

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (Review from real analysis)

- Suppose that $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence of real numbers, and that for some $M \in \mathbb{R}$, $a_n \leq M$ for all n . Show that $\lim_{n \rightarrow \infty} a_n \leq M$.
 - Let S be a bounded set of real numbers, and L its supremum (i.e. its least upper bound). Show that there exist a sequence of elements $s_n \in S$ such that $\lim_{n \rightarrow \infty} s_n = L$.
 - Conversely, let S be a bounded set of real numbers, and L its supremum. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of elements of S that converge to a real number s_∞ . Show that $s_\infty \leq L$. In particular, if s_∞ is an upper bound, then $s_\infty = L$.
 - Let S be a bounded set of real numbers. Show that, if S is closed, it contains its supremum.
2. Let X be a set, and let d_1 and d_2 be two topologically equivalent metrics on X (Assignment Problem 6). In this question we verify that topologically equivalent metrics have the same convergent sequences, and the same continuous functions. Both of these conditions only depend on the open subsets of X , not on the exact metric!
- Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in X . Show that $(x_n)_{n \in \mathbb{N}}$ converges in (X, d_1) if and only if it converges in (X, d_2) .
 - Let (Y, d) be a metric space, and let $f : X \rightarrow Y$ be a function. Show that $f : (X, d_1) \rightarrow (Y, d)$ is continuous if and only if $f : (X, d_2) \rightarrow (Y, d)$ is continuous.

Worksheet problems

(Hand these questions in!)

- Worksheet #8 Problems 1, 2.

Assignment questions

(Hand these questions in!)

1. Let X and Y be (nonempty) metric spaces. Let $X \times Y$ be their Cartesian product endowed with the product metric. Let $S \subseteq X$ be a subset, viewed as a metric space itself under the restriction of the metric on X .
- For each of the following statements: prove the statement, or give a counterexample. You may state the counterexample without proof.
- If X is a finite set, then X is sequentially compact.
 - If X and Y are sequentially compact, then so is $X \times Y$.
 - If $X \times Y$ is sequentially compact, then so are X and Y .
 - If X is sequentially compact, then so is S .
 - If $A, B \subseteq X$ are sequentially compact, then so is $A \cup B$.
 - If some family of subsets $A_i \subseteq X$, ($i \in I$), are all sequentially compact, then so is $\bigcup_{i \in I} A_i$.

2. (a) Prove the following theorem.

Theorem (Continuous images of sequentially compact subsets). Let $f : X \rightarrow Y$ be a continuous function of metric spaces X and Y . If $C \subseteq X$ is sequentially compact, then $f(C) \subseteq Y$ is sequentially compact.

- (b) Let $f : X \rightarrow Y$ be a continuous function of metric spaces X and Y . State an example showing that, if for some $C \subseteq X$ the image $f(C)$ is sequentially compact, C need not be sequentially compact. No justification needed.
- (c) State an example showing by example that, if $D \subseteq Y$ is sequentially compact, then $f^{-1}(D)$ need not be sequentially compact. No justification needed.
3. (a) Suppose that B is a nonempty closed and bounded subset of $\mathbb{R}$. Explain why the Least Upper Bound Property of $\mathbb{R}$ implies that the supremum of B exists, and show that B contains its supremum.
- (b) Prove the following theorem. This theorem is one of the important reasons we care about sequential compactness!

Theorem (Extreme value theorem for metric spaces). Let (X, d) be a metric space and C a nonempty, sequentially compact subset of X . Let $f : X \rightarrow \mathbb{R}$ be a continuous function. Then there is a point $c \in C$ so that

$$f(c) = \sup_{x \in C} f(x).$$

In other words, the restriction $f|_C$ achieves its supremum.

4. In this problem, we will give a classification of subsets of $\mathbb{R}^n$ that are sequentially compact, and that are complete.

- (a) Recall the following result from real analysis (which you do not need to prove):

Theorem (Bolzano–Weierstrass). Let $A \subseteq \mathbb{R}^n$ be a bounded infinite set. Then A has an accumulation point $x \in \mathbb{R}^n$.

Prove the following result (which is sometimes also called the Bolzano–Weierstrass theorem):

Theorem (Sequential compactness in $\mathbb{R}^n$). Consider the space $\mathbb{R}^n$ with the Euclidean metric. Let $S \subseteq \mathbb{R}^n$ be a subset. Then S is sequentially compact if and only if S is closed and bounded.

- (b) Prove the following theorem.

Theorem ($\mathbb{R}^n$ is complete). The space $\mathbb{R}^n$ is complete with respect to the Euclidean metric.

Hint: Assignment Problems 5 & 6 on Homework #4.

- (c) Formulate and prove necessary and sufficient conditions on a metric subspace S of $\mathbb{R}^n$ (with the Euclidean metric) for it to be complete.
5. (a) Let (X, d) be a metric space. Suppose that for some $\epsilon > 0$, for all $x \in X$ the “closed ball”

$$C_\epsilon(x) := \{y \mid d(x, y) \leq \epsilon\}$$

is sequentially compact. Show that X is complete.

- (b) Suppose that X is a set with the discrete metric $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$.

Deduce that X is complete.

- (c) Consider the open interval $(0, 1)$ with the Euclidean metric. Show that, for every $x \in (0, 1)$, there exists some $\epsilon > 0$ so that $C_\epsilon(x)$ is sequentially compact. Deduce that this condition (where the choice of ϵ is allowed to depend on x) is not strong enough to guarantee completeness.

6. **Definition (Equivalent Metrics).** Let X be a set. Two metrics d_1 and d_2 on X are called *equivalent* (or *topologically equivalent*) if they induce the same topology on X . In other words, they are topologically equivalent if they satisfy the following condition: a subset $U \subseteq X$ is open in the metric space (X, d_1) if and only if it is open in the metric space (X, d_2) .

- (a) Let X be a nonempty finite set. Show that all metrics on X are topologically equivalent.
- (b) Consider the natural numbers $\mathbb{N}$. Show that $(\mathbb{N}, \text{Euclidean})$ and $(\mathbb{N}, \text{discrete})$ are topologically equivalent. Conclude that a bounded metric can be topologically equivalent to an unbounded metric.
- (c) Give an example of a set X and two metrics on X that are not topologically equivalent. No justification needed.
- (d) For a metric d on the set X , use the notation $B_r^d(x)$ to denote the open ball of radius r about a point x . Show that the metrics d_1 and d_2 are equivalent if and only if open balls “nest” in the following sense: for any point $x \in X$ and radius $r > 0$, there exists $r_1 > 0$ such that

$$B_{r_1}^{d_1}(x) \subseteq B_r^{d_2}(x)$$

and there exists $r_2 > 0$ such that

$$B_{r_2}^{d_2}(x) \subseteq B_r^{d_1}(x).$$

- (e) Use your solution to Homework #1 Problem 3 to give an informal argument that the following metrics are $\mathbb{R}^2$ are all equivalent.

$$d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

(In fact, the analogous metrics on $\mathbb{R}^n$ are all equivalent).