

Worksheet problems

(Hand these questions in!)

- Worksheet #19 Problems 2, 3.
- Worksheet #20 Problems 1, 2.

Assignment questions

(Hand these questions in!)

1. Prove the following result. This theorem is a major reason we care about compactness!

Theorem (Generalized Extreme Value Theorem). Let X be a nonempty compact topological space, and let $f : X \rightarrow \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has the standard topology). Then $\sup(f(X)) < \infty$, and there exists some $z \in X$ such that $f(z) = \sup(f(X))$. That is, f achieves its supremum on X .

Hint: See Worksheet #19 Theorem 1.7, Homework #5 Problem 3(a), and Worksheet #20 Theorem 1.3.

2. Let X be a metric space. Let $f : X \rightarrow X$ a continuous function such that $f(x) \neq x$ for all $x \in X$. Prove that, if X is compact, there exists some $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.
3. **Definition (Normal space).** A topological space X is *normal* if, for any disjoint closed subsets $C, D \subseteq X$, there exist disjoint open subsets U_C and U_D of X with $C \subseteq U_C$ and $D \subseteq U_D$.
 - (a) Suppose that X is metrizable. Prove that X is normal.
 - (b) Suppose that X is a compact Hausdorff space. Prove that X is normal.
4. **Definition (Limit point compactness)** A topological space X is called *limit point compact* if every infinite subset of X has an accumulation point.
 - (a) Suppose that a space X is limit point compact. Show that any closed subset C of X is limit point compact with respect to its subspace topology.
 - (b) Show that the union of finitely many limit point compact subspaces is limit point compact.
5. You proved on Quiz #10 that every compact space is limit point compact. In this problem, we will see that the converse to this statement fails.
 - (a) Let $X = \{a, b\}$ with the indiscrete topology, and let $\mathbb{N}$ have the usual Euclidean topology (i.e., the discrete topology). Describe the product topology on $X \times \mathbb{N}$.
 - (b) Verify that every nonempty subset of $X \times \mathbb{N}$ has an accumulation point.
 - (c) Show that $X \times \mathbb{N}$ is not compact.