Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Rigorously prove that the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous. (Here, $\mathbb{R}$ implicitly has the Euclidean metric.)

   (a) $f(x) = 5$ (b) $f(x) = 2x + 3$ (c) $f(x) = x^2$

   (d) $f(x) = g(x) + h(x)$, for continuous functions g and h.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 + 2$. Find the preimages of the following sets, and verify that they are open.

   (a) $\mathbb{R}$ (b) $(-1, 1)$ (c) $(2, 3)$ (d) $(6, \infty)$

3. Let (X, d) be a metric spaces. Show that the identity function

$$\begin{aligned} g : X &\longrightarrow X \\ g(x) &= x \qquad \text{for all } x \in X \end{aligned}$$

   is always continuous.

4. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $y_0 \in Y$. Show that the constant function

$$\begin{aligned} f : X &\longrightarrow Y \\ f(x) &= y_0 \qquad \text{for all } x \in X \end{aligned}$$

   is always continuous.

Worksheet Problems

(Hand these questions in!)

- Worksheet 5 Problem 1.

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $A, B \subseteq X$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f(A \cap B) \quad \square \quad f(A) \cap f(B)$ (b) $f(A \cup B) \quad \square \quad f(A) \cup f(B)$

   (c) For $A \subseteq B$, $\quad f(B \setminus A) \quad \square \quad f(B) \setminus f(A)$

2. **(Closed subsets of product spaces).** Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $C \subseteq X$ and $D \subseteq Y$ are closed subsets. Prove or find a counterexample: the subset $C \times D \subseteq X \times Y$ is closed with respect to the product metric.

3. **(Restrictions of functions).** Let $f : X \to Y$ be a function between metric spaces. We proved that a subset $S \subseteq X$ inherits a metric space structure from the metric on X. Recall that the *restriction* of f to S, often written $f|_S$, is the function

$$f|_S : S \longrightarrow Y$$
$$f|_S(s) = f(s).$$

   Prove the following result.

   Theorem (Restrictions of continuous functions). Let X and Y be metric spaces, and $S \subseteq X$ a metric subspace. If $f : X \to Y$ is a continuous function, then $f|_S : S \to Y$ is continuous.

4. Prove the equivalent definition of continuity.

   Theorem (Equivalent definition of continuity via open balls.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if it satisfies the following property: given any open ball $B = B_r(y) \subseteq Y$, its preimage $f^{-1}(B)$ is an open subset of X.

5. **Definition (Projection maps).** For a product of sets $X \times Y$, the maps

$$\pi_X : X \times Y \to X \qquad \pi_Y : X \times Y \to Y$$
$$\pi_X(x, y) = x \qquad \pi_Y(x, y) = y$$

   are called the *projection onto* X and the *projection onto* Y, respectively.

   Let (X, d_X) and (Y, d_Y) be metric spaces, and endow their product $X \times Y$ with the product metric $d_{X \times Y}$. Show that the projection map

$$\pi_X : (X \times Y, d_{X \times Y}) \longrightarrow (X, d_X)$$

   is continuous. (The same argument, which you do not need to repeat, shows that the map π_Y is continuous).

6. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces. View the product $X \times Y$ as a metric space with the product metric $d_{X \times Y}$. Suppose that $f_X : Z \to X$ and $f_Y : Z \to Y$ are functions. Show that the function

$$F : Z \longrightarrow X \times Y$$
$$z \longmapsto (f_X(z), f_Y(z))$$

   is continuous if and only if $f_X : Z \to X$ and $f_Y : Z \to Y$ are continuous.

7. Let $f : X \to Y$ be a function between metric spaces. Prove that, if X is a **finite** metric space (this means that X is finite as a set), then the function f must be continuous.