Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. See Assignment Problem 1 for the definition of the diameter of a set.
   (a) Show that a nonempty subset A of a metric space X has diameter 0 if and only if A is a single point.
   (b) Let X be a metric space, $x \in X$, and $r > 0$. Show that the open ball $B_r(x)$ has diameter at most $2r$.
   (c) Give an example of a metric spaces X and open balls $B_r(x) \subseteq X$ where $\operatorname{diam}(B_r(x)) = 2r$, and an example where $B_r(x) \subseteq X$ where $\operatorname{diam}(B_r(x)) < 2r$.
2. Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions.
   (a) Show that if f and g are both surjective, then $g \circ f$ is surjective.
   (b) Show that if f and g are both injective, then $g \circ f$ is injective.
   (c) Show by example that, if we only assume one of f and g is surjective, then $g \circ f$ need not be surjective.
   (d) Show by example that, if we only assume one of f and g is injective, then $g \circ f$ need not be injective.
3. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show that $\mathcal{T}$ is a basis for $\mathcal{T}$.
   (b) Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Show that any collection of open sets in X containing $\mathcal{B}$ is also a basis for $\mathcal{T}$.
4. Verify that the set of open intervals $(a, b) \subseteq \mathbb{R}$ is a basis for $\mathbb{R}$ (with the Euclidean topology).

Worksheet problems

(Hand these questions in!)

- Worksheet #10 Problem 3.
- Worksheet #11 Problem 1.

Assignment questions

(Hand these questions in!)

1. (a) Prove the following theorem.

   Theorem (distance is continuous). Let (X, d) be a metric space. Then the distance function
   $$d : X \times X \to \mathbb{R}$$
   is continuous with respect to the product metric on $X \times X$ and the Euclidean metric on $\mathbb{R}$.

(b) Given a metric space (X, d), the *diameter* $\operatorname{diam}(S)$ of a nonempty subset $S \subseteq X$ is the value

$$\operatorname{diam}(S) = \sup_{s,t \in S} d(s,t).$$

Note that the diameter $\operatorname{diam}(S)$ can be either a nonnegative real number or ∞. Prove that, if S is sequentially compact, then there exists $x, y \in S$ so that $d(x, y) = \operatorname{diam}(S)$.

2. Prove the following theorem.

   Theorem. A metric space X is complete if and only if it satisfies the following condition: For every nested sequence of nonempty closed subsets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ such that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$, the intersection $\bigcap_n A_n$ is nonempty.

3. Consider the following sets X and collection of subsets $\mathcal{T}$. For each $(X, \mathcal{T})$, either prove that $\mathcal{T}$ is a valid topology on X, or show that one of the axioms fails.
   (a) Let X be an infinite set. Let $\mathcal{T} = \{U \mid U \text{ is finite}\} \cup \{X\}$.
   (b) Let X be a set. The *cofinite* topology on X is the topology $\mathcal{T} = \{U \mid X \setminus U \text{ is finite}\} \cup \{\varnothing\}$. The word "cofinite" is a contraction of "complements are finite".
   (c) Let $X = \mathbb{R}$ and $\mathcal{T} = \{\, (a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$.
   (d) Let $X = \mathbb{R}$ and $\mathcal{T} = \{\, [a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$.
   (e) Let $X = \mathbb{R}$ and $\mathcal{T} = \{U \mid 0 \in U\} \cup \{\varnothing\}$.
   (f) Let $X = \mathbb{R}$ and $\mathcal{T} = \{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$.
   (g) Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\varnothing, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}\}$.

4. Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be a subset. Let $\mathcal{T}_S$ denote the subspace topology on S.
   (a) Show by example that an open subset of S (in the subspace topology $\mathcal{T}_S$) may not be open as a subset of X. In other words, show there could be a subset $U \subseteq S$ with $U \in \mathcal{T}_S$, $U \notin \mathcal{T}_X$.
   (b) Conversely, suppose that $U \subseteq S$ and U is open in X. Show that U **is** open in the subspace topology on S. In other words, for $U \subseteq S$, if $U \in \mathcal{T}_X$ then $U \in \mathcal{T}_S$.
   (c) Suppose that S is a an open subset of X. Show that a subset $U \subseteq S$ is open in S (with the subspace topology) if and only if it is open in X. In other words, whenever S is open and $U \subseteq S$, $U \in \mathcal{T}_S$ if and only if $U \in \mathcal{T}_X$.

5. (a) In our definition of a basis, we began with a space with a given topology, and defined a basis to be a collection of open subsets satisfying certain properties. In many cases, however, we will wish to topologize our set X by first specifying a basis, and using the basis to define a topology on X. Prove the following theorem, which gives conditions on a collection of subsets $\mathcal{B}$ of X that ensure it will generate a valid topology on X.

   Theorem (An extrinsic definition of a basis). Let X be a set and let $\mathcal{B}$ be a collection of subsets of X such that
   - $\displaystyle\bigcup_{B \in \mathcal{B}} B = X$
   - If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq (B_1 \cap B_2)$.

Let

$$\mathcal{T} = \{ \, U \mid U \text{ is a (possibly empty) union of elements of } \mathcal{B} \, \}.$$

Then $\mathcal{T}$ is a topology on X, and $\mathcal{B}$ is a basis for $\mathcal{T}$. We say that $\mathcal{T}$ is the topology *generated by* the basis $\mathcal{B}$.

(b) Let (X, d) be a metric space. Verify that the collection of open balls in X satisfies the criteria of the theorem. This gives a second proof that the collection of open sets that they generate does indeed satisfy the axioms of a topology.

6. **Definition (Dense subset).** Let X be a topological space and $A \subseteq X$. The subset A is called *dense* in X if every nonempty open subset of X contains a point of A.

(a) Suppose that X is metrizable. Show that $A \subseteq X$ is dense if and only if it satisfies the following condition:

For any x in X, there is a sequence of points in A converging to x.

It turns out that this equivalence does not hold for all topological spaces!

(b) Let $f : X \to Y$ be a continuous function of metric spaces. If $A \subseteq X$ is a dense subset, show that its image $f(A)$ is dense in $f(X)$.

(c) Show by example that the preimage of a dense set of Y under a continuous map need not be dense in X.

(d) Consider the set $\mathbb{R}$ and the subset $A = \mathbb{N}$. For which of the following topologies on $\mathbb{R}$ is the subset $\mathbb{N}$ dense? No justification needed.

(i) the Euclidean topology
(ii) the discrete topology
(iii) the indiscrete topology
(iv) the cofinite topology
(v) the topology $\{ \, (a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$
(vi) the topology $\{ \, (-\infty, a) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$
(vii) the topology $\{U \mid 1 \in U\} \cup \{\varnothing\}$
(viii) the topology $\{U \mid \frac{1}{2} \in U\} \cup \{\varnothing\}$
(ix) the topology $\{U \mid 1 \notin U\} \cup \{\mathbb{R}\}$