

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements in a metric space X , and let $a_\infty \in X$. Show that the following statement is equivalent to the statement that $\lim_{n \rightarrow \infty} a_n = a_\infty$.

For every $\epsilon > 0$, the ball $B_\epsilon(a_\infty)$ about a_∞ contains a_n for all but finitely many n .

2. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements in a metric space X .
 - (a) Let $x \in X$. Negate the definition of convergence to state what it means for the sequence to **not** converge to x .
 - (b) Formally state what it means for the sequence $(a_n)_{n \in \mathbb{N}}$ to be non-convergent.
3. Rigorously determine the limits of the following sequences of real numbers, or prove that they do not converge.

(a) $a_n = 0$ (b) $a_n = \frac{1}{n^2}$ (c) $a_n = n$ (d) $a_n = (-1)^n$

4. Suppose that $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are sequences of real numbers that converge to a_∞ and b_∞ , respectively. Prove that the sequence $(a_n + b_n)_{n \in \mathbb{N}}$ converges to $(a_\infty + b_\infty)$.
5. Show that a sequence is Cauchy if and only if it satisfies the following condition: For every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that $a_n \in B_\epsilon(a_N)$ for all $n \geq N$.
6. Consider the sequence of real numbers $1, 2, 3, 4, 5, \dots$. Which of the following are subsequences?
 - (a) $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$
 - (b) $1, 1, 1, 1, 1, 1, 1, \dots$
 - (c) $2, 4, 6, 8, 10, 12, \dots$
 - (d) $1, 3, 2, 4, 5, 7, 6, 8, 9, 11, \dots$

7. Let (X, d) be a metric space, and let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X . Recall that we proved that, if $\lim_{n \rightarrow \infty} a_n = a_\infty$, then any subsequence of $(a_n)_{n \in \mathbb{N}}$ also converges to a_∞ .
 - (a) Suppose that $(a_n)_{n \in \mathbb{N}}$ has a subsequence that does not converge. Prove that $(a_n)_{n \in \mathbb{N}}$ does not converge.
 - (b) Suppose that $(a_n)_{n \in \mathbb{N}}$ has a subsequence converging to $a \in X$, and a different subsequence converging to $b \in X$, with $a \neq b$. Prove that $(a_n)_{n \in \mathbb{N}}$ does not converge.
8. Let (X, d) be a metric space, and let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X . Show that, if $\{a_n \mid n \in \mathbb{N}\}$ is a finite set, then $(a_n)_{n \in \mathbb{N}}$ must have a subsequence that is constant (and, in particular, convergent).
9. Let (X, d) be a metric space, and let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X . Suppose that the set $\{a_n \mid n \in \mathbb{N}\}$ is unbounded. Explain why $(a_n)_{n \in \mathbb{N}}$ cannot converge.
10. Find examples of sequences $(a_n)_{n \in \mathbb{N}}$ of real numbers with the following properties.
 - (a) $\{a_n \mid n \in \mathbb{N}\}$ is unbounded, but $(a_n)_{n \in \mathbb{N}}$ has a convergent subsequence
 - (b) $(a_n)_{n \in \mathbb{N}}$ has no convergent subsequences

- (c) $(a_n)_{n \in \mathbb{N}}$ is not an increasing sequence, but it has an increasing subsequence
 (d) $(a_n)_{n \in \mathbb{N}}$ has four subsequences that each converge to a distinct limit point

11. We defined *bounded* as follows.

Definition (Bounded subset.) Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $R \in \mathbb{R}$ such that $d(x, y) < R$ for all $x, y \in S$.

Show this is equivalent to the following definitions.

Definition (Bounded subset.) Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.

Definition (Bounded subset.) Let (X, d) be a metric space, and $S \subseteq X$ a subset. Then S is *bounded*, for **every** $x \in S$, there is some $R_x > 0$ such that $S \subseteq B_{R_x}(x)$.

12. (a) Negate the definition of *bounded* to state what it means for a subset S of a metric space to be *unbounded*.
 (b) Is $\emptyset$ a bounded set?
 (c) Show that any **finite** subset of a metric space is bounded.
13. Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set. Show that any subset of S is bounded.
14. Let (X, d_X) and (Y, d_Y) be bounded metric spaces. Show that $(X \times Y, d_{X \times Y})$ is bounded. What if we only assumed X is bounded?
15. Give examples of subsets of $\mathbb{R}$ (with the Euclidean metric) that satisfy the following.
- | | |
|-------------------------|---------------------------|
| (a) open, and bounded | (c) open, and unbounded |
| (b) closed, and bounded | (d) closed, and unbounded |
16. Write down a bounded sequence in $\mathbb{R}$. Can you identify a convergent subsequence?
17. Give an example of a metric space (X, d) and a continuous function $f : X \rightarrow \mathbb{R}$ such that f has a finite supremum on X , but f does not achieve its supremum at any point $x \in X$.

Worksheet problems

(Hand these questions in!)

- Worksheet #6, Problem 1.
- Worksheet #7, Problem 2.

Assignment questions

(Hand these questions in!)

- Let $f : X \rightarrow Y$ be a continuous function of metric spaces. For each of the following statements, either prove the statement, or state a counterexample. You may state the counterexample without proof.
 - If $U \subseteq X$ is open, then $f(U) \subseteq Y$ is open.
 - If $C \subseteq X$ is closed, then $f(C) \subseteq Y$ is closed.
 - If $B \subseteq X$ is bounded, then $f(B) \subseteq Y$ is bounded.
 - If $D \subseteq Y$ is bounded, then $f^{-1}(D) \subseteq X$ is bounded.

Hint: Below are some continuous functions that are useful sources of counterexamples. Subsets of $\mathbb{R}$ have the Euclidean metric unless otherwise noted.

$$\begin{array}{ll}
 (\mathbb{R}, \text{discrete metric}) \longrightarrow (\mathbb{R}, \text{Euclidean}) & \mathbb{R} \longrightarrow (-1, 1) \\
 x \longmapsto x & x \longmapsto \frac{\arctan(x)}{\pi/2} \\
 \\
 (0, \infty) \longrightarrow (0, \infty) & \mathbb{R} \longrightarrow \mathbb{R} \\
 x \longmapsto \frac{1}{x} & x \longmapsto 0
 \end{array}$$

- This question describes a general strategy to construct a sequence that converges to a given point in a metric space—a strategy that we will use frequently. This observation will be useful in the upcoming problems!

Lemma. Let X be a metric space and $x \in X$. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence of points in X with the property that $a_n \in B_{\frac{1}{n}}(x)$. Then $(a_n)_{n \in \mathbb{N}}$ converges to x .

- In this question, we will prove the following result.

Theorem (Another characterization of closed subsets). Let (X, d) be a metric space, and let $A \subseteq X$. Then A is closed if and only if it satisfies the following condition: If $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence of points in A converging to a point $a_\infty \in X$, then the limit a_∞ is contained in A .

- Suppose that $A \subseteq X$ is closed. Let a_∞ be the limit of a convergent sequence $(a_n)_{n \in \mathbb{N}}$ of points in A . Show that $a_\infty \in A$.
 - Suppose that A is not closed. Show that there exists a sequence of elements $(a_n)_{n \in \mathbb{N}}$ in A that converges to a point $x \in X$ not contained in A .
- Prove the following result:

Theorem (Another definition of continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \rightarrow Y$ be a function. Then f is continuous if and only if the following condition holds: given any convergent sequence $(a_n)_{n \in \mathbb{N}}$ in X , then $(f(a_n))_{n \in \mathbb{N}}$ converges in Y , and

$$\lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right).$$

5. Let (X, d) be a metric space, and let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X . Prove that, if $(a_n)_{n \in \mathbb{N}}$ is Cauchy, then $\{a_n \mid n \in \mathbb{N}\}$ is a bounded subset of X . This shows in particular that the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded for any convergent sequence $(a_n)_{n \in \mathbb{N}}$ in X .
6. Let (X, d) be a metric space. Prove that any bounded subset S of X is contained in a *closed* and bounded subset.
Hint: You proved on Quiz #2 that the subset $\{x \in X \mid d(x, x_0) \leq r\}$ is closed for any $r > 0$ and $x \in X$.
7. Prove the following proposition.

Proposition (Closed subsets of sequentially compact spaces). Suppose that (X, d) is a sequentially compact metric space, and $C \subseteq X$ a closed subset. Then C is sequentially compact.

8. Prove the following.

Theorem (sequentially compact $\implies$ complete). If a metric space is sequentially compact, then it is complete.