $\mathcal{T}$.
   (b) Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Show that any collection of open sets in X containing $\mathcal{B}$ is also a basis for $\mathcal{T}$.

2. Verify that the set of open intervals $(a, b) \subseteq \mathbb{R}$ is a basis for $\mathbb{R}$ (with the Euclidean topology).

3. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Find examples of subsets A of $\mathbb{R}$ with the following properties. Can you find more than one?
   (a) $\mathrm{Int}(A) = \varnothing$
   (b) $\mathrm{Int}(A) = \mathbb{R}$
   (c) $\mathrm{Int}(A) = A$
   (d) $\mathrm{Int}(A) \neq A$
   (e) $\overline{A} = \varnothing$
   (f) $\overline{A} = \mathbb{R}$
   (g) $\overline{A} = A$
   (h) $A \neq \overline{A}$
   (i) $\partial A = \varnothing$
   (j) A has a nonempty boundary, and A contains its boundary ∂A.
   (k) A has a nonempty boundary, and A contains no points in its boundary
   (l) A has a nonempty boundary, and A contains some but not all of the points in its boundary.
   (m) $A = \partial A \neq \varnothing$.
   (n) A is a **proper** subset of ∂A.
   (o) $\partial A = \mathbb{R}$

Worksheet problems

(Hand these questions in!)

- Worksheet # 12 Problem 1.

Assignment questions

(Hand these questions in!)

1. For each of the following topological spaces X and subsets $A \subseteq X$, find (with brief justification) the interior $\mathrm{Int}(A)$, closure $\overline{A}$, boundary ∂A, and set of accumulation points A' of A.
   (a) $X = \mathbb{R}$ with the Euclidean metric, and A is the set $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.
   (b) $X = \mathbb{R}$ with the Euclidean metric, and A is the set of rational numbers $\mathbb{Q}$.
   (c) $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$, and $A = \{a, b\}$.
   (d) $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$, and $A = \{a, c, d\}$.
   (e) $X = (\mathbb{R}, \text{cofinite})$, and $A = \{0\}$.

   (f) $X = (\mathbb{R}, \text{cofinite})$, and $A = \mathbb{N}$.

2. Prove Worksheet #12 Theorem 1.8. *Note:* You may, but do not have to, prove the results in the order they are listed in the theorem statement.

3. Prove Worksheet #12 Theorem 1.9.

4. Let A, B be subsets of a topological space X.
   (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (b) Show by example that $\overline{A \cap B}$ need not equal $\overline{A} \cap \overline{B}$.
   (c) Show that $\partial(\mathrm{Int}(A)) \subseteq \partial A$, but show by example that these sets need not be equal.
   (d) Show that $\partial(\overline{A}) \subseteq \partial A$, but show by example that these sets need not be equal.

5. We proved (by Homework #2 Problem 5) that if A is a subset of a **metric** space X, and x is an accumulation point of A, then every neighbourhood of x contains infinitely many points of A. Is the same statement true for subsets of any topological space? Give a proof, or state a counterexample.

6. Prove the following. We will use this result for our study of *paths* in topological spaces.

   Lemma (The pasting lemma). Let $(X, \mathcal{T}_X)$ be a topological space, and suppose $X = A \cup B$ for two **closed** subsets $A, B \subseteq X$. Let $(Y, \mathcal{T}_Y)$ be a topological space and $f : X \to Y$ a function. If $f|_A$ and $f|_B$ are continuous with respect to the subspace topologies on A and B respectively, then f is continuous.